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A REMARK ON KULESZA'S EXAMPLE

MICHAEL LEVIN

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ABSTRACT. In this note we simplify the proof of some properties of Kulesza's metric space Z with ind$Z = 0$ and Ind$Z = 1$.

In [1] J. Kulesza constructed a remarkable example of a completely metrizable space Z with ind$Z = 0$ and Ind$Z = 1$. In this note we would like to simplify some proofs from [1]. Let us recall the construction of Z.

Let $\Omega = [0, \omega_1)$ be the space of countable ordinals with the order topology and let $L \subset \Omega$ be the subset of limit ordinals. Denote $S = \Omega \setminus L$ and $S_k = \{\alpha + k : \alpha \in L\}$, $k = 1, 2, 3, ...,$ and define $Z = \{\sigma \in \Omega^\omega : \sigma(0) \in S$ and if $\sigma(k) \in L$, then $\sigma(k+1) = \sigma(k) + k$ and $\sigma(k+i) \in S_k$ for $i \geq 2\} \subset \Omega^\omega$.

Let us briefly describe the approach of [1] for proving some properties of Z. The metrizability of Z is shown by choosing a basis satisfying Frink's metrization theorem, the upper bound Ind$Z \leq 1$ is derived from a decomposition of Z into two subsets of Ind$= 0$ and the completeness of Z follows from the observation that Z is G_δ in $[0, \omega_1]^\omega$. For details see [1].

We will present short proofs for the following properties of Z.

1. Ind $Z > 0$. Since Z is metrizable, Ind$Z =$dim$Z$. Let $\mathcal{A}$ be the open cover of Z by the sets $A_\alpha = \{\sigma \in Z : 0 \leq \sigma(1) \leq \alpha\}, \alpha \in \Omega$, and, aiming at a contradiction, assume that $\mathcal{A}$ can be refined by a disjoint open cover $\mathcal{B}$.

For $s_0, s_1, ..., s_n \in S$ denote $V(s_0, ..., s_n) = \{\sigma \in S^\omega : \sigma(i) = s_i, i = 0, 1, ..., n\}$. By a regulator we mean a function from the family of finite sequences of the elements of S into S. Let us say that $V = V(s_0, ..., s_n)$ is regular if there are a regulator f and a set $B \in \mathcal{B}$ such that if $\sigma \in V$ satisfies $f(\sigma(0), ..., \sigma(j)) \leq \sigma(j+1)$ for every $j \geq n$, then $\sigma \in B$. Otherwise we say that V is irregular.

Note that since each element of $\mathcal{B}$ is contained in some element of $\mathcal{A}$, $V(s_0)$ is irregular for every $s_0 \in S$. We will show below that if $V(s_0, ..., s_n)$ is irregular, then there exists $s_{n+1} \in S$ such that $V(s_0, ..., s_n, s_{n+1})$ is also irregular. This enables us to construct a point $\sigma \in S^\omega$ such that $V(\sigma(0), ..., \sigma(k))$ is irregular for every $k \in \omega$. On the other hand note that if V is contained in an element of $\mathcal{B}$, then it is regular; it follows that for every point $\sigma \in S^\omega$ there is $k \in \omega$ such that $V(\sigma(0), ..., \sigma(k))$ is regular; this contradiction shows that $\mathcal{A}$ does not have a disjoint open refinement.

To complete the proof let us show that if $V(s_0, ..., s_n)$ is irregular, then there exists $s_{n+1} \in S$ such that $V(s_0, ..., s_n, s_{n+1})$ is also irregular. Assume the contrary

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and let a regulator f_s and $B_s \in \mathcal{B}$ witness the regularity of $V(s_0,...,s_n,s)$ for $s \in S$. Consider two cases.

Case (a). Assume that there are $\alpha \in S$ and $B \in \mathcal{B}$ such that $B_s = B$ for every $s \geq \alpha$, $s \in S$. Define a regulator f such that $f(s_0,...,s_n) = \alpha$ and $f(s_0,...,s_n,s_{n+1},...,s_{n+j}) = f_{s_{n+1}}(s_0,...,s_n,s_{n+1},...,s_{n+j})$ for $j > 0$ and $s_{n+1},...,s_{n+j} \in S$. Then f and B witness the regularity of $V(s_0,...,s_n)$. Contradiction.

Case (b). Assume that (a) does not hold. Then one can find an infinite sequence $\alpha_1, \alpha_2, ... \in S$ such that $B_{\alpha_i} \cap B_{\alpha_{i+1}} = \emptyset$, $\alpha_{i+1} > \alpha_i$ and $\alpha_{i+1} \geq f_{\alpha_i}(s_0,...,s_n,\alpha_i)$. Let $\alpha = \lim\alpha_i$, $\beta_0 = \alpha + (n+1)$ and define $\beta_1, \beta_2, ... \in S_{n+1}$ such that $\beta_{j+1} \geq \sup\{f_{\alpha_i}(s_0,...,s_n,\alpha_i,\beta_0,\beta_1,...,\beta_j) : i = 1,2,...\}$. Denote $\sigma = (s_0,...,s_n,\alpha,\beta_0,\beta_1,...)$ and $\sigma_i = (s_0,...,s_n,\alpha_i,\beta_0,\beta_1,...)$. Note that $\sigma \in Z$, $\sigma_i \in S^\omega$, $\beta_0 \geq f_{\alpha_i}(s_0,...,s_n,\alpha_i)$ and $\beta_{j+1} \geq f_{\alpha_i}(s_0,...,s_n,\alpha_i,\beta_0,\beta_1,...,\beta_j)$ for every i and j. Then the regularity of $V(s_0,...,s_n,\alpha_i)$ implies $\sigma_i \in B_{\alpha_i}$. Clearly $\lim\sigma_i = \sigma$ and this contradicts the fact that $\mathcal{B}$ is discrete.

2. A separable closed subset C of Z has a basis of clopen sets. It is easy to see that every separable closed set in Ω^ω is compact and every compact set in Ω^ω has a countable basis of clopen neighborhoods. Let K be closed in Z and disjoint from C. Denote by C' and K' the closures of C and K in Ω^ω and let $M' = C' \cap K'$. Then C' and M' are compact and hence there exists a decreasing basis G'_i, $i = 1,2,...$, of clopen neighborhoods of M'. Let $C'_0 = C' \setminus G'_1$, $C'_i = G'_i \cap (C' \setminus G'_{i+1})$ for $i = 1,2,...$ and let D'_i be a clopen neighborhood of C'_i which does not intersect K'. Define $O' = D'_0 \cup (\bigcup\{D'_i \cap G'_i : i = 1,2,...\})$. Then $O = O' \cap Z$ is clopen in Z and $C \subset O \subset Z \setminus K$.

References

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Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan
E-mail address: mlevin@math.tsukuba.ac.jp