PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 2, Pages 419–426
S 0002-9939(99)05546-X
Article electronically published on October 12, 1999

A FIXED POINT THEOREM AND ITS APPLICATION TO INTEGRAL EQUATIONS IN MODULAR FUNCTION SPACES

A. AIT TALEB AND E. HANEBALY

(Communicated by Dale Alspach)

ABSTRACT. In this paper we present a fixed point theorem of Banach type in modular spaces. Also, we give some applications of this result to a nonlinear integral equation in Musielak-Orlicz space.

0. INTRODUCTION

It is well known that one of the standard proofs of Banach's fixed point theorem is based on Cantor's theorem in complete metric spaces [3, 4]. To this end, using some convenient constants in the contraction assumption, we present a generalization of Banach's fixed point theorem in some classes of modular spaces, where the modular is s-convex, having the Fatou property and satisfying the Δ_2-condition.

As an application we study the existence of solutions for an integral equation of Lipschitz type in a Musielak-Orlicz space.

We begin by recalling some basic concepts of modular spaces; for more information, we refer to the books by Musielak [8] and Kozlowski [7].

Definition 0.1. Let X be an arbitrary vector space over K $(= \mathbb{R}$ or $\mathbb{C})$.

(a) A functional $\rho : X \to [0, +\infty]$ is called modular if:

(i) $p(x) = 0 \Leftrightarrow x = 0$.
(ii) $p(\alpha x) = p(x)$ for $\alpha \in K$ with $|\alpha| = 1$, $\forall x \in X$.
(iii) $p(\alpha x + \beta y) \leq p(x) + p(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$, $\forall x, y \in X$.

(b) If (iii) is replaced by:

(iii′) $p(\alpha x + \beta y) \leq \alpha^s p(x) + \beta^s p(y)$ for $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with an $s \in]0, 1]$, the modular p is called an s-convex modular; and if $s = 1$, ρ is called convex modular.

(c) A modular ρ defines a corresponding modular space, i.e. the space X_ρ given by:

$$X_\rho = \{x \in X \mid \rho(\lambda x) \underset{\lambda \to 0}{\to} 0\}.$$

Received by the editors October 30, 1997.
1991 *Mathematics Subject Classification.* Primary 46A80, 47H10, 45G10, 46E30.
Key words and phrases. Modular space, fixed point, integral equation.

Remarks. 1. Note that in general there is no reason to expect the subadditivity of a modular ρ. Nevertheless, in view of (iii) from Definition 0.1 the inequality $p(x+y) \leq p(2x) + \rho(2y)$ holds.

2. If ρ is convex modular, the modular space X_ρ can be equipped with a norm called the Luxemburg norm defined by:

$$|x|_\rho = \inf \left\{ \alpha > 0, p\left(\frac{x}{\alpha}\right) \leq 1 \right\}.$$

3. As a classical example, we would like to mention the Musielak-Orlicz space denoted by L^φ [8] and the modular function space denoted by L^ρ [7].

Definition 0.2. Let X_ρ be a modular space sequence.

(a) A sequence $(x_n)_n$ in X_ρ is said to be

(i) ρ-convergent to x if $p(x_n - x) \to 0$ as $n \to \infty$.
(ii) ρ-Cauchy if $p(x_n - x_m) \to 0$ as $n, m \to \infty$.

(b) X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(c) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_n \subset B$, if: $x_n \to x$ then: $x \in B$. $\overline{B}^\rho$ denotes the closure of B in the sense of ρ.

(d) A subset $B \subset X_\rho$ is called ρ-bounded if

$$\delta_\rho(B) = \sup_{x,y \in B} \rho(x-y) < \infty;$$

$\delta_\rho(B)$ is called the ρ-diameter of B.

(e) We say that ρ has the Fatou property if:

$$\rho(x-y) \leq \underline{\lim}\, \rho(x_n - y_n)$$

whenever $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$ as $n \to \infty$.

(f) ρ is said to satisfy the Δ_2-condition if:

$$\rho(2x_n) \to 0 \text{ as } n \to \infty \quad \text{whenever} \quad \rho(x_n) \to 0 \text{ as } n \to \infty.$$

I. Fixed point theorem

I.1. **Theorem I.1.** *Let X_ρ be a ρ-complete modular space. Assume that ρ is an s-convex modular satisfying the Δ_2-condition and has the Fatou property. Let B be a ρ-closed subset of X_ρ. $T : B \to B$ is a mapping such that*:

$$(*) \qquad \exists c, k \in \mathbb{R}^+ : c > \max(1,k), \rho(c(Tx - Ty)) \leq k^s \rho(x-y) \quad \forall x, y \in B.$$

Then T has a fixed point.

Remarks. 1. It is natural to introduce the constants c and k in the assumption of strict contraction in modular spaces. Note also that Theorem I.1 and its proof become simpler in the particular case where $s = 1$ (ρ is convex) and $c = 2 > k > 0$.

2. The contraction $(*)$ in Theorem I.1 is also true for any constant c_0 such that $1 < c_0 \leq c$:

$$\rho(c_0(Tx - Ty)) = \rho\left(\frac{c_0}{c} c(Tx - Ty)\right) \leq \left(\frac{c_0}{c}\right)^s \rho(c(Tx - Ty))$$
$$\leq k_0^s \rho(x-y) \quad \text{where } k_0 = \frac{c_0}{c} k < c_0, \text{ since } \frac{k}{c} < 1.$$

I.2. **Proof of Theorem I.1.** 1*st step.* The proof of Theorem I.1 is based on the next result which is the modular formulation of Cantor's theorem:

Theorem I.2. *Let X_ρ be a ρ-complete modular space. Let $(F_n)_n$ be a decreasing sequence of nonempty, ρ-closed subsets of X_r with: $\delta_\rho(F_n) \to 0$ as $n \to \infty$. Then $\bigcap_n F_n$ is reduced to one point.*

The proof of this result uses the same ideas as in complete metric spaces and it suffices to replace the distance by the modular ρ.

2*nd step.* Let $(\varepsilon_n)_n$ be a decreasing sequence of positive numbers such that: $\lim_{n\to\infty} \varepsilon_n = 0$, and consider the sets defined by:

$$M_{\varepsilon_n} = \{x \in B \mid \rho(L(x - Tx)) \le \varepsilon_n\},$$

where $L = \max\{c, 2\alpha\}$ and a is the s-conjugate of c, i.e. $\frac{1}{c^s} + \frac{1}{\alpha^s} = 1$.

Then the sequence (M_{ε_n}) has the following property:

1) $M_{\varepsilon_n} \neq \varnothing$, $\forall n$.

We assume without any loss of generality that $\exists x \in B$ such that $\rho(x - Tx) < \infty$. Then for $p \in \mathbb{N}^*$ we have:

$$\begin{aligned}\rho(c(Tx^{p+1} - Tx^p)) &\le k^s \rho(Tx^p - Tx^{p-1}) = k^s \rho\left(\frac{1}{c} c(T^p x - T^{p-1}x)\right)\\ &\le \left(\frac{k}{c}\right)^s k^s \rho(T^{p-1}x - T^{p-2}x).\end{aligned}$$

By induction we deduce

$$\rho(c(T^{p+1}x - T^p x)) \le \left(\frac{k}{c}\right)^{(p-1)s} \rho(x - Tx)$$

and since $\frac{k}{c} < 1$ we have $\rho(c(T^{p+1}x - T^p x)) \to 0$ as $p \to \infty$. Thus by (Δ_2) $\rho(L(T^{p+1}x - T^p x)) \to 0$ as $p \to \infty$.

Hence $\exists q \in \mathbb{N}^*$ such that $\rho(L(T^{q+1}x - T^q x)) \le \varepsilon_n$. Then $y = T^q x \in M_{\varepsilon_n}$.

2) M_{ε_n} is ρ-closed.

Let $(x_p)_p$ be a sequence in M_{ε_n}. Assume that $(x_p)_p$ is ρ-convergent to $x \in X_\rho$. Since $(x_p)_p \subset B$ and B is ρ-closed, it follows that $x \in B$

$$\rho(x_p - x) \to 0 \text{ as } p \to \infty, \text{ by } (\Delta_2) \quad \rho(L(x_p - x)) \to 0 \text{ as } p \to \infty.$$

On the other hand

$$\rho(c(Tx_p - Tx)) \le k^s \rho(x_p - x) \le k^s \rho(L(x_p - x)).$$

Then $\rho(c(Tx_p - Tx)) \to 0$ as $p \to \infty$.

Again by (Δ_2) $\rho(L(Tx_p - Tx)) \to 0$ as $p \to \infty$. The Fatou property implies that

$$\rho(L(Tx - x)) \le \lim \rho(L(Tx_p - x_p)) \le \varepsilon_n.$$

Therefore $x \in M_{\varepsilon_n}$ and hence M_{ε_n} is ρ-closed.

3) $\delta_\rho(M_{\varepsilon_n}) \to 0$ as $n \to \infty$.

Let $x, y \in M_{\varepsilon_n}$; we have

$$\begin{aligned}\rho(x-y) &= \rho\left[\frac{\alpha(x-Tx)}{\alpha} + \frac{c(Tx-Ty)}{c} + \frac{\alpha(Ty-y)}{\alpha}\right] \quad \text{where } \alpha^s = \frac{c^s}{c^s-1} \\ &\leq \frac{1}{\alpha^s}\rho(\alpha(x-Tx)+\alpha(Ty-y)) + \frac{1}{c^s}\rho(c(Tx-Ty)) \\ &\leq \frac{1}{2^s\alpha^s}\rho(2\alpha(x-Tx)) + \frac{1}{2^s\alpha^s}\rho(2\alpha(Ty-y)) + \frac{1}{c^s}\rho(c(Tx-Ty)) \\ &\leq \frac{1}{2^s\alpha^s}(\rho(L(x-Tx)) + \rho(L(Ty-y))) + \frac{1}{c^s}\rho(c(Tx-Ty)).\end{aligned}$$

Then

$$\rho(x-y) \leq \frac{1}{2^s\alpha^s}2\varepsilon_n + \left(\frac{k}{c}\right)^s \rho(x-y).$$

Hence $\rho(x-y) \leq \frac{2\varepsilon_n}{2^s\alpha^s}\frac{c^s}{c^s-k^s}$ and we deduce that

$$\delta_\rho(M_{\varepsilon_n}) = \sup_{x,y\in M_{\varepsilon_n}} \rho(x-y) \leq \varepsilon_n 2^{1-s}\frac{c^s-1}{c^s-k^s} \quad \text{and} \quad \delta_\rho(M_\varepsilon) \to 0 \text{ as } n \to \infty.$$

4) (M_{ε_n}) is decreasing; this is an immediate consequence of the fact that (ε_n) is decreasing.

It follows that the conditions of Cantor's theorem are satisfied and hence we have $\bigcap_n M_{\varepsilon_n} = \{x_0\}$.

But $x_0 \in M_{\varepsilon_n}\ \forall n \Rightarrow \rho(L(x_0 - Tx_0)) \leq \varepsilon_n \to 0 \Rightarrow Tx_0 = x_0$.

Remark. We are unable to prove whether the conclusion of Theorem I.1 is true if we have $c = 1$ and $0 < k < 1$. To this end, recall the results by Khamsi-Kozlowski-Reich:

Theorem I.3 ([5])**.** *Let ρ be a function modular satisfying the Δ_2-condition and let B be a $|\cdot|_\rho$-closed subset of L_ρ. $T : B \to B$ is a mapping such that*:

$$\exists 0 < k < 1 \,|\, \rho(Tf - Tg) \leq k\rho(f-g)\ \forall f, g \in B.$$

Then T has a fixed point if one of the following assumptions is satisfied:

(i) $\exists f_0 \in B$; $\sup_n(2T^n f_0) < \infty$.

(ii) *B is ρ-bounded.*

If (ii) is satisfied, the fixed point is unique. Note also that the modular ρ in Theorem I.3 has the Fatou property and the Δ_2-condition as in Theorem I.1. Also by (Δ_2) we have B is ρ-closed $\Leftrightarrow B$ is $|\cdot|\rho$-closed.

On the other hand the strict inequality $(*)$ in Theorem I.1 implies the inequality of Theorem I.3.

$$\rho(Tx-Ty) = \rho\left(\frac{1}{c}(c(Tx-Ty))\right) \leq \frac{1}{c^s}\rho(c(Tx-Ty)) \leq \frac{k^s}{c^s}\rho(x-y).$$

Consequently with some reinforced assumptions in Theorem I.1, namely the s-convexity of ρ and the strict contraction $(*)$, we prove the existence of a fixed point for T without restrictive conditions concerning domain of T $(\operatorname{dom} T)$: (i) or (ii) in Theorem I.3.

The following example shows that Theorem I.1 can be more appropriate for applications.

II. Application

II.1. **General frame.** Consider the following integral equation:

$$u(t) = e^{-t} f + \int_0^t e^{s-t} Tu(s)\, ds \tag{I}$$

where:

(i) $T : B \to B$ with B a ρ-closed, convex subset of a MUSIELAK-ORLICZ space L^φ satisfying the Δ_2-condition.

(ii) T is p-Lipschitz:

$$\exists \gamma > 0, \rho(Tu - Tv) \le \gamma \rho(u - v), \qquad u, v \in B.$$

(iii) $f \in B$.

Theorem II.1. *Under these conditions, for all $A > 0$ the integral equation* (I) *has a solution $u \in C^\varphi = C([0, A], L^\varphi)$. C^φ is the modular space of continuous mappings from $[0, A]$ into L^φ.*

By iterative techniques, Khamsi [6] has shown this result under supplementary conditions: B is ρ-bounded and T is ρ-Lipschitz with constant $\gamma = 1$.

To delete all restrictive assumptions on the Lipschitz constant γ we introduce the space C^φ equipped with a convenient modular. Our method follows the standard technique of the resolution of differential equations or integral equations of Lipschitz type in Banach spaces. See Deimling [2, p. 39].

II.2. **Functional frame: the modular space C^φ.**

Definition II.1. $u : I \to L^\varphi$, where $I = [0, A]$, is said to be continuous at $t_0 \in I$ if: for $t_n \in I$, $t_n \to t_0$ as $n \to 0 \Rightarrow \rho(u(t_n) - u(t_0))_\rho \to 0$ as $n \to 0$.

Since ρ satisfies the Δ_2-condition, it is equivalent to:

$$t_n \to t_0 \text{ as } n \to 0 \Rightarrow |u(t_n) - u(t_0)|_\rho \to 0 \text{ as } n \to 0.$$

Let $C^\varphi = C(I, L^\varphi)$ be the space of all continuous mappings from $I = [0, A]$ into L^φ.

Proposition II.1. *Suppose that the modular ρ satisfies* (Δ_2) *and $B \subset L^\varphi$ is a ρ-closed, convex subset of L^φ. For $a \ge 0$ let $\rho_a(u) = \sup_{t \in I} e^{-at} \rho(u(t))$ for $u \in C^\varphi$. Then*:

(1) *(C^φ, ρ_a) is a modular space, and ρ_a is a convex modular satisfying the Fatou property and the Δ_2-condition.*

(2) *C^φ is ρ_a-complete.*

(3) *$C_0^\varphi = C(I, B)$ is a ρ_a-closed, convex subset of C^φ.*

Proof. (1) (i) C^φ is a real vector space. Let $u, v \in C^\varphi$, $t_0 \in I$. Then for $t_n \in I$ such that $t_n \to t_0$ as $n \to \infty$ we have:

$$\rho\left(\frac{(u+v)(t_n) - (u+v)(t_0)}{2}\right) \le \frac{1}{2}\rho(u(t_n) - u(t_0)) + \frac{1}{2}\rho(v(t_n) - v(t_0)).$$

Hence

$$\rho\left(\frac{(u+v)(t_n) - (u+v)(t_0)}{2}\right) \to 0 \text{ as } n \to \infty,$$

and by the (Δ_2)-condition: $\rho((u+v)(t_n) - (u+v)(t_0)) \to 0$ as $n \to \infty$ which implies that $(u + v)$ is continuous at t_0.

Again by (Δ_2) $\rho(\lambda(u(t_n) - u(t_0))) \to 0$ as $n \to \infty$ for all $\lambda \in \mathbb{R}$; then λu is also continuous at t_0.

(ii) ρ_a is well defined. Since ρ satisfies the (Δ_2)-condition, the domain of ρ is $\{f \in L^\varphi, \rho(f) < \infty\} = L^\varphi$ and since ρ is convex, $|\cdot|_\rho$-continuous at 0 it follows that ρ is $|\cdot|_\rho$-continuous on L^φ. See Zeidler [9, p. 383].

Consequently for all $u \in C^\varphi$, $\rho_a(u)$ has a meaning.

(iii) ρ_a is a convex modular. This is a simple consequence of the fact that ρ is a convex modular.

(iv) ρ_a satisfies the Fatou property. Let $(u_n)_n$ (resp. $(v_n)_n$) be a sequence in C^φ, ρ_a convergent to u (resp. to v) $\in C^\varphi$. Then $\forall t \in I$ $\rho(u(t_n) - u(t)) \to 0$ as $n \to \infty$ and $\rho(v_n(t) - v(t)) \to 0$ as $n \to \infty$.

Since ρ satisfies the Fatou property we have:

$$\rho(u(t) - v(t)) \le \underline{\lim}\, \rho(u_n(t) - v_n(t)) \quad \forall t \in I,$$
$$e^{-at}\rho(u(t) - v(t)) \le \underline{\lim}\, e^{-at}\rho(u_n(t) - v_n(t)) \le \underline{\lim}\, \rho_a(u_n - v_n)$$

and then $\rho_a(u - v) \le \underline{\lim}\, \rho_a(u_n - v_n)$.

(v) ρ_a has the (Δ_2)-condition. Since ρ satisfies the (Δ_2)-condition, one has:

$$\begin{aligned} \rho_a(u_n) \to 0 \text{ as } n \to \infty &\Leftrightarrow \forall t \in I, e^{-at}\rho(u_n(t)) \to 0 \text{ as } n \to \infty \\ &\Leftrightarrow \forall t \in I, e^{-at}\rho(2u_n(t)) \to 0 \text{ as } n \to \infty \\ &\Leftrightarrow \rho_a(2u_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

(2) It is known that $(L^\varphi, |\cdot|_\rho)$ is a Banach space. Then the space $(C^\varphi, |\cdot|\rho_a)$ is also Banach. If $(u_n)_n$ is a ρ_a-Cauchy sequence in C^φ, by the Δ_2-condition it is $|\cdot|_\rho$-Cauchy. Hence $\exists u \in C^\varphi$ such that $(u_n) \to u$ in $(C^\varphi, |\cdot|\rho_a)$.

Consequently $(u_n) \to u$ in (C^φ, ρ_a). It follows that: C^φ is ρ_a-complete.

(3) $C_0^\varphi = C(I, B)$ is convex and ρ_a-closed. The convexity of C_0^φ and its ρ_a-closedness are clearly obtained by convexity and ρ-closedness of B in L^φ. □

II.3. **Proof of Theorem II.1.** Define the operator S over C_0^φ by:

$$\forall u \in C_0^\varphi, \quad Su(t) = e^{-t} f + \int_0^t e^{s-t} Tu(s)\, ds \quad \forall t \in I.$$

1*st step.* We show that $S : C_0^\varphi \to C_0^\varphi$.

(i) Su is continuous from I into $(L^\varphi, |\cdot|_\rho)$. Let $t_n, t_0 \in I$ with $t_n \to t_0$ as $n \to \infty$. T is ρ-Lipschitz $\rho(Tu(t_n) - Tu(t_0)) \le \gamma\rho(u(t_n) - u(t_0))$. Since u is ρ-continuous, Tu is then ρ-continuous at t_0 and by (Δ_2) Tu is $|\cdot|_\rho$-continuous at t_0. Hence Su is $|\cdot|_\rho$-continuous at t_0.

(ii) $Su(t) \in B$, $\forall t \in I$. It is well known that in the Banach space $(L^\varphi, |\cdot|_\rho)$

$$\begin{aligned} \int_0^t e^{s-t} Tu(s)\, ds &\in \left(\int_0^t e^{s-t}\, ds\right) \overline{\text{co}}\{Tu(s),\ 0 \le s \le t\} \\ &\le (1 - e^{-t})\overline{\text{co}} B, \end{aligned}$$

where $\overline{\text{co}}B$ is the closed convex hull of B in $(L^\varphi, |\cdot|_\rho)$. But B is convex and ρ-closed; then $\overline{\text{co}}B = \overline{B} \subset \overline{B}^\rho = B$. Hence $Su(t) \in e^{-t}B + (1 - e^{-t})B \subseteq B$ $\forall t \in I$.

2*nd step.* For $u, v \in C_0^\varphi$ and $\lambda > 0$ we have:

$$\lambda(Su(t) - Sv(t)) = \int_0^t \lambda e^{s-t}(Tu(s)Tv(s))\, ds.$$

Lemma II.1 ([6])**.** *Let $x \in C^\varphi$ and $0 < \lambda \leq \frac{e^A}{e^A - 1}$. Then*

$$\rho\left(\int_0^t \lambda e^{s-t} x(s)\,ds\right) \leq \lambda \frac{e^{at} - e^{-t}}{1+a} \rho_a(x).$$

Note that in [6] this result was shown with $\lambda = 1$ and $a = 0$.

Proof. Let $T = \{t_0, t_1, \ldots, t_n\}$ be any subdivision of $[0, t]$. $\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i - t} x(t_i)$ is $|\cdot|_\rho$-convergent, and consequently, ρ-convergent to $\int_0^t \lambda e^{s-t} x(s)\,ds$ in L^φ when $|T| = \sup\{|t_{i+1} - t_i|, i = 0, \ldots, n-1\} \to 0$ as $n \to \infty$. By Fatou we have

$$\rho\left(\int_0^t \lambda e^{s-t} x(s)\,ds\right) \leq \underline{\lim}\, \rho\left(\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i - t} x(t_i)\right).$$

On the other hand

$$\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i - t} \leq \int_0^t \lambda e^{s-t}\,ds = (1 - e^{-t})\lambda \leq \lambda(1 - e^{-A}).$$

Since $0 < \lambda \leq \frac{e^A}{e^A - 1}$ it follows that $\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i - t} \leq 1$, and the convexity of ρ implies

$$\begin{aligned}
\rho\left(\sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i - t} x(t_i)\right) &\leq \sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i - t} \rho(x(t_i)) \\
&= \sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{t_i - t} e^{at_i} e^{-at_i} \rho(x(t_i)) \\
&\leq \sum_{i=0}^{n-1} \lambda(t_{i+1} - t_i) e^{(1+a)^{t_i - t}} \rho_a(x) \\
&\leq \left(\int_0^t \lambda e^{(1+a)s - t}\,ds\right) \rho_a(x).
\end{aligned}$$

Then

$$\rho\left(\int_0^t \lambda e^{s-t} x(s)\,ds\right) \leq \lambda \frac{e^{at} - e^{-t}}{1+a} \rho_a(x).$$

3rd step.

$$\rho(\lambda(Su(t) - Sv(t))) \leq \lambda \frac{e^{at} - e^{-t}}{1+a} \rho_a(Tu - Tv),$$

but $\rho_a(Tu - Tv) = \sup_{t \in I} e^{-at} \rho(Tu(t) - Tv(t)) \leq \gamma \rho_a(u - v)$. Then

$$\begin{aligned}
e^{-at} \rho(\lambda(Sa(t) - Sv(t))) &\leq \frac{\lambda\gamma}{1+a}(1 - e^{-(1+a)t}) \rho_a(u - v) \\
&\leq \frac{\lambda\gamma}{1+a}(1 - e^{-(1+a)A}) \rho_a(u - v), \quad \forall t \in I.
\end{aligned}$$

Hence $\rho_a(\lambda(Su - Sv)) \leq \frac{\lambda\gamma}{1+a}(1 - e^{-(1+a)A}) \rho_a(u - v)$. Consider λ, $1 < \lambda \leq \frac{e^A}{e^A - 1}$. Then S has a fixed point if

$$\frac{\lambda\gamma}{1+a}(1 - e^{-(1+a)A}) < \lambda \Leftrightarrow \frac{\gamma}{1+a}(1 - e^{-(1+a)A}) < I.$$

The last inequality is satisfied if we take for example $a \geq \gamma$. In the end, by Theorem I.1, S has a fixed point which is a solution of the integral equation (I). $\square$

References

1. A. Ait Taleb, *Points fixes et Applications aux equations intégrales dans les espaces modulaires.* Thèse de 3ème cycle, Département de Mathématiques et Informatique, Rabat (1996).
2. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985. MR **86j:**47001
3. K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry and nonexpansive mappings*, Marcel Dekker, New York, 1984. MR **86d:**58012
4. K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990. MR **92c:**47070
5. M. A. Khamsi, W. M. Kozlowski, and S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Anal. **14** (1990), 935–953. MR **91d:**47042
6. M. A. Khamsi, *Nonlinear semigroups in modular function space*, thèse d'état. Département de Mathématiques, Rabat (1994).
7. W. M. Kozlowski, *Modular function spaces*, Marcel Dekker, 1988. MR **99d:**46043
8. J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, 1983. MR **85m:**46028
9. E. Zeidler, *Nonlinear functional analysis and its applications.* III, Springer-Verlag, 1985. MR **90b:**49005

Department of Mathematics and Informatic, University of Mohammed V, BP 1014, Rabat, Morocco