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ERRATUM TO "ON THE RELATIONSHIP BETWEEN CONVERGENCE IN DISTRIBUTION AND CONVERGENCE OF EXPECTED EXTREMES"

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As pointed out to us by Slawomir Kolodynski, Theorem 1.1 of [1] is true as stated for nonnegative random variables. The statement "The argument to show ≥ 0 is analogous" which appears on page 1238 is false however, and results in the erroneous conclusion of uniform integrability based on $\alpha_k = o(k)$ alone. The needed uniform integrability does follow by adding a strictly analogous argument to the one given, but utilizing $\beta_k = o(k)$ for the expected minima m_k. The corrected version of Theorem 1.1 is as follows.

Theorem 1.1′. *Suppose $\{X_n\}$, $n \geq 1$, are integrable random variables such that $\lim_{n\to\infty} M_k(X_n) = \alpha_k$ exists and is finite for all $k \geq 1$. Then*

(i) $\lim_{n\to\infty} m_k(X_n) = \beta_k := \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \alpha_k$ *for all $k \geq 1$; and*

(ii) *there exists a random variable X with $M_k(X) = \alpha_k$ and $m_k(X) = \beta_k$ for all $k \geq 1$ and $X_n \xrightarrow{\mathcal{L}} X$ if and only if $\alpha_k = o(k)$ and $\beta_k = o(k)$, in which case $\{X_n\}$ is also uniformly integrable.*

Proof. Conclusion (i) follows from the hypothesis $M_k(X_n) \to \alpha_k$ and from Lemma 2.2(ii). For (ii) the argument in [1] shows that

$$\alpha_k = o(k) \Rightarrow \lim_{\lambda\to\infty} \sup_n \int_{X_n > \lambda} X_n \, dP = 0, \tag{1}$$

and the analogous argument (or using symmetry: $M_k(X_n) = -m_k(-X_n)$) yields

$$\beta_k = o(k) \Rightarrow \lim_{\lambda\to\infty} \sup_n \int_{X_n < -\lambda} -X_n \, dP = 0. \tag{2}$$

Together (1) and (2) imply that $\{X_n\}$ is uniformly integrable, and the argument in [1] completes the proof. □

Kolodynski's example [2] to show that the condition $\alpha_k = o(k)$ does not suffice for the conclusion of Theorem 1.1 if the random variables X_n take negative values is this: Let $\{X_n\}$ be independent with $P(X_n = -n) = 1/n = 1 - P(X_n = 0)$; then $M_1(X_n) \to -1$, and $M_k(X_n) \to 0$ for all $k > 1$, but there is no random variable with maximal moment sequence $(-1, 0, 0, \dots)$ since $M_k(X) = M_{k+1}(X)$ for some

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k iff X is constant a.s., in which case $EX = -1$ implies $M_k(X) = -1$ for $k > 1$ as well.

Proposition 4.1 is still true as stated. Its proof shows that if $\gamma > 1$, then $\lim_{n\to\infty} M_k(Y_n) = M_k(H_\gamma)$. By Lemma 2.2(ii) it follows that $\lim_{n\to\infty} m_k(Y_n) = m_k(H_\gamma)$, so M_k and m_k are both $o(k)$.

Theorem 3.2 requires the analogous minimal-moment hypotheses; its corrected version is as follows.

Theorem 3.2′. *For all sequences of integrable random variables $\{X_n\}_{n\in\mathbb{N}}$ the following are equivalent:*

(i) $\lim_{n\to\infty} M_{k_j}(X_n) = \alpha_{k_j}$ *and* $\lim_{n\to\infty} m_{k_j}(X_n) = \beta_{k_j}$ *exist and are finite for all* j, *for some distinct* $\{k_j\} \subset \mathbb{N}$ *with* $k_1 = 1$ *and* $\sum 1/k_j = \infty$, *and* $\alpha_{k_j} = o(k_j)$ *and* $\beta_{k_j} = o(k_j)$;

(ii) $\lim_{n\to\infty} M_k(X_n) = \alpha_k$ *exists and is finite for all* $k \in \mathbb{N}$, *and* $\alpha_k = o(k)$ *and* $\beta_k = o(k)$;

(iii) *there exists an integrable random variable* X *such that* $X_n \xrightarrow{\mathcal{L}} X$ *and* $\lim_{n\to\infty} E|X_n| = E|X|$;

(iv) *there exists an integrable random variable* X *such that* $X_n \xrightarrow{\mathcal{L}} X$ *and* $\{X_n\}_{n\in\mathbb{N}}$ *is uniformly integrable.*

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References

[1] Hill, T. P. and Spruill, M. C., *On the relationship between convergence in distribution and convergence of expected extremes*, Proc. Amer. Math. Soc. **121** (1994), 1235–1243. MR **94j:**60070

[2] Kolodynski, S., Private communication.

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