

## BLOCH RADIUS, NORMAL FAMILIES AND QUASIREGULAR MAPPINGS

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ABSTRACT. Bloch's Theorem is extended to  $K$ -quasiregular maps  $\mathbf{R}^n \rightarrow \mathbf{S}^n$ , where  $\mathbf{S}^n$  is the standard  $n$ -dimensional sphere. An example shows that Bloch's constant actually depends on  $K$  for  $n \geq 3$ .

Let  $B(a, r) := \{x \in \mathbf{R}^n : |x - a| < r\}$  be an open ball,  $0 < r \leq \infty$ . Consider an open discrete map  $f : B(0, r) \rightarrow M$  where  $M$  is a Riemannian manifold of dimension  $n$ . For every  $x \in B(0, r)$  we define  $d_f(x)$  as the radius of the maximal open ball  $B \subset M$  centered at  $f(x)$ , such that a continuous right inverse  $\phi$  with the property  $\phi(f(x)) = x$  exists in  $B$ . If there is no such ball  $d_f(x) = 0$ . The Bloch radius of  $f$  is defined as

$$\mathfrak{B}(f) := \sup_{x \in B(0, r)} d_f(x).$$

The cases of principal interest are  $M = \mathbf{R}^n$  with the standard Euclidean metric  $|dx|$ , and  $M = \mathbf{S}^n$ , the sphere with the standard spherical metric  $2|dx|/(1 + |x|^2)$ , so that the diameter of the sphere in this metric is  $\pi$ . We denote the corresponding Bloch radii by  $\mathfrak{B}_e(f)$  and  $\mathfrak{B}_s(f)$ , respectively. Notice that  $\mathfrak{B}_s(f) \leq 2\mathfrak{B}_e(f)$ , if we consider the push forward of the spherical metric to  $\mathbf{R}^n$  via the stereographic projection. Notation  $\mathfrak{B}(f)$  will be used in statements which are true for both metrics.

A family  $\mathcal{F}$  of continuous maps  $B(0, r) \rightarrow M$  is called normal if every sequence in  $\mathcal{F}$  contains a subsequence which converges uniformly on compacta. In the case of compact  $M$  a family is normal if and only if its restriction to every compact in  $B(0, r)$  is equicontinuous [1]. The classical theorem of Bloch [3] can be stated as:

**Theorem A.** *Every family of holomorphic maps  $\mathbf{C} \supset B(0, 1) \rightarrow \mathbf{C}$  with bounded Euclidean Bloch radius is equicontinuous with respect to the Euclidean metric.*

A related result belongs to Valiron [13]:

**Theorem B.** *Every non-constant entire function has infinite Euclidean Bloch radius.*

The case of spherical metric was first considered by Minda [8]:

**Theorem C.** (i) *There exists an absolute constant  $b_0 > 0$ , such that the family of all meromorphic functions  $\mathbf{C} \supset B(0, 1) \rightarrow \overline{\mathbf{C}}$  whose spherical Bloch radius is at most  $b_0 - \epsilon$  is normal for every  $\epsilon \in (0, b_0)$ .*

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(ii) For every non-constant meromorphic function  $f : \mathbf{C} = B(0, \infty) \rightarrow \overline{\mathbf{C}}$  we have  $\mathfrak{B}(f) \geq b_0$ .

It is conjectured that  $b_0 = \arccos(1/3) \approx 70^\circ 32'$ . Some recent results about precise constants are contained in [5]. The proofs of Theorems A-C are usually based on specific properties of holomorphic functions, like Taylor series expansion ([3, 13]) or on Gauss curvature considerations [2, 8].

In this paper we show that all results mentioned above, as well as their  $n$ -dimensional generalizations, follow from a simple normal families argument, which is due to Zalcman [14] in dimension 2 and to Ruth Miniowitz [9] in arbitrary dimension. The survey [15] describes other applications of this tool.

The natural framework in dimensions greater than 2 is the class of quasiregular maps (see [10, 11] for the general theory of these maps). We recall that a continuous map is called  $K$ -quasiregular if its generalized partial derivatives are locally summable in degree  $n$  and the derivative  $D_f$  satisfies

$$\|D_f(x)\|^n \leq K |\det D_f(x)| \quad \text{almost everywhere.}$$

Non-constant quasiregular maps are open and discrete, according to [10, II §6.3]. Apparently they were first recognized by Bochner [4], as early as in 1946, as the appropriate class for extension of Geometric Function Theory to higher dimensions, the point of view widely shared today. Bochner showed that Theorem A extends to  $K$ -quasiregular maps  $\mathbf{R}^n \supset B(0, 1) \rightarrow \mathbf{R}^n$ , whose coordinates are *harmonic functions*. For further results in this direction see [7].

**Theorem 1.** (i) *There exists a constant  $b(n, K) > 0$ , such that the family of all  $K$ -quasiregular maps  $\mathbf{R}^n \supset B(0, 1) \rightarrow \mathbf{S}^n$ , whose spherical Bloch radii are at most  $b(n, K) - \epsilon$ , is normal for every  $\epsilon \in (0, b(n, K))$ .*

(ii) *For every non-constant  $K$ -quasiregular map  $f : \mathbf{R}^n \rightarrow \mathbf{S}^n$  we have  $\mathfrak{B}_s(f) \geq b(n, K)$ .*

(iii) *Every family of  $K$ -quasiregular maps  $\mathbf{R}^n \supset B(0, 1) \rightarrow \mathbf{R}^n$  with bounded Euclidean Bloch constant is equicontinuous with respect to the Euclidean metric.*

(iv) *Every non-constant  $K$ -quasiregular map  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  satisfies  $\mathfrak{B}_\epsilon(f) = \infty$ .*

*Remark.* The proof of Theorem 1 is a pure existence proof. It is desirable to find some way to estimate  $b(n, K)$  effectively for  $n \geq 3$ . For  $n = 2$  the spherical Bloch constant  $b_0 = b(2, K)$  is actually independent of  $K$ . This follows from the fact that every quasiregular map  $f : \mathbf{C} \rightarrow \overline{\mathbf{C}}$  can be factored as  $f = g \circ \phi$ , where  $g$  is meromorphic in  $\mathbf{C}$ , and  $\phi$  is a homeomorphism, and it is clear that  $\mathfrak{B}(f) = \mathfrak{B}(g)$ . The situation is different for  $n \geq 3$  as the following example shows. It is similar to the situation with ‘‘Picard’s constant’’ which is equal to 3 for quasiregular maps in dimension 2, but depends on  $K$  in higher dimensions, as the example of Rickman [12] shows. Smooth<sup>1</sup> quasiregular maps in dimension at least 3 are locally injective, and thus by a theorem of Zorich [11] smooth quasiregular maps  $\mathbf{R}^n \rightarrow \mathbf{R}^n$  are bijective, so (ii) and (iv) are immediate for smooth quasiregular maps.

**Example 1.** For every  $n \geq 3$  and every  $\epsilon > 0$  there exists a quasiregular map  $\mathbf{S}^n \rightarrow \mathbf{S}^n$  of degree 2, such that  $\mathfrak{B}_s(f) < \epsilon$ . There exists also an infinitely differentiable open discrete map  $f : \mathbf{S}^n \rightarrow \mathbf{S}^n$  with  $\mathfrak{B}_s(f) < \epsilon$ .

Consider the standard embedding  $\mathbf{S}^{n-2} \subset \mathbf{S}^n$ , and introduce the cylindrical coordinates on the complement  $D := \mathbf{S}^n \setminus \mathbf{S}^{n-2} = \{(r, \theta, y) : r > 0, \theta \in [0, 2\pi), y \in \mathbf{R}\}$ .

<sup>1</sup> $C^3$  for  $n = 3$  and  $C^2$  for  $n \geq 4$ ; see [11, p. 12].

$\mathbf{S}^{n-2}$ . The “winding map”  $D \rightarrow D$  given in cylindrical coordinates by  $(r, \theta, y) \mapsto (r, 2\pi\{\theta/\pi\}, y)$ , where  $\{\cdot\}$  stands for the fractional part, extends by continuity to the 2-quasiregular map  $g : \mathbf{S}^n \rightarrow \mathbf{S}^n$ , which is locally homeomorphic in  $D$ , and locally 2-to-1 at every point of  $\mathbf{S}^{n-2}$ . If  $B \subset \mathbf{S}^n$  is a ball, then a right inverse to  $g$  in  $B$  exists if and only if  $B \cap \mathbf{S}^{n-2} = \emptyset$ . Now we postcompose  $g$  with a diffeomorphism  $h : \mathbf{S}^n \rightarrow \mathbf{S}^n$  such that  $h(\mathbf{S}^{n-2})$  forms an  $\epsilon$ -net, that is,  $\text{dist}_s(x, h(\mathbf{S}^{n-2})) < \epsilon$  for every  $x \in \mathbf{S}^n$ . It is clear that  $\mathfrak{B}_s(h \circ g) < \epsilon$ . To construct an infinitely smooth map with similar properties, we replace the winding map by  $(r, \theta, y) \mapsto (r^2, 2\pi\{\theta/\pi\}, y)$ .  $\square$

The normality argument mentioned above is the following

**Lemma 1** ([9]). *Let  $\mathcal{F}$  be a family of  $K$ -quasiregular maps  $\mathbf{R}^n \supset B(0, 1) \rightarrow \mathbf{S}^n$  which is not normal. Then there exist  $r \in (0, 1)$  and sequences  $f_m \in \mathcal{F}$ ,  $x_m \in B(0, r)$  and  $\rho_m > 0$ ,  $\rho_m \rightarrow 0$ , such that  $g_m(x) := f_m(x_m + \rho_m x) \rightarrow f(x) \neq \text{const}$  uniformly on compacta in  $\mathbf{R}^n$ , and  $f : \mathbf{R}^n \rightarrow \mathbf{S}^n$  is  $K$ -quasiregular. Moreover, we have for  $x_1, x_2 \in B(0, R)$*

$$\text{dist}(f(x_1), f(x_2)) \leq 2(1 + R^2)^\alpha |x_1 - x_2|^\alpha, \text{ where } \alpha = (K)^{1/(1-n)}$$

and

$$\text{diam} f(B(0, 1)) \geq \delta > 0,$$

where  $\text{diam}$  is the diameter with respect to the spherical metric, and  $\delta$  is a constant depending only on  $K$  and  $n$ .

*Proof of Theorem 1.* First we notice that  $\mathfrak{B}_s(f) > 0$  for every open discrete map. Recall that  $x \in B(0, 1)$  is called *critical* if there is no neighborhood  $V$  of  $x$  such that  $f|_V$  is a homeomorphism onto its image. The set of all critical points is closed and its topological dimension is at most  $n - 2$  [6]. So there is a point  $a \in B(0, 1)$  such that the restriction of  $f$  onto some ball  $B(a, r)$  is a homeomorphism onto the image, so  $\mathfrak{B}_s(f) > 0$ .

Second we notice the following semicontinuity property of the Bloch radius: if  $g_m \rightarrow f$  uniformly on compacta, then

$$(1) \quad \mathfrak{B}(f) \leq \liminf \mathfrak{B}(g_m).$$

To prove this property we fix arbitrary  $\epsilon \in (0, \mathfrak{B}(f)/4)$  and put  $r := \mathfrak{B}(f) - 2\epsilon$ . Then there exists a ball  $B(a, r + \epsilon)$  in which a continuous right inverse  $\phi$  to  $f$  is defined. Put  $\bar{D} := \phi(\bar{B}(a, r))$ ; this is an imbedding of the closed ball. As  $g_m \rightarrow f$  uniformly on  $\bar{D}$ , we conclude for large values of  $m$  that  $g_m(\partial\bar{D})$  is contained in  $\epsilon$ -neighborhood of  $\partial B(a, r)$ . So the degree  $\mu(y, g_m, D)$  is defined for every  $y \in B(a, r - \epsilon)$  [10, II, §2]. Since the degree  $\mu(y, f, D)$  for  $y \notin f(\partial D)$  is continuous with respect to  $f$ , we conclude that  $\lim \mu(y, g_m, D) = \mu(y, f, D) = \pm 1$  for every  $y \in B(a, r - \epsilon)$ . This means that for large  $m$  the restrictions  $g_m|_D$  have continuous right inverses in  $B(a, r - \epsilon)$  so  $\mathfrak{B}(g_m) \geq r - \epsilon \geq \mathfrak{B}(f) - 3\epsilon$ , which proves (1).

To prove (i) in Theorem 1 by contradiction, we assume that for every  $\epsilon > 0$  the family  $\mathcal{F}_\epsilon$  consisting of all  $K$ -quasiregular maps  $f : \mathbf{R}^n \supset B(0, 1) \rightarrow \mathbf{S}^n$  with  $\mathfrak{B}_s(f) \leq \epsilon$  is not normal. Applying Lemma 1 to each  $\mathcal{F}_\epsilon$  we obtain a family  $\{f_\epsilon : \epsilon > 0\}$  of quasiregular maps  $\mathbf{R}^n \rightarrow \mathbf{S}^n$ . This family is normal and has no constant limit functions because of the uniform estimates in Lemma 1 and we have  $\mathfrak{B}(f_\epsilon) \leq \epsilon$ . So we can find a convergent sequence  $f_{\epsilon_k} \rightarrow f \neq \text{const}$  with  $\mathfrak{B}(f_{\epsilon_k}) \rightarrow 0$ . We have  $\mathfrak{B}_s(f) > 0$  because  $f$  is open, and this contradicts (1).

To prove (ii) it is enough to notice that the family  $\{x \mapsto f(2^n x) : n \in \mathbf{N}\}$  with a non-constant function  $f$  is never normal in  $B(0, 1)$ .

To prove (iii) we fix arbitrary  $M > 0$  and consider the family  $\mathcal{F}_M$  consisting of all  $K$ -quasiregular maps  $\mathbf{R}^n \supset B(0, 1) \rightarrow \mathbf{R}^n$  with the property  $\mathfrak{B}_e(f) \leq M$ . Put  $\kappa = b(n, K)/(3M)$ , where  $b(n, K)$  is the constant from (i). Then all maps from the new family  $\mathcal{F}^* = \{\kappa f : f \in \mathcal{F}_M\}$  satisfy  $\mathfrak{B}_s(f) \leq 2\mathfrak{B}_e(f) < b(n, K)$ , and thus by (i)  $\mathcal{F}^*$  is equicontinuous with respect to the spherical metric.<sup>2</sup> Now we fix a compact  $E \subset B(0, 1)$  and  $\epsilon \in (0, \pi)$ . We choose  $\delta > 0$  such that for  $x, y \in E$  from  $|x - y| < \delta$  follows  $\text{dist}_s(f(x), f(y)) < \epsilon$  for every  $f \in \mathcal{F}^*$ . Let  $f \in \mathcal{F}^*$ . Then  $g = f - f(x) \in \mathcal{F}^*$  because addition of a constant changes neither the Bloch radius nor  $K$ . So we have  $\text{dist}_s(g(x), g(y)) = \text{dist}_s(0, g(y)) < \epsilon$ , that is,  $|f(x) - f(y)| = |g(y)| \leq \tan(\epsilon/2)$ . So for the members of the original family  $\mathcal{F}_M$  we obtain  $|f(x) - f(y)| < (3M/b(n, K)) \tan(\epsilon/2)$ . This proves equicontinuity with respect to the Euclidean metric.

Now (iv) follows from (iii) exactly like (ii) follows from (i).  $\square$

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<sup>2</sup>Pushed forward via stereographic projection  $\mathbf{S}^n \setminus \{\text{point}\} \rightarrow \mathbf{R}^n$ .