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SOBOLEV SPACES, DIMENSION, AND RANDOM SERIES

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Abstract. We investigate dimension-increasing properties of maps in Sobolev spaces; we obtain sharp results with a random process somewhat like Brownian motion.

Introduction

We investigate dimension–increasing properties of mappings in the space $W^{1,p}(R^n)$, $p > n \geq 1$, obtaining exact results as far as the Hausdorff dimension (dim) and the packing dimension (Dim) are concerned. The upper bounds are consequences of Hölder's inequality; the lower bounds for $n > 1$ depend on inequalities from probability theory like those governing Brownian motion.

All functions in $W^{1,p}(R^n)$, $p > n \geq 1$, are assumed to be continuous. When $0 < \alpha < n$ and $n < p < +\infty$, we define $\beta = (p\alpha)(p - n + \alpha)^{-1}$, so that $\alpha < \beta < n$.

Theorem 1. *Let E be a subset of R^n of finite α-dimensional measure, and let $f \in W^{1,p}(R^n)$. (The range of f can have any dimension, even infinite.) Then $f(E)$ has β-measure* 0.

Theorem 2. *Let E be a closed subset of R, of positive α-measure, $0 < \alpha < 1$. Then there is a real function f on R, such that f' is in weak L^p, and $f(E)$ has positive β-measure.*

Theorem 3. *Let E be a closed subset of R^n, of positive α-measure, $0 < \alpha < n$. Then* dim $f(E) \geq \beta$ *for some $f \in W^{1,p}(R^n)$ mapping R^n to R^n.*

Similar bounds and existence theorems are obtained afterwards for the packing dimension [1], [2], [4], [5]. In this part the exposition is less formal.

1. Upper bounds for $\dim f(E)$

Constant use is made of the usual system Q_{Nk} ($N = 0, \pm 1, \pm 2, \ldots, k \geq 1$) of dyadic n-cubes in R^n. These can be used in estimating Hausdorff measures in R^n at the expense of a constant c_n. Let Q be a cube of side r, and $f \in W^{1,p}(R^n)$. Then $f(Q)$ has diameter at most $c_{n,p} r^{1-n/p} (\int_Q \|\Delta f\|^p dm)^{1/p}$. This is just an invariant form of a Sobolev inequality [3, p. 124]; it is valid for mappings into normed spaces.

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To prove Theorem 1 we cover E by cubes Q_j of side r_j, so that $\sum r_j^\alpha \le c(E)$ and $\sup(r_j)$ is as small as we please. Then

$$\operatorname{diam}(f(Q_j)) \le c r_j^{1-n/p} (\int_{Q_j} \|\Delta f\|^p)^{1/p},$$

$$\operatorname{diam}(f(Q_j))^\beta \le c^\beta r_j^{(1-n/p)\beta} (\int_{Q_j} \|\Delta f\|^p)^{\beta/p}.$$

We use the fact that the cubes Q_j are essentially disjoint. We sum the left side over j, using Hölder's inequality with exponent $q = p\beta^{-1}$ for the factors on the right. The conjugate exponent is $q' = p(p-\beta)^{-1}$; and finally $q'(1-np^{-1})\beta = \alpha$. Since the union $\bigcup Q_j$ has measure $0(1)\sup(r_j)^{n-\alpha}$, we obtain $o(1)$ for the sum $\sum \operatorname{diam}(f(Q_j))^\beta$.

2. Proof of Theorem 2

By Frostman's Theorem, E carries a probability measure μ such that $\mu(I) \le c|I|^\alpha$ for all intervals I and some constant c. We shall find a strictly increasing function f such that $|f(I)| \ge \mu(I)^{1/\beta}$ for all intervals I. Let $f^*(\mu)$ be the image of μ by the mapping $f : \mu^*(S) \equiv \mu(f^{-1}(S))$ for open sets S. When S is an interval, so is $f^{-1}(S)$ and it is evident that $M^*(I) \le |I|^\beta$ for all intervals I, so that$f(E)$ has positive β-measure.

To find f we require that $f' \ge \mu(I)^{1/\beta}|I|^{-1}$ on every open interval I, and prove that this can be done with some f' in weak L^p. We define $h(x)$ to be the supremum of $\mu(I)^{1/\beta}|I|^{-1}$ over all intervals I containing x. To prove that h is in weak $L^p(R)$, we introduce $h(x,t) = \sup\ \mu(I)^t|I|^{-1}$, where $\alpha^{-1} \le t \le 1$. Then $h(x,1)$ is the Hardy-Littlewood maximal function of μ, and is in weak L^1. Again, $h(x,\alpha^{-1})$ is bounded, so that $h(x) = h(x,\beta^{-1})$ is in weak L^p, because $\beta^{-1} = p^{-1} + \alpha^{-1}(1-p^{-1})$. This completes Theorem 2; to obtain $f' \in L^p$ and $\dim\ f(E) \ge \beta$, we decrease h slightly to $h^\sim(x) = \sup\ \mu(I)^{1/\beta}|I|^{-1}\log^{-1}(e+|I|^{-1})$.

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For Theorem 3 we use the system Q_{Nk} of dyadic squares of side 2^{-N}, where $N = 0, \pm1, \pm2, \dots$ and $k = 1, 2, 3, \dots$. These define a sequence of bump functions ψ_{Nk}, equal to 1 on Q_{Nk}, and vanishing outside the cube $Q^\sim_{Nk}$ obtained by expanding Q_{Nk} on its center by a factor 5/4; moreover $0 \le \psi_{Nk} \le 1$ and $\|\nabla\psi_{Nk}\| \le c_n 2^N$. For a sum $u = \sum_k c_k \psi_{Nk}$ we find $\|\nabla u\|_p^p \approx 2^{Np}2^{-nN}\sum |c_k|^p$. Let μ be a probability measure in R^n, satisfying the Hölder condition in exponent α and $c_k = \mu(Q_{Nk})^{1/\beta}$. We claim that $\|\nabla u\|_p^p$ is bounded by a number independent of k; here α, β, n, p are the numbers defined before. In fact $\sum_k \mu(Q_{Nk}) \le C_1$ and $\sup\ \mu(Q_{Nk}) \le C_2 2^{-N\alpha}$. Thus $\sum_k c_k^p \le C_3 2^{-N\gamma}$ where $\gamma = \alpha(p\beta^{-1} - 1)$. To complete our claim we observe that $p - n = \gamma$.

In fact we use a more involved variant of the function u. First we replace $\mu(Q_{Nk})$ by $\mu(Q^{**}_{Nk})$, where Q^{**}_{Nk} is obtained from Q_{Nk} by expansion in a ratio of 8. Moreover we introduce a sequence of random multipliers ξ_{Nk}, independent random variables uniformly distributed on the unit ball in R^n. We call the variant so obtained v_N and observe that the estimates on ∇v_N are almost the same as the ones above.

Suppose now that E is contained in the unit cube of R^n, and that μ is concentrated in E. We define $v = \sum_0^\infty (N+1)^{-2} v_N$, and prove that the support of $v^*(\mu)$

has Hausdorff dimension at least β a.s., as follows. Let $k(t) = t^{-\beta}\log^{-2n-2}(e+t^{-1})$ for $t > 0$. Then the energy of $v^*(\mu)$ with respect to the kernel k is

$$I = \iint k(v(x) - v(y))\mu(dx)\mu(dy).$$

If $I < +\infty$, then $v^*(\mu)$ vanishes on sets of dimension $< \beta$; we prove that $\mathcal{E}(I) < +\infty$, whence our assertion on the support of $v^*(\mu)$ follows.

In fact we prove that $\int \mathcal{E}(k(v(x) - v(y)))\mu(dy) \leq c < +\infty$ for all x in the unit cube, whence $\mathcal{E}(I) < +\infty$ by Fubini's Theorem. For each x and y, $v(x) - v(y) = \sum\sum a_{Nk}(x,y)\xi_{Nk}$; the variables ξ_{Nk} are those described above. Let $\rho(x,y) = \sup|a_{Nk}(x,y)|$ so that $0 \leq \rho \leq 2$. We easily see that the expectation of $k(v(x)-v(y))$ is $\leq ck(\rho)$, since $0 < \beta < n$. Thus we seek a uniform upper bound for $\int k(\rho(x,y))\mu(dy)$.

Let $\nu(x,y)$ be the largest integer $N \geq 0$ such that x and y belong to a single square Q_{Nk} or to touching squares of side 2^{-N}. (When $x \neq y$, then $\nu(x,y) < +\infty$.) Then there will be a bump function ψ_{N+1}, of the next generation, such that $\psi_{N+1}(x) = 1$, $\psi_{N+1}(y) = 0$. More precisely, $x \in Q_{N+1,\ell}$, $y \notin Q^{\sim}_{N+1,\ell}$; however $y \in Q^{**}_{N+1,\ell}$. Thus, if we partition the domain of integration into the sets defined by $\nu(x,y) = 0$, $\nu(x,y) = 1, \dots, \nu(x,y) = N, \dots$, then the μ-measure of the N^{th} set is $< \mu(Q^{**}_{N+1,\ell}) \equiv \mu_{N+1}$ but for each y in the set $\rho(x,y) \geq (N+1)^{-2}\mu^{1/\beta}_{N+1}$. Moreover $\mu_{N+1} < c2^{-2N\alpha}$. Observing the logarithm in $k(t)$ we see that the integral of $k(\rho(x,y))$ on the N^{th} set is $0(N^{-2})$, since the logarithm adds a factor cN^{-2n-2}, which balances the factor $N^{2\beta}$ arising from the formula for v. We observe also that $\nu(x,y) \geq 0$ because E is in the unit cube, and when $\nu(x,y) = 0$, then $\rho(x,y) = 1$. Thus we have verified the estimate on $\int \mathcal{E}(k(v(x) - v(y)))\mu(dy)$ and Theorem 3 is completely proved.

4. Packing dimension

This concept is derived from the more precise notion of *packing measure* in much the same way as Hausdorff dimension is derived from Hausdorff measure; thus the following definition is part of a larger picture, [1], [2, pp. 82–86], [4]. Let S be a set in a metric space and $r > 0$; then $\nu(S,r)$ is the smallest number of sets of diameter at most r, sufficient to cover S. The *packing exponent* $\delta(S) = \limsup \nu(S,r)/-\log\ r,\ r \to 0+$. (This notion is standard, but the name is not.) A variant concept is the number $\nu^*(S,r)$, the largest number of elements of S separated by more than r. Then $\nu^*(S,R) \leq \nu(S,r) \leq \nu^*(S, \frac{1}{2}r-)$, so that ν^* can be used to define $\delta(S)$. We say that S has *packing dimension* $\leq \alpha$, $\mathrm{Dim} S \leq \alpha$, if for each $\epsilon > 0$, $S = \bigcup_1^\infty S(m,\epsilon)$ where $\delta(S(m,\epsilon)) < \alpha + \epsilon$.

Theorem 4. *Let* $f \in W^{1,p}(R^n)$, $E \subseteq R^n$, *and* $\mathrm{Dim} E \leq \alpha$. *Then* $\mathrm{Dim} f(E) \leq \beta$.

Proof. In calculating the exponent of packing of a set in E^n, we can use dyadic squares of the same side 2^{-N}. Let $\gamma = \alpha\beta^{-1}$ and $r = 2^{-N\gamma}$. A cube Q of side 2^{-s}, $s \geq N$, is called *major* if diam $f(Q) \geq r$, *minor* in the opposite case, and *critical* if it is major, but each of its descendants is minor. Thus a critical cube is the union of 2^n minor cubes. Therefore critical cubes play almost the same role as minor cubes, except that they can be counted by counting the number of major cubes. Since f is uniformly continuous in R^n, major cubes can be subdivided successively until critical cubes are encountered. We shall first count *all* major cubes; in spite of its apparent inefficiency, this gives the correct estimate of packing dimension.

The number m_p of major cubes of side 2^{-s} is estimated by Sobolev's inequality: $m_p \leq cr^{-p}2^{-s(p-n)}$. Since $p > n$, the sum is at most $c'r^{-p}2^{-N(p-n)} = c'2^{\alpha N}$ since $\alpha p\beta^{-1} - (p-n) = \alpha$.

Suppose that a set B has packing exponent $< \eta$, so that for large N B is covered by at most $2^{\eta N}$ cubes of side 2^{-N}. Then B is covered by the minor cubes of side 2^{-N}, augmented by all of the critical cubes of side 2^{-N} or smaller. The number of cubes in this covering is $0(2^{\eta N} + 2^{\alpha N})$, whence $f(B)$ has packing exponent at most $\max(\beta, \eta\alpha^{-1}\beta)$. For each $\eta > \alpha$, E is contained in a sequence of sets of packing exponent $< \eta$, whence $\mathrm{Dim} f(E) \leq \max(\beta, \eta\alpha^{-1}\beta)$, or $\mathrm{Dim} f(E) \leq \beta$.

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Theorem 5. *Let E be a closed set in R^n and $\mathrm{Dim}E > \alpha$. Then there is a mapping h of class $W^{1,p}(R^n)$ into R^n, such that $\mathrm{Dim}h(E) \geq \beta$.*

With more attention to inequalities, Theorem 4 can be proved with a weaker hypothesis, $\mathrm{Dim}E \geq \alpha$. Using the separability of R^n we can find a closed subset $E_0 \neq \emptyset$, such that every relatively open subset $W_0 \neq \emptyset$ of E_0 has $\mathrm{Dim}W_0 > \alpha$. Henceforth we replace E by E_0 in the argument. In place of energy integrals, appearing in Theorem 3, we employ the following consequence of Cauchy's inequality. A set B has m distinct elements $x_1, \ldots, x_m$ and is partitioned into k subsets $T_1, \ldots, T_k$. Then the number N of pairs (x_i, x_j) such that x_i and x_j belong to the same set T_ν, $1 \leq \nu \leq k$, is at least m^2/k. Counting the number N_0 of pairs in which $x_i \neq x_j$, we obtain $N_0 \geq m^2/k - m$.

Next we describe a basic step in Theorem 5, with some heuristics. The set B is located in the unit cube of R^n, and its elements are separated by r, $0 < r < e^{-2}$, while the size m of B is $[r^{-\alpha}]$. We seek a random function f so that nearly all of the images $f(x_i)$, $1 \leq i \leq m$, are separated by r^γ, $\gamma = \alpha\beta^{-1}$. (This would reverse the inequalities obtained in Theorem 4.) A cube Q containing $\ell \geq 2$ of the points x_i should be mapped on a set of diameter $\geq cr^\gamma \ell^{1/n}$. This suggests the following choice of coefficients in the random function f; only numbers N such that $1 \geq 2^{-N} \geq r/8n$ are used in the summation. Let λ be the counting measure of the set B, and the coefficient of c_{Nk} of ψ_{Nk} be $r^\gamma \lambda(Q^{**})^{1/n}$. First we have to verify the $W^{1/p}$-type inequality for the indicated values of N, i.e. $\sum_k c_{Nk}^p \leq c2^{N(n-p)}$ or $\sum_k \lambda(Q_{Nk}^{**})^{p/n} \leq cr^{-\gamma p}2^{N(n-p)}$. Now the total mass of λ is $< r^{-\alpha}$ and $\lambda(Q_{Nk}^{**}) \leq c(2^{-N}r^{-1})^n$ since $2^{-N} \geq r/8n$. We obtain $\sum_k \lambda(Q_{Nk}^{**})^{p/n} \leq cr^{-\alpha}(2^{-N}r^{-1})^{(p-n)}$, and this suffices because $-\alpha + n - p = -\gamma p$. (We don't use the factors $(N+1)^{-2}$ in the sum, so there is a small correction later.)

By the method of estimation used in Theorem 3, we find that the expected number of pairs (x_i, x_j), such that $i \neq j$ and $|f(x_i) - f(x_j)| < r^\gamma$, is $0(m \log\ 1/r)$. Here we used the requirement that $|x_i - x_j| \geq r$ when $i \neq j$. The $W^{1,p}$-norm of the sum $\sum_N \sum_k c_{Nk}\ \xi_{Nk}\ \psi_{Nk}$ is $0(\log\ 1/r)$. Outside a set of measure $c\ \log^{-2}(1/r)$, the number of pairs referred to above is $\leq m \log^3 1/r$. Thus

$$\nu(f(B), r^\gamma) \geq \frac{1}{2} r^{-\alpha} \log^{-3}(1/r).$$

The same estimate for the exceptional set is valid for a sum $f + g$, provided f and g are independent.

Returning to the set E_0, which we can assume is contained in the unit square, we find a sequence of sets B_j of E_0, and numbers r_j such that $r_j < \exp -j^{-2}$, B_j has size $[r_j^{-\alpha}]$, and its elements are separated by r_j. Moreover, every open set $W \neq \emptyset$ of E_0 contains infinitely many of the sets B_j. For each j we define the random function f_j as above and then define $h = \sum_1^\infty j^{-2} \log^{-1}(1/r_j) f_j$, a series converging in $W^{1,p}(R^n)$, with independent terms. Except for a set of measure $0(j^{-4})$, we have $\nu(h(B_j),\ r_j^\gamma j^{-2} \log^{-1}(1/r_j)) > c r_j^{-\alpha} \log^{-3}(1/r_j)$.

We can now prove that Dim $h(E_0) \geq \beta$ almost surely, using an observation from [1]. In the opposite case $h(E_0)$ is a countable union of sets A_j, of packing exponent $< \beta$. Since A_j and its closure have the same exponents, we conclude with the aid of the Baire Category Theorem that some open set $V \neq \emptyset$ in $h(E)$ has packing exponent $< \beta$. But then $h^{-1}(V)$ is relatively open in E_0, so that V contains infinitely many of the sets $h(B_j)$. This contradicts the almost-sure estimates on the packing numbers of the sets $h(B_j)$, proving Theorem 5.

References

1. Haase, H., *Non-σ-finite sets for packing measure*, Mathematika **33** (1986), 129–136. MR **88a:**28003
2. Mattila, P., *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge studies 44, Cambridge University Press (1995). MR **96h:**28006
3. Stein, E. *Singular integrals and differentiability properties of functions*, Princeton University Press (1970). MR **44:**7280
4. Taylor, S.J. and Tricot, C., *Packing measure and its evaluation for a Brownian path*, Trans. Amer. Math. Soc. **288** (1985), 679–699. MR **87a:**28002
5. Tricot, C., *Two definitions of fractional dimension*, Math. Proc. Camb. Phil. Soc. **91** (1982), 57–74. MR **84d:**28013

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