SOBOLEV SPACES, DIMENSION, AND RANDOM SERIES

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ABSTRACT. We investigate dimension-increasing properties of maps in Sobolev spaces; we obtain sharp results with a random process somewhat like Brownian motion.

INTRODUCTION

We investigate dimension–increasing properties of mappings in the space \( W^{1,p}(\mathbb{R}^n) \), \( p > n \geq 1 \), obtaining exact results as far as the Hausdorff dimension (dim) and the packing dimension (Dim) are concerned. The upper bounds are consequences of Hölder’s inequality; the lower bounds for \( n > 1 \) depend on inequalities from probability theory like those governing Brownian motion.

All functions in \( W^{1,p}(\mathbb{R}^n) \), \( p > n \geq 1 \), are assumed to be continuous. When \( 0 < \alpha < n \) and \( n < p < +\infty \), we define \( \beta = (p\alpha)(p - n + \alpha)^{-1} \), so that \( \alpha < \beta < n \).

**Theorem 1.** Let \( E \) be a subset of \( \mathbb{R}^n \) of finite \( \alpha \)-dimensional measure, and let \( f \in W^{1,p}(\mathbb{R}^n) \). (The range of \( f \) can have any dimension, even infinite.) Then \( f(E) \) has \( \beta \)-measure 0.

**Theorem 2.** Let \( E \) be a closed subset of \( \mathbb{R}^n \), of positive \( \alpha \)-measure, \( 0 < \alpha < 1 \). Then there is a real function \( f \) on \( \mathbb{R} \), such that \( f' \) is in weak \( L^p \), and \( f(E) \) has positive \( \beta \)-measure.

**Theorem 3.** Let \( E \) be a closed subset of \( \mathbb{R}^n \), of positive \( \alpha \)-measure, \( 0 < \alpha < n \). Then \( \dim f(E) \geq \beta \) for some \( f \in W^{1,p}(\mathbb{R}^n) \) mapping \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

Similar bounds and existence theorems are obtained afterwards for the packing dimension [1], [2], [4], [5]. In this part the exposition is less formal.

1. UPPER BOUNDS FOR \( \dim f(E) \)

Constant use is made of the usual system \( Q_{Nk} \) (\( N = 0, \pm 1, \pm 2, \ldots, k \geq 1 \)) of dyadic \( n \)-cubes in \( \mathbb{R}^n \). These can be used in estimating Hausdorff measures in \( \mathbb{R}^n \) at the expense of a constant \( c_n \). Let \( Q \) be a cube of side \( r \), and \( f \in W^{1,p}(\mathbb{R}^n) \). Then \( f(Q) \) has diameter at most \( c_n r^{1-n/p} (\int_Q \|\Delta f\|^p dm)^{1/p} \). This is just an invariant form of a Sobolev inequality [3, p. 124]; it is valid for mappings into normed spaces.
To prove Theorem 1 we cover $E$ by cubes $Q_j$ of side $r_j$, so that $\sum r_j^\beta \leq c(E)$ and $\text{sup}(r_j)$ is as small as we please. Then

$$\text{diam}(f(Q_j)) \leq c r_j^{1-n/p} \left( \int_{Q_j} \| \Delta f \|^p \right)^{1/p},$$

$$\text{diam}(f(Q_j))^\beta \leq c r_j^{(1-n/p)\beta} \left( \int_{Q_j} \| \Delta f \|^p \right)^{\beta/p}.$$  

We use the fact that the cubes $Q_j$ are essentially disjoint. We sum the left side over $j$, using Hölder’s inequality with exponent $q = p\beta^{-1}$ for the factors on the right. The conjugate exponent is $q' = p(p - \beta)^{-1}$; and finally $q'(1 - np^{-1})\beta = \alpha$. Since the union $\bigcup Q_j$ has measure 0(1) sup($r_j)^{n-\alpha}$, we obtain $o(1)$ for the sum $\sum \text{diam}(f(Q_j))^\beta$.

2. Proof of Theorem 2

By Frostman’s Theorem, $E$ carries a probability measure $\mu$ such that $\mu(I) \leq c|I|^{\alpha}$ for all intervals $I$ and some constant $c$. We shall find a strictly increasing function $f$ such that $|f(I)| \geq \mu(I)^{1/\beta}$ for all intervals $I$. Let $f^*(\mu)$ be the image of $\mu$ by the mapping $f : \mu^*(S) = \mu(f^{-1}(S))$ for open sets $S$. When $S$ is an interval, so is $f^{-1}(S)$ and it is evident that $M^*(I) \leq |I|^\beta$ for all intervals $I$, so that $f(E)$ has positive $\beta$-measure.

To find $f$ we require that $f' \geq \mu(I)^{1/\beta}|I|^{-1}$ on every open interval $I$, and prove that this can be done with some $f'$ in weak $L^p$. We define $h(x)$ to be the supremum of $\mu(I)^{1/\beta}|I|^{-1}$ over all intervals $I$ containing $x$. To prove that $h$ is in weak $L^p(R)$, we introduce $h(x, t) = \sup \mu(I)^{t}|I|^{-1}$, where $\alpha^{-1} \leq t \leq 1$. Then $h(x, 1)$ is the Hardy-Littlewood maximal function of $\mu$, and is in weak $L^1$. Again, $h(x, \alpha^{-1})$ is bounded, so that $h(x) = h(x, \beta^{-1})$ is in weak $L^p$, because $\beta^{-1} = p^{-1} + \alpha^{-1}(1 - p^{-1})$.

This completes Theorem 2; to obtain $f' \in L^p$ and dim $f(E) \geq \beta$, we decrease $h$ slightly to $h^-(x) = \sup \mu(I)^{1/\beta}|I|^{-1} \log^{-1}(e + |I|^{-1})$.

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For Theorem 3 we use the system $Q_{Nk}$ of dyadic squares of side $2^{-N}$, where $N = 0, \pm 1, \pm 2, \ldots$ and $k = 1, 2, 3, \ldots$. These define a sequence of bump functions $\psi_{Nk}$, equal to 1 on $Q_{Nk}$, and vanishing outside the cube $Q_{Nk}^\gamma$ obtained by expanding $Q_{Nk}$ on its center by a factor $5/4$; moreover $0 \leq \psi_{Nk} \leq 1$ and $\| \nabla \psi_{Nk} \| \leq c_n 2^N$. For a sum $u = \sum c_k \psi_{Nk}$ we find $\| \nabla u \|^p_p \approx 2^{Np} 2^{-nN} \sum |c_k|^p$. Let $\mu$ be a probability measure in $R^n$, satisfying the Hölder condition in exponent $\alpha$ and $c_k = \mu(Q_{Nk})^{1/\beta}$. We claim that $\| \nabla u \|^p_p$ is bounded by a number independent of $k$; here $\alpha, \beta, n, p$ are the numbers defined before. In fact $\sum_k \mu(Q_{Nk}) \leq C_1$ and $\mu(Q_{Nk}) \leq C_2 2^{-N^\gamma}$. Thus $\sum_k c_k \leq C_3 2^{-N^\gamma}$ where $\gamma = \alpha(p\beta^{-1} - 1)$. To complete our claim we observe that $p - n = \gamma$.

In fact we use a more involved variant of the function $u$. First we replace $\mu(Q_{Nk})$ by $\mu(Q_{Nk}^\gamma)$, where $Q_{Nk}^\gamma$ is obtained from $Q_{Nk}$ by expansion in a ratio of 8. Moreover we introduce a sequence of random multipliers $\xi_{Nk}$, independent random variables uniformly distributed on the unit ball in $R^n$. We call the variant so obtained $v_N$ and observe that the estimates on $\nabla v_N$ are almost the same as the ones above.

Suppose now that $E$ is contained in the unit cube of $R^n$, and that $\mu$ is concentrated in $E$. We define $v = \sum_0^\infty (N + 1)^{-2} v_N$, and prove that the support of $v^*(\mu)$
has Hausdorff dimension at least $\beta$ a.s., as follows. Let $k(t) = t^{-\beta} \log^{-2n-2}(e+t^{-1})$ for $t > 0$. Then the energy of $v^*(\mu)$ with respect to the kernel $k$ is

$$I = \int \int k(v(x) - v(y))\mu(dx)\mu(dy).$$

If $I < +\infty$, then $v^*(\mu)$ vanishes on sets of dimension $< \beta$; we prove that $\mathcal{E}(I) < +\infty$, whence our assertion on the support of $v^*(\mu)$ follows.

In fact we prove that $\int \mathcal{E}(k(v(x) - v(y)))\mu(dy) \leq c < +\infty$ for all $x$ in the unit cube, whence $\mathcal{E}(I) < +\infty$ by Fubini’s Theorem. For each $x$ and $y$, $v(x) - v(y) = \sum \sum a_{nk}(x,y)\xi_{nk}$; the variables $\xi_{nk}$ are those described above. Let $\rho(x,y) = \sup |a_{nk}(x,y)|$ so that $0 \leq \rho \leq 2$. We easily see that the expectation of $k(v(x) - v(y))$ is $\leq c\kappa(\rho)$, since $0 < \beta < n$. Thus we seek a uniform upper bound for $\int k(\rho(x,y))\mu(dy)$.

Let $\nu(x,y)$ be the largest integer $N \geq 0$ such that $x$ and $y$ belong to a single square $Q_N$ or to touching squares of side $2^{-N}$. (When $x \neq y$, then $\nu(x,y) < +\infty$.) Then there will be a bump function $\psi_{N+1}$, of the next generation, such that $\psi_{N+1}(x) = 0, \nu(x,y) = 0$. More precisely, $x \in Q_{N+1}, y \notin Q_{N+1},$ however $y \in Q_{N+1}$. Thus, if we partition the domain of integration into the sets defined by $\nu(x,y) = 0$, $\nu(x,y) = 1, \nu(x,y) = N, \nu(x,y) = N, \ldots$, then the $\mu$-measure of the $N^{th}$ set is $< \mu(Q_{N+1}^{*}) = \mu_{N+1}$ but for each $y$ in the set $\rho(x,y) \geq (N+1)^{-2}\mu_{N+1}$. Moreover $\mu_{N+1} < c2^{-2N\alpha}$. Observing the logarithm in $k(t)$ we see that the integral of $k(\rho(x,y))$ on the $N^{th}$ set is $0(N^{-2})$, since the logarithm adds a factor $cN^{2\alpha-2}$, which balances the factor $N^{2\beta}$ arising from the formula for $v$. We observe also that $\nu(x,y) \geq 0$ because $E$ is in the unit cube, and when $\nu(x,y) = 0$, then $\rho(x,y) = 1$. Thus we have verified the estimate on $\int \mathcal{E}(k(v(x) - v(y)))\mu(dy)$ and Theorem 3 is completely proved.

4. Packing dimension

This concept is derived from the more precise notion of packing measure in much the same way as Hausdorff dimension is derived from Hausdorff measure; thus the following definition is part of a larger picture, [1], [2, pp. 82–86], [4]. Let $S$ be a set in a metric space and $r > 0$; then $\nu(S,r)$ is the smallest number of sets of diameter at most $r$, sufficient to cover $S$. The packing exponent $\delta(S) = \limsup \nu(S,r)/-\log r, r \to 0$. (This notion is standard, but the name is not.) A variant concept is the number $\nu^*(S,r)$, the largest number of elements of $S$ separated by more than $r$. Then $\nu^*(S,R) \leq \nu(S,r) \leq \nu^*(S,1/r)$, so that $\nu^*$ can be used to define $\delta(S)$. We say that $S$ has packing dimension $\leq \alpha$, $\dim S \leq r$, if for each $\epsilon > 0, S = \bigcup \infty S(m,\epsilon)$ where $\delta(S(m,\epsilon)) < \alpha + \epsilon$.

**Theorem 4.** Let $f \in W^{1,p}(\mathbb{R}^n), E \subset \mathbb{R}^n$, and $\dim E \leq \alpha$. Then $\dim f(E) \leq \beta$.

**Proof.** In calculating the exponent of packing of a set in $E^n$, we can use dyadic squares of the same side $2^{-N}$. Let $\gamma = \alpha\beta^{-1}$ and $r = 2^{-N\gamma}$. A cube $Q$ of side $2^{-s}$, $s \geq N$, is called major if $\text{diam } f(Q) \geq r$, minor in the opposite case, and critical if it is major, but each of its descendants is minor. Thus a critical cube is the union of $2^s$ minor cubes. Therefore critical cubes play almost the same role as minor cubes, except that they can be counted by counting the number of major cubes. Since $f$ is uniformly continuous in $\mathbb{R}^n$, major cubes can be subdivided successively until critical cubes are encountered. We shall first count all major
Let $\alpha \leq T$.

The same estimate for the exceptional set is valid for a sum of the images $\sum_{i} f_{i}$ and this suffices because $\sum_{i} m_{i} \leq m$.

Henceforth we replace $E$ by $E_{0}$ in the argument. In place of energy integrals, appearing in Theorem 3, we employ the following consequence of Cauchy's inequality. With more attention to inequalities, Theorem 4 can be proved with a weaker hypothesis, Dim $E > \alpha$. We obtain the following consequence of Cauchy's inequality. A set $B$ has $m$ distinct elements $x_{1}, \ldots, x_{m}$ and is partitioned into $k$ subsets $T_{1}, \ldots, T_{k}$. Then the number $N$ of pairs $(x_{i}, x_{j})$ such that $x_{i}$ and $x_{j}$ belong to the same set $T_{\nu}$, $1 \leq \nu \leq k$, is at least $m^{2}/k$. Counting the number $N_{0}$ of pairs in which $x_{i} \neq x_{j}$, we obtain $N_{0} \geq m^{2}/k - m$.

Next we describe a basic step in Theorem 5, with some heuristics. The set $B$ is located in the unit cube of $R^{n}$, and its elements are separated by $r$, $0 < r < e^{-2}$, while the size $m$ of $B$ is $[r^{-\alpha}]$. We seek a random function $f$ so that nearly all of the images $f(x_{i})$, $1 \leq i \leq m$, are separated by $r^{\gamma}$, $\gamma = \alpha\beta^{-1}$. (This would reverse the inequalities obtained in Theorem 4) A cube $Q$ containing $\ell \geq 2$ of the points $x_{i}$ should be mapped on a set of diameter $\geq cr^{-\ell^{1/n}}$. This suggests the following choice of coefficients in the random function $f$; only numbers $N$ such that $1 \geq 2^{-N} \geq r/8n$ are used in the summation. Let $\lambda$ be the counting measure of the set $B$, and the coefficient of $c_{Nk}$ of $\psi_{Nk}$ be $r^{\gamma} \lambda(Q^{**})^{1/n}$. First we have to verify the $W^{1,p}$-type inequality for the indicated values of $N$, i.e. $\sum_{k} c_{Nk} \leq c_{2}^{N(n-p)}$ or $\sum_{k} \lambda(Q_{Nk}^{**})^{p/n} \leq c_{2}^{N(n-p)}$. Now the total mass of $\lambda$ is $< r^{-\alpha}$ and $\lambda(Q_{Nk}^{**}) \leq c(2^{-N}r^{-1})^{n}$ since $2^{-N} \geq r/8n$. We obtain $\sum_{k} \lambda(Q_{Nk}^{**})^{p/n} \leq c_{2}^{N(n-p)} - \gamma p$. (We don’t use the factors $(N + 1)^{-2}$ in the sum, so there is a small correction later.)

By the method of estimation used in Theorem 3, we find that the expected number of pairs $(x_{i}, x_{j})$, such that $i \neq j$ and $|f(x_{i}) - f(x_{j})| < r^{\gamma}$, is $0(m \log 1/r)$. Here we used the requirement that $|x_{i} - x_{j}| \geq r$ when $i \neq j$. The $W^{1,p}$-norm of the sum $\sum_{Nk} c_{Nk} \xi_{Nk} / \psi_{Nk}$ is $0(\log 1/r)$. Outside a set of measure $c \log^{-2}(1/r)$, the number of pairs referred to above is $\leq m \log^{3} 1/r$. Thus

$$\nu(f(B), r^{\gamma}) \geq \frac{1}{2} r^{-\alpha} \log^{-3}(1/r).$$

The same estimate for the exceptional set is valid for a sum $f + g$, provided $f$ and $g$ are independent.
Returning to the set $E_0$, which we can assume is contained in the unit square, we find a sequence of sets $B_j$ of $E_0$, and numbers $r_j$ such that $r_j < \exp -j^{-2}$, $B_j$ has size $[r_j^{-\alpha}]$, and its elements are separated by $r_j$. Moreover, every open set $W \neq \emptyset$ of $E_0$ contains infinitely many of the sets $B_j$. For each $j$ we define the random function $f_j$ as above and then define $h = \sum_{j=1}^{\infty} j^{-2} \log^{-1}(1/r_j)f_j$, a series converging in $W^{1,p}(R^n)$, with independent terms. Except for a set of measure $0(j^{-4})$, we have $\nu(h(B_j), r_j^{-2} \log^{-1}(1/r_j)) > c r_j^{-\alpha} \log^{-3}(1/r_j)$.

We can now prove that $\text{Dim } h(E_0) \geq \beta$ almost surely, using an observation from [1]. In the opposite case $h(E_0)$ is a countable union of sets $A_j$, of packing exponent $< \beta$. Since $A_j$ and its closure have the same exponents, we conclude with the aid of the Baire Category Theorem that some open set $V \neq \emptyset$ in $h(E)$ has packing exponent $< \beta$. But then $h^{-1}(V)$ is relatively open in $E_0$, so that $V$ contains infinitely many of the sets $h(B_j)$. This contradicts the almost-sure estimates on the packing numbers of the sets $h(B_j)$, proving Theorem 5.

References


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