$KD_\infty$ IS A CS-ALGEBRA

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Abstract. In this paper, it is shown that the group algebra $KD_\infty$ is right CS if and only if $\text{char}(K) \neq 2$. Moreover, when $\text{char}(K) \neq 2$, then $KD_\infty$ is also CS as a module over its center.

1. Introduction

Rings whose complement right ideals are direct summands are called right CS-rings. The class of CS-rings includes selfinjective rings, continuous rings etc. and have been of interest to many authors. However, there is hardly any literature on CS-group algebras. It is well known that the group algebra $KG$, where $K$ is a field, is selfinjective if and only if $G$ is a finite group. But the group algebra $KG$ may be CS without the finiteness condition on the group $G$. For example if $G$ is a torsion-free solvable-by-finite group, then $KG$ is an Ore domain and hence is a CS-algebra. On the other hand if $G$ is a finite group, then the group ring $RG$ over the ring $R = M_n(Z)$ is not CS for any $n \geq 1$. It is, therefore, of interest to study when a given group algebra is CS. In this paper we study the group algebra $S = KD_\infty$ over a field $K$ for its being CS or not. It is proved that $S$ is a right CS-algebra if and only if $\text{char}(K) \neq 2$ (Theorem 3.6). It is further shown that the center $Z(S)$ of $S$ is a Dedekind domain (Lemma 3.8) and that $S$ is also a CS-module over $Z(S)$ (Theorem 3.9).

2. Notation and preliminaries

Throughout, unless otherwise stated, $K$ will denote a field and $D_\infty$, the infinite dihedral group, that is, the group generated by two elements $a$ and $b$ where $a$ is of infinite order, $b$ is of order 2 and $ab = ba^{-1}$. A module will always mean a right unital module. A nonzero submodule $N$ of a module $M$ is said to be essential in $M$, denoted by $N \subset_e M$, if, for every nonzero submodule $L$ of $M$, $L \cap N \neq 0$. $N$ is called closed in $M$ if $N$ has no proper essential extensions in $M$. A module $M$ is said to be CS if every nonzero submodule of $M$ is essential in a summand of $M$, or equivalently, if every closed submodule of $M$ is a summand of $M$. CS-modules are also commonly known as extending modules ([1]). A ring $R$ is called right CS if it is CS as a right module over itself.
If $A$ is an algebra over a ring $R$, then an element $u \in A$ is called integral over $R$ if it satisfies a polynomial equation with coefficients in $R$ and leading coefficient 1. $A$ is called integral if all its elements are integral.

3. Group ring $KD_\infty$

Throughout this section, $S$ will denote the group algebra $KD_\infty$, and $R$ will denote the group algebra $KA$ where $A = \langle a \rangle$. It is well known that $KA$ is a PID ([5], Exercise 2, p.28). Also, by ([6], Theorem 5.1 and [3], Proposition 9, p.165), $S$ is a prime PI-ring. We begin with some lemmas which will be useful to prove our main result. Our first lemma is well known. We state it here without proof for convenience.

**Lemma 3.1** ([1], Corollary 12.8). For a commutative domain $R$, the following are equivalent:

(a) $R$ is a Prüfer domain.
(b) $(R \oplus R)_{R}$ is a CS-module.

**Lemma 3.2.** $S$ is CS as a right $R$-module.

**Proof.** Since $S = R \oplus Rb \simeq R \times R$ and $R$ is a Prüfer domain, the result follows by Lemma 3.1. 

**Lemma 3.3.** If $\text{char}(K) \neq 2$ and if $U$ is a right ideal of $S$ such that $S_R = U_R \oplus X_R$ for some $R$-submodule $X$ of the right $R$-module $S_R$, then there exists a right ideal $V$ of $S$ such that $S = U \oplus V$.

**Proof.** Let $\pi_1, \pi_2$ be the projections of $S_R$ onto $U_R$ and $X_R$ respectively. Define $\gamma : S \to S$ by $\gamma(s) = \frac{1}{2}[\pi_2(s) + \pi_2(sb)b]$. Since $\text{char}(K) \neq 2$, $\gamma$ is well-defined. Clearly, $\gamma(s_1 + s_2) = \gamma(s_1) + \gamma(s_2)$. Also,

$$
\gamma(sa) = \frac{1}{2}[\pi_2(sa) + \pi_2(sab)b] = \frac{1}{2}[\pi_2(s)a + \pi_2(sba^{-1})b] \\
= \frac{1}{2}[\pi_2(s)a + \pi_2(sb)a^{-1}b] = \frac{1}{2}[\pi_2(s)a + \pi_2(sb)ba] = \gamma(sa).
$$

Similarly, $\gamma(sb) = \gamma(sb)$. Thus $\gamma \in \text{Hom}_S(S, S)$. Let $V = \gamma(S)$. Then $V$ is a right ideal of $S$. We will prove that $S = U \oplus V$. So, let $s \in S$. Write $s = (s - \gamma(s)) + \gamma(s)$. Since

$$
s - \gamma(s) = s - \frac{1}{2}[\pi_2(s) + \pi_2(sb)b] = \frac{1}{2}[(s - \pi_2(s)) + (s - \pi_2(sb)b)] \\
= \frac{1}{2}[(s - \pi_2(s)) + (sb - \pi_2(sb))b] = \frac{1}{2}[\pi_1(s) + \pi_1(sb)b],
$$

and $U$ is a right ideal of $S$, we have $s - \gamma(s) \in U$. Thus $S = U \oplus V$. Also since for every $s \in S$, $s - \gamma(s) \in U$, we have $\gamma(s - \gamma(s)) = 0$, that is, $\gamma(s) = \gamma^2(s)$ for every $s \in S$.

To prove $U \cap V = (0)$, let $x \in U \cap V$. Then $x = \gamma(s)$ for some $s \in S$ and $\gamma(x) = 0$. Thus $\gamma^2(s) = 0$ and consequently, $x = \gamma(s) = \gamma^2(s) = 0$, as desired. 

**Lemma 3.4.** If $U$ is a closed right ideal of $S$, then $U$ is a closed submodule of the right $R$-module $S_R$. 

Proof. Suppose \( x \in cl(U_R) \), the closure of \( U_R \) in \( S_R \). Then \( x \in S \) and \( xE \subset U \) for some essential right ideal \( E \) of \( R \). Consequently, \( x(ES) \subset US \subset U \). Also \( ES \) is an essential right ideal of \( S \) ([5], Exercise 27, p. 467). Thus \( x \in cl(U_S) = U \), because \( U \) is a closed right ideal of \( S \). This completes the proof.

Lemma 3.5. If \( char(K) = 2 \), then \( S \) has no nontrivial idempotents.

Proof. For \( \alpha = \sum k_i a_i \in R \), let \( \alpha^* = \sum k_i a_i^{-1} \). Since \( ab = ba^{-1} \) for every \( a \in R \), \( ab = ba^* \) for every \( a \in R \). Now if \( \alpha + b \beta \in S \) is a nontrivial idempotent in \( S \), then \( ab = ba^* \) and \( (\alpha + b \beta) \beta = \alpha + b \beta \) we get \( \alpha^2 + \beta^2 = (\alpha + b \beta)(\alpha^* + b \beta) = \alpha + b \beta \). Thus \( \alpha^2 + \beta^2 = \alpha + \beta \alpha + \beta = \beta \). Since \( R \) is a PID and \( \alpha + b \beta \) is an ideal in \( S \), \( \beta \neq 0 \). Consequently, using \( R \) is a domain, the relation \( \alpha^* + \beta = \beta \) yields \( \alpha^* + \alpha = 1 \). Since \( \alpha \in R \), \( \alpha = \sum k_i a_i \) where \( k_i \in K \). Since \( \alpha^* + \alpha = 1 \) and \( char(K) = 2 \), we have \( 0 = 1 \), a contradiction. Thus \( S \) has no nontrivial idempotents.

Theorem 3.6. \( S \) is a right CS-ring if and only if \( char(K) \neq 2 \).

Proof. First assume that \( char(K) \neq 2 \). Let \( U \) be a closed right ideal of \( S \). By Lemma 3.4, \( U_R \) is a closed submodule of the right \( R \)-module \( S_R \). Since \( S_R \) is CS (Lemma 3.2), \( U_R \) is a direct summand of \( S_R \). But then by Lemma 3.3, \( U \) is a summand of \( S \). Hence \( S \) is a right CS-ring. Conversely, let \( S \) be right CS. Since \( S \) is right CS, every nonzero right ideal of \( S \) is essential in \( S \). Thus \( S \) and hence the right maximal quotient ring \( Q_{max}^*(S) \) of \( S \) is uniform. Since \( S \) is right nonsingular, it follows that \( Q_{max}^*(S) \) is a division ring. Hence \( S \) is a domain, a contradiction because \( 1 + b \neq 0 \) and \( 1 + b \neq 0 \). Thus \( char(K) \neq 2 \).

In what follows \( Z(S) \) will denote the center of the ring \( S \). Unless otherwise stated \( char(K) \neq 2 \) and \( e = \frac{1}{2} + \frac{1}{2}b \). Notice that \( e \) is an idempotent in \( S \). For \( \alpha = \sum k_i a_i \), we will write \( \alpha^* = \sum k_i a_i^{-1} \). In the following lemma we determine \( Z(S) \).

Lemma 3.7. For any field \( K \), \( Z(S) = \{ \alpha \in R \mid \alpha = \alpha^* \} \).

Proof. Clearly, \( \{ \alpha \in R \mid \alpha = \alpha^* \} \subset Z(S) \). To prove the reverse inclusion, let \( s = \alpha + b \beta \in Z(S) \). Then \( sx = xs \) for every \( x \in S \). In particular, \( sa = as \) and \( sb = bs \). Now \( sa = as \) gives \( \alpha a + \beta ba = a \alpha + a b \beta \). Since \( \alpha a = a \alpha \), we get \( \beta a^{-1}b = a \beta \), that is, \( (a^2 - 1) \beta = 0 \). Thus, \( \beta = 0 \). Consequently, \( s = \alpha \in R \). Again as \( sb = bs \), we have \( \alpha b = ba = \alpha^* b \). Thus \( \alpha = \alpha^* \) and the proof is complete.

Lemma 3.8. \( Z(S) \) is a Dedekind domain.

Proof. Clearly, \( Z(S) \simeq eZ(S) = eSe \). Since \( S \) is right noetherian, \( eSe \) is right noetherian ([7], Lemma 2.7.12). Thus \( Z(S) \) is right noetherian. Since \( S \) is a prime PI ring, by ([4], Corollary 6.14, p. 467), \( S \) is a finitely generated \( Z(S) \)-module. Let \( S = s_1 Z(S) + s_2 Z(S) + ... + s_k Z(S) \) and let for \( 1 \leq i \leq k \), \( s_i = \alpha_i + \beta_i b \) where \( \alpha_i, \beta_i \in R \). Then \( R \oplus R = (\alpha_1 + \beta_1 b)Z(S) + (\alpha_2 + \beta_2 b)Z(S) + ... + (\alpha_k + \beta_k b)Z(S) \). Consequently, \( R = \alpha Z(S) + \alpha Z(S) + ... + \alpha Z(S) \), that is, \( R \) is a finitely generated noetherian \( Z(S) \)-module. Thus by ([2], Theorem 17) \( R \) is integral over \( Z(S) \), that is, \( R \mid Z(S) \) is an integral extension of \( Z(S) \).

Let \( L \) denote the quotient field of \( Z(S) \). We will show that \( R \cap L = Z(S) \). Let \( \gamma \in R \cap L \). Then \( \gamma = \alpha \beta^{-1} \) for some \( \alpha, \beta \in Z(S) \). Thus \( \alpha = \gamma \beta \). Since
\[\alpha, \beta \in Z(S), \alpha = \alpha^*, \beta = \beta^*.\] Hence \[\gamma^* = \alpha^* = (\gamma \beta)^* = \beta^* \gamma^* = \beta \gamma^*\] so that \[\gamma^* = \alpha \beta^{-1} = \gamma.\] Since \[\gamma \in R,\] we have \[\gamma \in Z(S).\] Hence \[R \cap L = Z(S).\] Since \[R\] is a Dedekind domain, it follows, by ([8], Theorem 20, p.283), that \[Z(S)\] is a Dedekind domain.

**Theorem 3.9.** \(R\) and \(S\) are CS as right \(Z(S)\)-modules.

**Proof.** Clearly, \[Z(S) \cap aZ(S) = (0).\] Further, since \[a^{-n} = (a^n + a^{-n}) - a^n\] and \[a^n = a^{n-1}(a + a^{-1}) - a^{n-2}\] we have \[R = Z(S) \oplus aZ(S) \simeq Z(S) \times Z(S).\] Also as \[S = R \oplus bR,\] we have \[S \simeq Z(S) \times Z(S) \times Z(S) \times Z(S).\] By Theorem 3.8, \[Z(S)\] is a Dedekind domain. The result now follows by Lemma 3.1.

**Remark 1.** The uniform dimension of \(R\) as a \(Z(S)\)-module is 2 and that of \(S\) as a \(Z(S)\)-module is 4.

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