MINIMALLY ALMOST PERIODIC
TOTAL DISCONNECTED GROUPS

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Abstract. In this paper we prove that every closed noncompact group $G$ of isometries of a homogeneous tree which acts transitively on the tree boundary contains a normal closed cocompact subgroup $G'$ which is minimally almost periodic. Moreover we prove that $G'$ is a topologically simple group.

1. Introduction

Let $X$ be a homogeneous tree of finite order $q + 1 \geq 3$. We denote by Aut$(X)$ the locally compact group of all isometries of $X$ with respect to the natural distance of $X$ ($d(x, y)$ is the length of the unique geodesic connecting $x$ to $y$). We refer the reader to [2] for undefined notions and terminology. We fix $x_0 \in X$; then the sets $X^+ = \{x \in X : d(x, x_0) \text{ is even}\}$ and $X^- = \{x \in X : d(x, x_0) \text{ is odd}\}$ are the equivalence classes of the relation “$d(x, y)$ is an even number”. Therefore this partition of $X$ into the sets $X^+$ and $X^-$ is independent of the choice of $x_0$. If $G$ is a closed noncompact subgroup of Aut$(X)$ acting transitively on the tree boundary $\Omega$, then either $G$ acts transitively on $X$ or $G$ has exactly the orbits $X^+$ and $X^-$ [4, Prop. 2, pg. 143]. In particular if $G$ has two orbits $X^+$ and $X^-$, then every closed noncompact subgroup of $G$ acting transitively on $\Omega$ has the same orbits of $G$. A notable example of this type is the subgroup Aut$^+(X)$ of Aut$(X)$ generated by all rotations of $X$. More generally, let $G$ be a closed subgroup of Aut$(X)$ acting transitively on $X$ and $\Omega$. Then the subgroup $G^+$ generated by all rotations of $G$ is an open normal subgroup of $G$ of index 2 acting transitively on $\Omega$ and having two orbits ($X^+$ and $X^-$) on $X$. In [8] J. Tits has proved that Aut$^+(X)$ is an algebraically simple group. Furthermore, J. Tits proved that the group $G^+$ is algebraically simple for a larger class of groups with property (P) (see [8, 4.2, pg. 197]).

Let $G$ be a locally compact group; then $G$ is said to be minimally almost periodic (briefly: m.a.p.) if every finite-dimensional unitary representation is trivial. This is equivalent to the fact that there is no continuous almost periodic function except constant functions.

In the present paper we consider the class $\mathcal{G}$ of all closed subgroups $G$ of Aut$(X)$ with the following properties: $G$ acts transitively on $\Omega$ and $G$ has two orbits on $X$. We prove that every group $G \in \mathcal{G}$ contains one and only one nontrivial normal closed subgroup $G' \in \mathcal{G}$ which is m.a.p., cocompact and topologically simple. This
implies that a group $G \in \mathcal{G}$ is m.a.p. if and only if $G$ is topologically simple. If $G$ acts transitively on $X$ and $\Omega$, then $G^+ \in \mathcal{G}$, therefore also $G$ contains one and only one topologically simple m.a.p. subgroup in $\mathcal{G}$.

2. The results

Let $G$ be a closed subgroup of $\text{Aut}(X)$. Let $H$ be a closed subgroup of $G$. Let $v \in X$ be a fixed vertex of $X$ and $K_v = \{g \in G : g(v) = v\}$. $K_v$ is a compact open subgroup of $G$.

**Proposition 1.** The space $G/H$ is compact if and only if the orbit $G(v)$ is the union of finitely many orbits of $H$; that is, there exist $x_1, x_2, \ldots, x_n \in G(v)$, such that $G(v) = H(x_1) \cup H(x_2) \cup \cdots \cup H(x_n)$.

**Proof.** Let $\{K_v gH\}$ be the partition of $G$ into the double cosets $K_v gH$ for $g \in G$. Since $K_v gH$ is an open set of $G$ and $p(K_v gH) = p(K_v g)$ is a compact open subset of $G/H$ for every $g \in G$ (where $p : G \rightarrow G/H$ is the canonical map), then it is easy to see that $G/H$ is compact if and only if the partition $\{K_v gH\}_{g \in G}$ has only finitely many sets. Therefore the proposition follows from the fact that the map $\Lambda(K_v gH) = H(g^{-1}v)$ is a bijective map of the double cosets of the partition onto the set of $H$-orbits contained in $G(v)$.

**Remark.** In particular Proposition 1 implies that if $G$ acts transitively on $X$, then $G/H$ is a compact space if and only if $H$ has finitely many orbits on $X$.

**Definition 1.** Let $\mathcal{G}$ be the class of all closed subgroups $G$ of $\text{Aut}(X)$ with the following properties:

1. $G$ acts transitively on the tree boundary $\Omega$,
2. $G$ has exactly two orbits on $X$, that is, $X^+$ and $X^-$.

If $W \subseteq K_v$ is a subset acting transitively on $\Omega$ and $g$ is a translation of even step, then the closed subgroup generated by $W$ and $g$ is in $\mathcal{G}$. On the other hand, as observed in the introduction, if $G$ acts transitively on $\Omega$ and on $X$, then $G^+ \in \mathcal{G}$. The reader is referred to [2, pg. 31–32, 133–134] for examples. By [4, Prop. 2, pg. 143] it follows that if $G \in \mathcal{G}$ and $H$ is a closed noncompact subgroup of $G$ acting transitively on $\Omega$, then $H \in \mathcal{G}$.

**Lemma 1.** Let $G$ be in the class $\mathcal{G}$, and let $H$ be a closed nontrivial normal subgroup of $G$; then $H \in \mathcal{G}$. In particular $G/H$ is a compact group. If in addition $H$ is open, then $G/H$ is a finite group.

**Proof.** First we observe that $H$ is not compact. In fact since $G$ has two orbits on $X$, then $G$ contains no inversion (an inversion interchanges $X^+$ and $X^-$). Therefore every compact subgroup of $G$ fixes a vertex $v \in X$ [2, Theorem 5.2, pg. 12]. But if $H \subseteq K_v$, then $H = gHg^{-1} \subseteq gK_v g^{-1} = K_{g(v)}$ for all $g \in G$, which means that $H$ fixes $X^+$ or $X^-$ and so $H$ fixes $X$. This is impossible because $H$ is not trivial. We prove now that $H$ acts transitively on $\Omega$. Since $H$ is not trivial, then there exist $h \in H$ and $\omega \in \Omega$ such that $\omega \neq h(\omega)$. Let $\omega'$ be an end of $\Omega$ such that $\omega' \neq h(\omega)$. Since $G$ acts doubly transitively on $\Omega$ (see [2, pg. 29–30]) there exists $g \in G$ such that $g(\omega) = \omega$ and $g(h(\omega)) = \omega'$. Therefore $ghg^{-1}(\omega) = \omega'$ and $\omega' \in H(\omega)$ because $ghg^{-1} \in H$. So $H(\omega) = \Omega$. The lemma follows from [2, Prop. 10.2, pg. 27] and Proposition 1.
Lemma 2. Let $G$ be in the class $\mathcal{G}$; let $\{H_n\}$ be a sequence of open normal subgroups of $G$. Then $\bigcap_{n=1}^{\infty} H_n$ is a nontrivial subgroup of $G$.

Proof. The proof is similar to the proof of [1, Prop. 16.4.4, pg. 302]. We suppose, on the contrary, that $\bigcap_{n=1}^{\infty} H_n$ is trivial. Replacing, if necessary, $H_n$ by $H_1 \cap H_2 \cap \cdots \cap H_n$ we may assume that $H_{n+1} \subseteq H_n$. $G$ contains a translation $w$ [2, Th. 8.1, p. 20]; let $K_v$ be the stability subgroup of a vertex $v$; let $U = K_v \cup wK_v \cup K_vw^{-1}$. Therefore $U$ is a compact open symmetric neighborhood of the identity $e$ of $G$. Since $U^n \subseteq U^{n+1}$, then $\bigcup_{n=1}^{\infty} U^n$ is a noncompact open subgroup of $G$, in fact the subgroup of $G$ generated by $U$. Since $K_v \subseteq U$, then $\bigcup_{n=1}^{\infty} U^n$ acts transitively on $\Omega$ [4, Prop. 1, pg. 143]. Therefore $\bigcup_{n=1}^{\infty} U^n \in \mathcal{G}$ [4, Prop. 2, pg. 143], that is, $\bigcup_{n=1}^{\infty} U^n$ has the same orbits of $G$. The fact that $K_v \subseteq U$ implies that $G = \bigcup_{n=1}^{\infty} U^n$. The sequence $H_n \cap U^3$ is a sequence of compact open subgroups of $U^3$ such that $\bigcap_{n=1}^{\infty} (H_n \cap U^3) = \{e\}$. Since $e \in U \subseteq U^3$, it follows that there exists $m$ such that $H_m \cap U^3 \subseteq U$. This implies that $H = H_m \cap U^3$ is a compact open subgroup of $G$. We prove now that $H$ is a normal subgroup of $G$. Indeed, if $t \in U$ and $h \in H \subseteq U$, then $tht^{-1} \in U^3 \cap H_m = H$ and $tHt^{-1} \subseteq H$. But $U$ is symmetric and $tHt^{-1} = H$ for every $t \in U$. Since $G = \bigcup_{n=1}^{\infty} U^n$, then $gHg^{-1} = H$ for every $g \in G$. As observed in the first part of the proof of Lemma 1, this is impossible because $G$ is not discrete and $H$ is not trivial.

Let $\hat{G}$ be the set of equivalence classes of unitary continuous irreducible representations of $G$. If $G$ is a totally disconnected group, then $\text{Ker} \pi$, the kernel of the representation $\pi$, is a normal open subgroup of $G$ for every unitary continuous finite dimensional representation $\pi$ [2, Prop. 1.2, pg. 86].

Definition 2. For $G \in \mathcal{G}$, we define:

$\mathcal{A}(G) = \{H: H$ is a nontrivial normal closed subgroup of $G\}$,

$\mathcal{B}(G) = \{H: H$ is a normal open subgroup of $G\}$,

$\mathcal{C}(G) = \{\text{Ker} \pi: \pi \in \hat{G}$ and $\dim \pi < +\infty\}$.

We have $\mathcal{C}(G) \subseteq \mathcal{B}(G) \subseteq \mathcal{A}(G)$. If $G \in \mathcal{G}$ and $H \in \mathcal{A}(G)$, then $G/H$ is a compact group. In particular, if $p: G \to G/H$ is the natural homomorphism, then $p \circ \pi$ is a finite dimensional irreducible representation of $G$ for every $\pi \in (G/H)$. Therefore $\text{Ker}(p \circ \pi) \in \mathcal{C}(G)$ and $\bigcap_{\pi \in (G/H)} \text{Ker}(p \circ \pi) = H$. This means that

$$\bigcap_{H \in \mathcal{C}} H = \bigcap_{H \in \mathcal{B}} H = \bigcap_{H \in \mathcal{A}} H.$$

We put $G' = \bigcap_{H \in \mathcal{C}} H = \bigcap_{H \in \mathcal{B}} H = \bigcap_{H \in \mathcal{A}} H$. $G'$ is a closed normal subgroup of $G$ and, by Lemma 2, $G' \neq \{e\}$, hence $G' \in \mathcal{G}$. In fact $G$ is separable; therefore there exists a sequence $K_{n+1} \subseteq K_n$ of compact open subgroups of $G$ which is a basis of neighborhoods of the identity of $G$. Let $H_n$ be the open normal subgroup of $G$ generated by $K_n$. $G/H_n$ is finite and every $H \in \mathcal{B}(G)$ contains $H_n$ for $n$ sufficiently large. This proves that $\mathcal{B}(G)$ is finite or countable. In particular $G' \neq \{e\}$ by Lemma 2.

We summarize the above facts in the following proposition.

Proposition 2. Let $G' = \bigcap_{H \in \mathcal{A}} H$; then $G' \in \mathcal{A}(G)$. In particular $G' \subseteq H$ for every $H \in \mathcal{A}(G)$. Moreover $G' = \bigcap_{H \in \mathcal{C}} H = \bigcap_{H \in \mathcal{B}} H = \bigcap_{H \in \mathcal{A}} H$. 

Theorem. Let $G \in \mathcal{G}$, and let $G'$ be as in Proposition 2. Then $G' \in \mathcal{G}$, $G'$ is a topologically simple group.

Proof. The fact that $G' \in \mathcal{G}$ follows from Lemma 1. Let $H$ be a nontrivial closed normal subgroup of $G'$. We prove now that $H = G'$. $G'$ is a normal subgroup of $G$, therefore, for every $g \in G$, $gHg^{-1}$ is a nontrivial closed normal subgroup of $G'$. Proposition 2 implies that $G'' \subseteq gHg^{-1}$ for every $g \in G$ where $G'' = (G')'$ (we recall that if $G \in \mathcal{G}$, then $G' \in \mathcal{G}$ and $G'' \neq \{e\}$). In particular $\{e\} \neq G'' \subseteq \bigcap_{g \in G} gHg^{-1}$. This means that $\bigcap_{g \in G} gHg^{-1}$ is a normal closed nontrivial subgroup of $G$, and so $G' \subseteq \bigcap_{g \in G} gHg^{-1} \subseteq H \subseteq G'$. Hence $H = G'$ and theorem follows.

Remarks. 1) $G'$ is topologically simple, therefore $G'' = G'$. 2) The compact group $G/G'$ is the compact group associated with $G$ in the sense of [1, Th. 16.1.1, pg. 296] and the canonical surjection $p: G \to G/G'$ is the canonical morphism of $G$ [1, Th. 16.1.1, pg. 296]. In particular a bounded continuous function $f$ on $G$ is almost periodic iff $f$ is $G'$-invariant. As $\pi$ varies among all finite dimensional representations of $G/G'$ (as $\pi$ varies in $(G/G')^\sim$), $p \circ \pi$ describes all finite dimensional representations of $G$ (all finite dimensional irreducible representations of $G$). 3) Obviously $G''$ is open in $G$ iff $G$ has only finitely many classes of finite dimensional irreducible representations.

Corollary 1. Let $G \in \mathcal{G}$; then the following are equivalent.

1) $G$ is m.a.p.
2) $G$ is topologically simple.
3) $G = G'$.

Proof. The corollary follows from Remark 2) above and the fact that $G$ is topologically simple iff $G = G'$, that is, iff $(G/G')^\sim$ is trivial.

Corollary 2. Let $G$ be a closed subgroup of $\text{Aut}(X)$ acting transitively on $X$ and $\Omega$. Let $G^+ = \{g \in G : d(x, g(x))$ is even\}. Then $G^+ \in \mathcal{G}$ is topologically simple iff $G$ has exactly two (classes of) unitary irreducible finite dimensional representations, that is, the trivial character $(\chi(g) = 1 \text{ for every } g \in G)$ and the character $\chi^+$ (where $\chi^+(g) = 1 \text{ for every } g \in G^+$ and $\chi^+(g) = -1 \text{ for every } g \notin G^+$).

Proof. Because $G/G^+$ is the cyclic group of order 2, then an elementary argument of induced representations proves that the set of irreducible finite dimensional representations of $G$ is $\{\chi, \chi^+\}$ iff every irreducible finite dimensional representation of $G^+$ is trivial. Therefore Corollary 2 follows from Corollary 1.

Remarks. 1) If $G$ acts transitively on $X$ and $\Omega$ and it satisfies the property (P) of Tits [8], then $G^+ = G'$.

2) Let $\mathbb{Q}_p$ be the field of the $p$-adic numbers; the group $\text{PGL}(2, \mathbb{Q}_p)$ can be embedded into the group of all isometries of some homogeneous tree in such a way that it acts transitively on $X$ and $\Omega$ [7]. In this case $\text{PGL}(2, \mathbb{Q}_p)^+$ is not simple but $$(\text{PGL}(2, \mathbb{Q}_p))^+ = \text{PSL}(2, \mathbb{Q}_p) = [\text{PGL}(2, \mathbb{Q}_p), \text{PGL}(2, \mathbb{Q}_p)]$$

where $[\cdot, \cdot]$ means the commutator subgroup. The finite dimensional irreducible representations of $\text{PGL}(2, \mathbb{Q}_p)$ are in fact characters [3, Prop. 2.7, pg. 31].
3) For $G = \text{Aut}(X)$ or $G = \text{PGL}(2, \mathbb{Q}_p)$ we have that $G' = [G, G]$ and $G'$ is open in $G$. This means that every finite dimensional irreducible representation of $G$ is a character and the set of characters must be finite. For $G = \text{PGL}(2, \mathbb{Q}_p)$ this is a consequence of the fact that every open normal subgroup of $\text{PGL}(2, \mathbb{Q}_p)$ contains the group $\langle \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \rangle$ (see the first part of the proof of [3, Prop. 2.7, pg. 31]).

4) Let $G$ be a closed subgroup of $\text{Aut}(X)$ acting transitively on $X$ and $\Omega$, and let $\omega \in \Omega$. We define $B^G_\omega$ as the subgroup of all rotations of $G$ such that $g(\omega) = \omega$. In [5] we consider the group $B^G_\omega$ for $G = \text{Aut}(X)$ and we prove that $B^G_\omega$ is minimally almost periodic. A curious fact is that if $B^G_\sigma$ is minimally almost periodic for a general $G$ acting transitively on $X$ and $\Omega$, then $G^+$ is topologically simple. This is a consequence of the following claim: if an open normal subgroup $H$ of $G^+$ contains $B^G_\sigma$, then $H = G^+$. We prove now, briefly, the claim. Since $H$ is normal and $B^G_\sigma \subseteq H$, it follows that $gB^G_\sigma g^{-1} = B^G_{g(\omega)} \subseteq gHg^{-1} = H$ for every $g \in G^+$. Hence $B^G_\sigma \subseteq H$ for every $\sigma \in \Omega$ because $G^+$ acts transitively on $\Omega$. We recall that $G^+$ is the subgroup generated by all rotations of $G$; therefore it is enough to prove that $K_v \subseteq H$ for every $v \in X$. As observed $K_v \cap B^G_\sigma \subseteq H$ for every $v \in X$ and $\sigma \in \Omega$. Let $k$ be in $K_v$ and $\sigma, \sigma' \in \Omega$ such that $k(\sigma) = \sigma'$. The subgroup $H$ in $G$ and so $H \cap K_v$ acts transitively on $\Omega$ [4, Prop. 1, pg. 143]; therefore there exists $h \in H \cap K_v$ such that $h(\sigma') = \sigma$. This means that $hk \in B^G_\sigma \subseteq H$ and $k \in H$.

References

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