AN OBSTRUCTION TO 3-DIMENSIONAL THICKENINGS

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Abstract. In this paper we give a characterization of those locally finite 2-dimensional simplicial complexes that have an orientable 3-manifold thickening. This leads to an obstruction for a fake surface $X$ to admit such a thickening. The obstruction is defined in $H^1(\Gamma; \mathbb{Z}_2)$, where $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order three.

1. Introduction

A simplicial complex $X$ “thickens” to a CW-complex $Y$ if $Y$ admits a CW-structure containing, as a subcomplex, a copy of a subdivision of $X$ onto which $Y$ collapses. For a standard 2-complex (a finite 2-complex with a single vertex) L. Neuwirth [6] exhibited an algorithm to decide whether the given 2-complex expands to an orientable 3-manifold. Later, H. Ikeda [4] introduced the concept of a fake surface (see §4). It is known that for this class of 2-complexes we always have a thickening to a singular 3-manifold [3, 7], i.e., a polyhedron in which the link of each point is a disk $D^2$ (boundary point), a sphere $S^2$ (inner point), or a projective plane $P^2$ (singular point). P. Wright [10] showed a sufficient condition for a (compact) fake surface $X$ to embed into a 3-manifold; namely, if every simple closed curve $C$ in the subgraph of all triple edges is untwisted (i.e., a regular neighborhood of $C$ in $X$ contains a $T$-bundle over $C$ which embeds in $\mathbb{R}^3$), then $X$ embeds into a 3-manifold.

In order to detect some kind of “twists” in a 2-dimensional locally finite simplicial complex $X$ (with planar links), we work with a family of embeddings $\Phi = \{\phi_v : lk(v, X) \to \mathbb{R}^2, v \in X^0\}$ and we consider certain cyclic orderings around each vertex, determined by the family $\Phi$. This way we associate to each family of embeddings $\Phi$ a cochain (cocycle) $\omega_\Phi \in C^1(\Gamma; \mathbb{Z}_2)$, where $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order $\geq 3$. From the study of these cochains we obtain the main results of this paper. More precisely

Theorem 1.1. Let $X$ be a 2-dimensional connected locally finite simplicial complex. Then, $X$ thickens to an orientable 3-manifold if and only if

(i) $lk(v, X)$ is planar, for every vertex $v$ of $X$, and
(ii) there exists a family of embeddings $\Phi = \{\phi_v : lk(v, X) \to \mathbb{R}^2, v \in X^0\}$ so that the associated cochain $\omega_\Phi$ is trivial.
A particular case of this result can be found in [1]: namely, for those finite 2-
complexes in which the vertices of the subgraphs $X^1$ and $lk(v, X)$ all have valence
at least 2, for every vertex $v \in X^0$.

If, in addition, $X$ is a fake surface, then $\Gamma \subset X$ is the subgraph consisting of all
the 1-simplexes of order 3, and Theorem 1.1 leads to the following theorem which
gives us an obstruction to thickening.

**Theorem 1.2.** Let $X$ be a fake surface, and let $\Gamma \subset X$ be as above. There exists a
well defined cohomology class $\xi_X \in H^1(\Gamma; \mathbb{Z}_2)$ with the property that $\xi_X = 0$ if and
only if $X$ thickens to an orientable 3-manifold.

On the other hand, the cocycle $\omega_\Phi \in C^1(\Gamma; \mathbb{Z}_2)$ can be regarded as a cochain in
the whole complex, i.e., $\omega_\Phi \in C^1(X; \mathbb{Z}_2)$. We will say that the family of embeddings
$\Phi$ is *admissible* if the cocycle $\omega_\Phi$ can be completed to a cocycle in $X$, via the
complementary graph $\Gamma'$ of $\Gamma$ in $X^1$ (see §4). We show that this property is intrinsic
to $X$, i.e., if a family of embeddings is admissible, then so is any other family of embeddings for $X$. This gives rise to a sufficient condition for a fake surface $X$ to
thicken to a 3-manifold (not necessarily orientable). More explicitly

**Theorem 1.3.** If a fake surface $X$ has an admissible family of embeddings $\Phi$, then
$X$ thickens to a 3-manifold $M$. If, in addition, we can choose $\eta \in C^1(\Gamma'; \mathbb{Z}_2)$ so
that $\omega_\Phi + \eta \in C^1(X; \mathbb{Z}_2)$ is in fact a coboundary, then $M$ is orientable.

2. (Co)homology of infinite CW-complexes

Let $R$ be a ring and let $X$ be an oriented locally finite CW-complex. Let $R(e)$
be the free left $R$-module generated by the cell $e$ in $X$, and let $C_n^\infty(X; R) = \prod_{\dim(e)=n} R(e)$. Elements in $C_n^\infty(X; R)$ will be denoted by infinite sums, and will be
called *infinite cellular $n$-chains with coefficients in $R$*. Note that the $R$-module of
ordinary cellular $n$-chains in $X$, $C_n(X; R)$, is a submodule of $C_n^\infty(X; R)$. Since $X$ is
locally finite, the ordinary boundary homomorphism $\partial : C_n(X; R) \rightarrow C_{n-1}(X; R)$
extends to a boundary homomorphism $\partial : C_n^\infty(X; R) \rightarrow C_{n-1}^\infty(X; R)$. This way we have a chain complex $(C_n^\infty(X; R), \partial)$ whose homology modules $H_n^\infty(X; R)$ are
called the *cellular homology modules of $X$ based on infinite chains* [2].

We can also define a coboundary homomorphism $\delta : C_n^\infty(X; R) \rightarrow C_{n+1}^\infty(X; R)$
as follows:

$$\delta \left( \sum_i \lambda_i e_i^n \right) = \sum_j \left( \sum_i \lambda_i [e_{i,j}^{n+1} : e_i^n] \right) e_j^{n+1},$$

where $[e_{i,j}^{n+1} : e_i^n]$ represents the incidence number corresponding to the (oriented)
cells $e_{i,j}^{n+1}$ and $e_i^n$. This gives us a cochain complex $(C_*^\infty(X; R), \delta)$ whose
cohomology modules $H^*(X; R)$ are, indeed, the ordinary cellular cohomology modules of
$X$ with coefficients in $R$ [2]. The cochain complex $(C_*^\infty(X; R), \delta)$ will also be
denoted by $(C_*(X; R), \delta)$. Again, since $X$ is locally finite, $\delta$ maps $C_n(X; R)$ into
$C_{n+1}(X; R)$, giving us another cochain complex $(C_*^\infty(X; R), \delta)$, whose cohomology modules $H^*_\delta(X; R)$ are called the *cellular cohomology modules of $X$ based on finite chains*
or, as they are usually referred to, the cellular cohomology modules of $X$ with compact support. The cochain complex $(C_*^\infty(X; R), \delta)$ will also be denoted by
$(C_\delta^*(X; R), \delta)$. 


For a subcomplex $A \subset X$ the corresponding relative (co)homology modules $H^n_c(X, A; R), H^*(X, A; R)$ and $H^*_c(X, A; R)$ of the pair $(X, A)$ are defined in the usual way.

3. A thickening theorem

Our purpose in this section is to prove Theorem 1.1. For this, we need the following definition.

**Definition 3.1.** Let $X$ be a 2-dimensional locally finite simplicial complex, and assume the link $lk(v, X)$ is planar for every vertex $v$ of $X$. Let $(v, w)$ be a 1-simplex of $X$. Given an embedding $\phi_v : lk(v, X) \to R^2$, we denote by $\theta_{\phi_v}(w)$ the cyclic ordering determined by $\phi_v$ on $lk((v, w), X)$ as we go around $\phi_v(w)$ following the orientation in $R^2$. Note that if the cardinality $|lk((v, w), X)|$ is $\leq 2$, then there is only one cyclic ordering $\theta_{\phi_v}(w)$. Moreover, if $|lk((v, w), X)| = 3$, then there are only two possible orderings.

We will denote by $\Gamma \subset X$ the graph consisting of those 1-simplexes of $X$ which are of order $\geq 3$ (i.e., those which are the face of at least three 2-simplexes of $X$). Consider the cochain complex of $\Gamma$ over $Z_2$

$$0 \to C^0(\Gamma; Z_2) \xrightarrow{\delta} C^1(\Gamma; Z_2) \to 0.$$ 

Given a family $\Phi = \{\phi_v : lk(v, X) \to R^2, v \in X\}$ of embeddings, we can associate it to a cochain (cocycle)

$$\omega_\Phi = \sum_{\sigma \in \Gamma} \omega_\Phi(\sigma) \cdot \sigma \in C^1(\Gamma; Z_2),$$

where $\omega_\Phi(\sigma) = 0$ if $\theta_{\phi_{t(\sigma)}}(t(\sigma))$ and $\theta_{\phi_{t(\sigma)}}(t(\sigma))$ are opposite, and $\omega_\Phi(\sigma) = 1$ otherwise. Here $\phi(\sigma)$ and $t(\sigma)$ are the vertices of $\sigma$. By extension, we define $\omega_\Phi(\sigma) = 0$ for every 1-simplex $\sigma$ of order $\leq 2$.

**Proof of Theorem 1.1.** We will concentrate on the “sufficient” part, since the converse can be checked making use of some results of p.l. topology. More explicitly, if $M$ collapses to a subdivision $X'$ of $X$, we choose orientation-preserving homeomorphisms $\psi_v : lk(v, M) \to S^2$ ($lk(v, M)$ inherits an orientation from $M$) whose restrictions $\phi_v : lk(v, X') \to S^2$ induce a family $\Phi$ of embeddings for $X$ satisfying $\omega_\Phi = 0$, since a regular neighborhood $N$ of any 1-simplex $(v, w)$ of $X$ is a 3-ball and we can apply the Annulus Theorem (see [8]) to get a product $S^1 \times [0, 1] \subset \partial N$ with $S^1 \times \{0\} \subset lk(v, M)$ and $S^1 \times \{1\} \subset lk(w, M)$, showing opposite cyclic orderings from the two vertices of $(v, w)$.

We now proceed to check that properties (i) and (ii) yield a thickening of $X$. Assume that $X$ is oriented, with all vertices having orientation +1. We may regard the family $\Phi$ as a family of embeddings into the sphere $S^2$. We are going to build an orientable 3-manifold $M$, and a CW-structure on it, from $X$ as follows. For every vertex $v \in X^0$, we consider a 3-ball $e^3_v \subset R^3$ with the usual orientation inherited from $R^3$. Given $v \in X^0$, we construct a regular neighborhood $N_v$ of $\phi_v(lk(v, X))$ in $\partial e^3_v$ (a 2-sphere) and give that neighborhood a CW-structure with a 2-cell $e^2_{\sigma,v}$ for each 1-simplex $\sigma \supset v$ in $X$, and a 1-cell $e^1_{\tau,v}$ for each 2-simplex $\tau \supset v$. For this, we thicken every vertex $v \in \phi_v(lk(v, X))$ to a disk $D_v \subset \partial e^3_v$ and complete the regular neighborhood by considering strips $C_\nu$, one for each 1-cell $\nu$ in $\phi_v(lk(v, X))$, joining the corresponding disks $D_v$ so that the core of $C_\nu$ is contained in $\nu$. Thus,
given a 1-cell $v$ in $\phi_v(lk(v, X))$, there is a 2-simplex $\tau$ in $X$ containing $v$ and the preimage of $v$; and $e^1_{\tau,v}$ is going to be a 1-cell dividing $C_v$ in half, transverse to its core. Given a 1-simplex $\sigma = (v, w)$ in $X$, the 2-cell $e^2_{\sigma,v}$ is going to be the union of $D_w$ together with those halves of the strips $C_v$ intersecting $D_w$. See Figure 1.

This way $\partial e^3_v$ has been given a CW-structure (and so has been $e^3_v$) for every vertex $v \in X^0$, with special cells of the type $e^2_{\sigma,v}$ and $e^1_{\tau,v}$, as described above. Observe that if $lk(v, X)$ was not connected, we can still achieve such a CW-structure by introducing extra 1-cells in the closure of $\partial e^3_v - N_v$. Note that a cell $e^1_{\tau,v}$ is a face of $e^2_{\sigma,v}$ if and only if $\sigma$ is a face of $\tau$. Let’s fix a vertex $v \in X^0$. Given a 1-simplex $\sigma \succ v$, we orient the 2-cell $e^2_{\sigma,v}$ in such a way that the incidence number $[e^3_v : e^2_{\sigma,v}]$ equals $[\sigma : v]$. Next, we orient the 1-cell $e^1_{\tau,v}$ so that $[e^2_{\sigma,v} : e^1_{\tau,v}]$ equals $[\tau : \sigma]$, where $\tau \succ \sigma \succ v$. We have to check that this orientation on $e^1_{\tau,v}$ does not depend on the choice of $\sigma$. For this, if $\sigma'$ is the other face of $\tau$ containing $v$, then from the identity $\partial^2(\tau) = 0$, where $\partial$ stands for the boundary homomorphisms in $C_*(X; \mathbb{Z})$, it follows that

$$[\tau : \sigma] [\sigma : v] = -[\tau : \sigma'] [\sigma' : v]$$

and hence

$$[\tau : \sigma] [e^3_v : e^2_{\sigma,v}] = -[\tau : \sigma'] [e^3_v : e^2_{\sigma',v}].$$

Also, $\partial^2(e^3_v) = 0$ (for the boundary in the chain complex of $e^3_v$) and similarly, assuming $e^1_{\tau,v}$ oriented as prescribed above with respect to $e^2_{\sigma,v}$, we get

$$[e^3_v : e^2_{\sigma,v} : e^1_{\tau,v}] = -[e^3_v : e^2_{\sigma',v} : e^1_{\tau,v}].$$

Thus, from (3.1) it readily follows that $[e^2_{\sigma,v} : e^1_{\tau,v}] = [\tau : \sigma']$.

Now let $Z = \bigcup_{v \in X^0} e^3_v$ (disjoint union). We form $M$, as a quotient of $Z$, as follows.

For every 1-simplex $\sigma = (v, w)$ of $X$, we glue $e^3_v$ and $e^3_w$ along the 2-cells $e^2_{\sigma,v}$ and $e^2_{\sigma,w}$ via a (cellular) homeomorphism which is orientation-reversing with respect to the orientations on these 2-cells inherited from $e^3_v$ and $e^3_w$, taking cells of the form $e^1_{\tau,v}$ to cells $e^1_{\tau,w}$, for $\tau \succ \sigma$, in such a way as to preserve the orientations defined above on these 1-cells. Observe that this homeomorphism is orientation-preserving with respect to the orientations defined on the 2-cells we glue along, since $[e^3_v : e^2_{\sigma,v}] = -[e^3_w : e^2_{\sigma,w}]$. Such a homeomorphism does exist since $\theta_{\phi_v}(w)$

\[\begin{figure}
\begin{center}
\includegraphics[scale=0.5]{example.png}
\end{center}
\end{figure}\]
and $\theta_{\partial_v}(v)$ are opposite, by hypothesis, i.e., the cyclic orderings under which the cells $e^1_{\tau,v}$ appear on $\partial e^2_{\sigma,v}$, and the cells $e^1_{\tau,w}$ appear on $\partial e^2_{\sigma,w}$ are opposite. Let $q : Z \to M$ be the quotient map. To simplify notation, we will write $e_1^\rho = q(e_1^\rho)$, $e_2^\sigma = q(e_2^\sigma)$, and $e_1^\tau = q(e_1^\tau)$. Notice that cells in $M$ of the form $e_1^\rho$, $e_2^\sigma$ and $e_1^\tau$ come with a well defined orientation from $Z$ (via the quotient map). We give any other cell of $M$ an arbitrary orientation. It can be checked that $M$ is a 3-manifold with boundary. Moreover, the boundary of $M$, $\partial M$, consists of those points in the closure of a 2-cell of $M$ which is not of the form $e_2^\sigma$, and the interior of $M$ consists of those points in the interior of a cell of the form $e_1^\rho$, $e_2^\sigma$ or $e_1^\tau$.

Next, we claim that $M$ contains a copy of the first barycentric subdivision $X'$ of $X$ on which $M$ collapses. For this, we take points $b_v$, $b_\sigma$ and $b_\tau$ in the interior of $e_1^v$, $e_2^\sigma$ and $e_1^\tau$ respectively, for every triple $\tau \succ \sigma \succ v$ in $X$, and denote by $C(v, \sigma, \tau) \subset e_2^\sigma$ the cone from $b_v$ over an arc in $e_2^\sigma$ from $b_\sigma$ to $b_\tau$. This cone can be identified with the 2-simplex $(b(v), b(\sigma), b(\tau))$ of $X'$ (where $b(\rho)$ stands for the barycenter of $\rho$). Then, the 2-complex $\bigcup_{\tau \succ \sigma \succ v} C(v, \sigma, \tau) \subset M$ is simplicially isomorphic to $X'$, by construction of $M$. The 3-cell $e_3^v$, endowed with the cone structure $b_v \cdot \partial e_3^v$, collapses onto the cone $b_v \cdot N_v$, using as free faces those 2-cells in $\partial e_3^\sigma - N_v$. Furthermore, since the regular neighborhood $N_v$ collapses to $\phi_v(\partial k(v, X)) \cong \bigcup_{\tau \succ \sigma} C(v, \sigma, \tau)$, for every $v \in X^0$, it follows that $M$ collapses to $\bigcup_{\tau \succ \sigma \succ v} C(v, \sigma, \tau) \subset M$. See Figure 2.

Finally, consider the chain map $\psi_k : C^k(X; \mathbb{Z}) \to C^\infty_{3-k}(M, \partial M; \mathbb{Z})$ determined by $\psi_0(v) = e_1^v$, $\psi_1(\sigma) = e_2^\sigma$ and $\psi_2(\tau) = e_1^\tau$. The map $\psi_*$ is indeed a chain map, because of the choice of the orientations on the cells $e_1^v$, $e_2^\sigma$ and $e_1^\tau$, for every triple $\tau \succ \sigma \succ v$ in $X$. Moreover, $\psi_*$ is a chain isomorphism, since the cells of the form $e_1^v$, $e_2^\sigma$ and $e_1^\tau$ are the only ones which are not in $\partial M$ and $C^\infty_0(M, \partial M; \mathbb{Z}) = 0$, by construction of $M$. Thus, $H^k(X; \mathbb{Z}) \cong H^\infty_{3-k}(M, \partial M; \mathbb{Z})$. In particular, $H^\infty_3(M, \partial M; \mathbb{Z}) \cong H^0(X; \mathbb{Z}) \cong \mathbb{Z}$, which proves that $M$ is an orientable 3-manifold (see [2, 5]).

4. FAKE SURFACES

The concept of a “fake surface” was introduced by H. Ikeda [4] in 1971. From now on, we will assume that $X$ is a (closed) fake surface. Explicitly
**Definition 4.1.** A 2-dimensional locally finite simplicial complex $X$ is called a *fake surface* if each vertex in $X$ has a neighborhood of one of the following three types:

![Types of vertices](image)

In particular, $\text{lk}(v, X)$ is planar for every vertex $v$ of $X$.

**Remark 4.2.** Although the combinatorial structure of a fake surface might appear as a quite simple structure, it is important to mention the fact that every compact 2-dimensional polyhedron has the (simple) homotopy type of a finite fake surface [9].

We will show that Theorem 1.1 together with the combinatorial structure of a fake surface (Lemma 4.3 and Proposition 4.4 below) lead to an obstruction to an orientable thickening (Theorem 1.2). For this, we recall that associated to a family of embeddings $\Phi = \{ \phi_v : \text{lk}(v, X) \to \mathbb{R}^2, v \in X^0 \}$ we have a cocycle $\omega_\Phi \in C^1(\Gamma; \mathbb{Z}_2)$. Observe that if $\sigma = (v, w)$ and $\omega_\Phi(\sigma) \neq 0$, then $\theta_{\phi_v}(w)$ and $\theta_{\phi_w}(v)$ coincide, since such a simplex $\sigma$ is always of order 3 in a fake surface. Given such a family $\Phi$ and a subset $S \subset X^0$, we can get another family of embeddings $\Phi^S = \{ \phi_v' : \text{lk}(v, X) \to \mathbb{R}^2, v \in X^0 \}$, where $\phi_v' = \phi_v, v \in X^0 - S$, and $\phi_v' = h \circ \phi_v$, if $v \in S$, where $h$ is a chosen orientation-reversing homeomorphism of $\mathbb{R}^2$. Note that if $v \in S$ and $(v, w)$ is a 1-simplex of $X$, then $\theta_{\phi_v'}(w)$ and $\theta_{\phi_v}(w)$ are opposite. In this situation one can readily check the following lemma.

**Lemma 4.3.** $\omega_\Phi + \omega_{\Psi}:S = \delta \left( \sum_{v \in S \cap \Gamma} v \right) \in C^1(\Gamma; \mathbb{Z}_2)$.

**Proposition 4.4.** Let $\Phi = \{ \phi_v, v \in X^0 \}$ and $\Phi' = \{ \phi_v', v \in X^0 \}$ be two families of embeddings, and let $S \subset X^0$ be the subset

$$S = \{ v \in X^0 \mid \exists w \in \text{lk}(v, X) \cap \Gamma \text{ with } \theta_{\phi_v}(w) \text{ opposite to } \theta_{\phi_v}(w) \}.$$ 

Then, $\omega_{\Phi'} = \omega_{\Psi}$.

**Proof.** We want to show that

$$S = \{ v \in X^0 \mid \forall w \in \text{lk}(v, X) \cap \Gamma; \theta_{\phi_v'}(w) \text{ is opposite to } \theta_{\phi_v}(w) \}.$$ 

For this, let $v \in S$ and $w \in \text{lk}(v, X)$ such that $\theta_{\phi_v'}(w)$ and $\theta_{\phi_v}(w)$ are opposite. Let $\mu : \phi_v(\text{lk}(v, X)) \to \phi_v'(\text{lk}(v, X)) \subset \mathbb{R}^2$ be a (cellular) homeomorphism satisfying $\mu \circ \phi_v = \phi_v'$. The homeomorphism $\mu$ can be extended to a (cellular) homeomorphism $\tilde{\mu} : S^2 \to S^2$, regarding $S^2$ as the one-point compactification of $\mathbb{R}^2$ with a cell structure determined by the type of $\text{lk}(v, X)$, i.e., two 2-cells if type I, three 2-cells if type II, or four 2-cells if type III. Either $\tilde{\mu}$ preserves the orientation of $S^2$ or it reverses it. The second is our case, since $\theta_{\phi_v'}(w)$ and $\theta_{\phi_v}(w)$ are opposite. Thus, $\tilde{\mu}$ is an orientation-reversing cellular homeomorphism of $S^2$, whence $\theta_{\phi_v'}(w)$ and $\theta_{\phi_v}(w)$ are opposite, for every vertex $w \in \text{lk}(v, X)$, as we wanted.
Let $h$ be a (chosen) orientation-reversing homeomorphism of $\mathbb{R}^2$, and $v \in S$. Then, $\theta_{h \circ \phi}(w)$ is opposite to $\theta_{h \circ \phi}(w)$, which makes the latter coincide with $\theta_{\phi_v}(w)$, for every $w \in lk(v, X)$. If the vertex $v$ is not in $S$, then $\theta_{\phi_v}(w)$ coincides with $\theta_{\phi_v}(w)$, for every $w \in lk(v, X)$. Therefore, $\omega_{\phi_v} = \omega_{\phi_v}$.

**Proof of Theorem 1.2.** We define $\xi_X = [\omega_{\phi}]$, where $\Phi$ is a family of embeddings as in Definition 3.1. We claim that $[\omega_{\phi}] = [\omega_{\phi}] \in H^1(\Gamma; \mathbb{Z}_2)$, for any two such families of embeddings $\Phi, \Phi'$. Indeed, by Proposition 4.4, there is a subset $S \subset X^0$ such that $\omega_{\phi'} = \omega_{\phi}$. Therefore, by Lemma 4.3,

$$\omega_{\phi} + \omega_{\phi} = \omega_{\phi} + \omega_{\phi} = \delta \left( \sum_{v \in S} v \right),$$

whence $[\omega_{\phi}] = [\omega_{\phi}] = \xi_X$.

Next, we want to prove Theorem 1.3. For this we will denote by $\Gamma'$ the complementary graph of $\Gamma$ in $X^1$. Observe that the cochain $\omega_{\phi}$ can be completed, via $\Gamma'$, to a cocycle in $X$; i.e., there is $\eta \in C^1(\Gamma'; \mathbb{Z}_2)$ so that $\delta(\omega_{\phi} + \eta) = 0$ in $C^2(X; \mathbb{Z}_2)$.

**Lemma 4.6.** (a) For a fake surface $X$, either every family of embeddings is admissible or none is.

(b) If $X$ has an admissible family of embeddings, then so does any subdivision obtained by deriving $X$ away from $X^1$.

**Proof.** (a) Let $\Phi, \Phi'$ be two families of embeddings, and assume $\Phi$ is admissible; i.e., there is $\eta \in C^1(\Gamma'; \mathbb{Z}_2)$ so that $\delta(\omega_{\phi} + \eta) = 0$ in $C^2(X; \mathbb{Z}_2)$. There is $\eta' \in C^1(\Gamma'; \mathbb{Z}_2)$ so that $\omega_{\phi} + \omega_{\phi'} + \eta'$ is a coboundary in $X$, since $\omega_{\phi}$ and $\omega_{\phi'}$ are cohomologous in $\Gamma$, by Lemma 4.3 and Proposition 4.4. Then,

$$\delta(\omega_{\phi} + \eta + \eta') = \delta(\omega_{\phi} + \eta + \eta' + \omega_{\phi} + \omega_{\phi'}) = \delta(\omega_{\phi} + \omega_{\phi'} + \eta') + \delta(\omega_{\phi} + \eta) = 0.$$ 

Furthermore, if $\omega_{\phi} + \eta$ is a coboundary, then so is $\omega_{\phi'} + (\eta + \eta')$.

(b) Let $\Phi$ be an admissible family of embeddings for $X$. A derived subdivision $Y$ of $X$ away from $X^1$ does not have any new vertices of type II or III. Thus, the embeddings $\phi_v \in \Phi$ induce embeddings $\phi'_v : lk(v, Y) \rightarrow \mathbb{R}^2$ having $\phi_v(lk(v, Y))$ as the image set. For the new vertices (all of type I) we choose arbitrary embeddings. This gives us a family of embeddings $\Phi'$ for $Y$ such that $\omega_{\phi} = \omega_{\phi'}$. Moreover, if there is $\eta \in C^1(\Gamma'; \mathbb{Z}_2)$ so that $\omega_{\phi} + \eta$ is a cocycle in $X$, we consider the complementary graph $\Gamma''$ of $X^1$ in $Y^1$ and we take $\eta' \in C^1(\Gamma''; \mathbb{Z}_2)$ as the sum of the new 1-simplex $\sigma''$ of $Y$ sharing a vertex with two 1-simplexes $\sigma, \sigma'$ of $X$ with coefficient 0 in $\omega_{\phi} + \eta$. It is not hard to check that $\omega_{\phi} + \eta + \eta'$ is a cocycle in $Y$. 

$\square$
Proof of Theorem 1.3. Let $Y$ be the first derived subdivision of $X$ away from $X^1$, and let $\Gamma \subset Y$ be the corresponding subgraph of all 1-simplices of order 3. By Lemma 4.6, if $X$ has an admissible family of embeddings, then so does $Y$. Observe that any 2-simplex of $Y$ contains at most one 1-simplex of order 3. This is why we subdivide.

Let $\Phi = \{ \phi_v : \text{lk}(v, Y) \to \mathbb{R}^2, \ v \in Y^0 \}$ be an admissible family of embeddings. We consider the (oriented) CW-structure on $Z = \bigsqcup_{v \in Y^0} e_3^v$ given in the proof of Theorem 1.1. Recall that every 3-ball $e_3^v \subset \mathbb{R}^3$ had the inherited orientation from $\mathbb{R}^3$, and so we have a field of normal vectors defined on each sphere $\partial e_3^v, v \in Y^0$.

In particular, each 2-cell $e_2^\sigma,v, \sigma \succ v$, comes with a normal vector (independently of the orientation defined on $e_2^\sigma,v$). Denote this normal vector by $\vec{n}(\sigma,v)$.

We claim that we can glue these 3-cells $e_3^v$ in such a way as to produce a 3-manifold $M$ which collapses to a copy of the first barycentric subdivision $Y'$ of $Y$. Let $\eta \in C^1(\Gamma'; \mathbb{Z}_2)$ so that $\delta(\omega_\Phi + \eta) = 0$ in $Y$, and let $\sigma = (v, w), \sigma' = (u, w)$ and $\sigma'' = (u, v)$ be the faces of the 2-simplex $(u, v, w)$ of $Y$. We follow some rules for the gluing.

Rule (1). If $\omega_\Phi(\sigma) \neq 0$ (which implies that $\sigma$ is of order 3), then we glue $e_3^v$ and $e_3^w$ along $e_2^\sigma,v$ and $e_2^\sigma,w$ via an orientation-preserving homeomorphism (with respect to the inherited orientations from the 3-balls), taking cells $e_1^{\tau,v}$ to cells $e_1^{\tau,w}$, $\tau \succ \sigma$. This way, we make the vectors $\vec{n}(\sigma,v)$ and $\vec{n}(\sigma,w)$ match. See Figure 3.

To complete the picture obtaining a 3-ball we could glue $e_3^u$ (along $e_2^\sigma,u$ and $e_2^{\sigma'',u}$) in such a way that $\vec{n}(\sigma',u)$ matches $\vec{n}(\sigma',w)$, and $\vec{n}(\sigma'',u)$ does not match $\vec{n}(\sigma'',v)$, or vice versa. Rule (3) will give the criterion to follow, since the 1-simplexes $\sigma'$ and $\sigma''$ are both of order 2.

Rule (2). If $\omega_\Phi(\sigma) = 0$ and $\sigma$ is of order 3, then we glue $e_3^v$ and $e_3^w$ along $e_2^\sigma,v$ and $e_2^{\sigma',v}$ via an orientation-reversing homeomorphism (with respect to the inherited orientations from the 3-balls), taking cells of the form $e_1^{\tau,v}$ to cells $e_1^{\tau,w}$, $\tau \succ \sigma$. Observe that the gluing is done so that $\vec{n}(\sigma,v)$ and $\vec{n}(\sigma,w)$ do not match. See Figure 4.

Notice that the gluing of the 3-cell $e_3^u$ (along $e_2^{\sigma',u}$ and $e_2^{\sigma'',u}$) can be done in two different ways to obtain a 3-ball, depending on whether the pair $(\vec{n}(\sigma',u), \vec{n}(\sigma'',u))$ matches $(\vec{n}(\sigma',u), \vec{n}(\sigma'',u))$ or it does not, in which case neither $\vec{n}(\sigma',u)$ matches
\[ \vec{n}(\sigma', u) \text{ nor } \vec{n}(\sigma'', v) \text{ matches } \vec{n}(\sigma'', w). \] Rule (3) will give the criterion to follow, since \( \sigma' \) and \( \sigma'' \) are again of order 2.

Rule (3). If \( \sigma \) is a 1-simplex of order 2 (and hence \( \omega_{\Phi}(\sigma) = 0 \)) and the coefficient of \( \sigma \) in \( \omega_{\Phi} + \eta \) is 0, then we glue \( e^3_{\tau,v} \) and \( e^3_{\tau,w} \) so that \( \vec{n}(\sigma,v) \) and \( \vec{n}(\sigma,w) \) do not match. Otherwise, we do the gluing so that \( \vec{n}(\sigma,v) \) and \( \vec{n}(\sigma,w) \) match.

This way we make sure we are building up a 3-manifold each time we consider a 2-simplex of \( Y \). Indeed, using the fact that \( \omega_{\Phi} + \eta \) is a cocycle, i.e., the number of faces of a 2-simplex \( \tau \) of \( Y \) which occur with non-zero coefficient in \( \omega_{\Phi} + \eta \) is either 0 or 2, it can be checked that the points on the 1-cells of the form \( e_1^3_{\tau,v} \) have euclidean neighborhoods in the space obtained after the gluing. Let \( M \) be the resulting 3-manifold built this way. The same argument used in the proof of Theorem 1.1 proves that \( M \) collapses to a copy of \( Y' \).

Finally, if \( [\omega_{\Phi} + \eta] = 0 \) in \( H^1(X;\mathbb{Z}_2) \), say \( \omega_{\Phi} + \eta = \delta(z), z \in C^0(X;\mathbb{Z}_2) \), then
\[
\delta(i^*(z)) = i^*(\delta(z)) = i^*(\omega_{\Phi}) + i^*(\eta) = \omega_{\Phi},
\]
where \( i^* \) is induced by the inclusion \( \Gamma \hookrightarrow X \). Therefore, the cohomology class \( \xi_X = [\omega_{\Phi}] \in H^1(\Gamma;\mathbb{Z}_2) \) is trivial and the conclusion follows from Theorem 1.2.

Remark 4.7. Notice that if \( \xi_X = 0 \), then any family of embeddings \( \Phi \) for \( X \) is admissible. For if \( \omega_{\Phi} = \delta(z) \) in \( \Gamma \), then the coboundary of \( z \) in \( X \), \( \delta(z) = \omega_{\Phi} + \eta \) (\( \eta \in C^1(\Gamma';\mathbb{Z}_2) \)), is in particular a cocycle in \( X \). If \( H^1(X;\mathbb{Z}_2) = 0 \) (e.g., \( X \) simply connected), then we have the converse, since every cocycle is then a coboundary.

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Added in Proof

Independently and essentially simultaneously these results were obtained by D. Repovš, N. B. Brodskij and A. B. Skopenkov; see A classification of 3-thickenings of 2-polyhedra, Topology and its Applications 94 (1999), 307-314.
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