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P-CONTINUITY ON CLASSICAL BANACH SPACES

P. HÁJEK AND J. G. LLAVONA

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ABSTRACT. Given a Banach space X and an integer n, the existence of an n-homogeneous polynomial which is not uniformly continuous with respect to the polynomial topology on B_X is investigated. We provide a complete characterization for some classical Banach spaces, while for others a surprising unresolved difficulty is encountered for a certain value of n (depending on X).

The notion of P-continuity (definition below) was introduced and investigated in [ACL], [GGL] and [GL] where it was shown that every P-continuous linear operator is weakly compact. It was observed there that if X has a separating polynomial, i.e. a real valued polynomial which separates the origin from the unit sphere, or X has DPP and $\ell_1 \not\hookrightarrow X$ (e.g. $C(K)$ Asplund spaces), then every polynomial on B_X is P-continuous. Examples of 2-homogeneous polynomials on $L_1[0,1]$, 3-homogeneous polynomials on $C[0,1]$ and $L_\infty[0,1]$ and n-homogeneous polynomials on ℓ_1, $n \geq 2$, which are not P-continuous were constructed.

In our present note we address the following general question: Given a (classical) Banach space X, for which $n \in \mathbb{N}$ does there exist an n-homogeneous non-P-continuous polynomial on B_X?

A simple method of construction of non-P-continuous polynomials gives us an almost complete answer (with the exception of at most one value of n). The remaining case turns out to be surprisingly difficult, and we are able to settle the problem only for some classical Banach spaces, using various additional properties. The complete answer is described below for $C(K)$, $L_p[0,1]$, $p \geq 2$, and ℓ_p, $p \in I := \mathbb{N} \cup \{(\frac{2n}{2n-1})\}$. In the remaining cases there is one value of n (indicated in the bracket) which remains unclear: $L_p[0,1]$, $1 < p < 2$ ($n = 2$); ℓ_p, $p \in [1,\infty) \setminus I$ ($n = [p]+1$).

In this note all polynomials and operators are assumed to be bounded, all spaces are real. Recall that, given Banach spaces X, Y, by $(X \oplus Y)_\infty$ (resp. $(X \oplus Y)_1$) we denote the direct sum together with the norm $\|(x,y)\|_\infty = \max(\|x\|, \|y\|)$ (resp. $\|(x,y)\|_1 = \|x\| + \|y\|$). By S_X we denote the unit sphere of X.

$\mathcal{P}({}^nX)$ (resp. $\mathcal{P}(X)$) denotes the space of all n-homogeneous (resp. all) polynomials on X.

Let us start with the definition of P-continuity.

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Definition 1. Let X, Y be Banach spaces, and let $f : X \to Y$ be a mapping. We say that f is *P-continuous* if given $\varepsilon > 0$ and a bounded set $B \subset X$, there exist scalar polynomials $P_1, \dots, P_n$ on X and $\delta > 0$ such that for all $x, y \in B$,

$$|P_i(x-y)| < \delta,\ i = 1, \dots, n, \quad \text{implies} \quad \|f(x) - f(y)\| < \varepsilon.$$

It is straightforward to observe that if f is a polynomial, we may restrict our attention only to $B = B_X$ (the closed unit ball of X) and instead of $P_1, \dots, P_n$ use only one homogeneous polynomial.

Given $r \in (0,1]$ we put $K_r = \{(x_i) \in B_{\ell_1}, x_i \geq 0, \sum x_k^r \leq 1\}$. The following is a slight improvement of a well-known fact.

Lemma 2. *Let $r \in (0,1]$, and let X be a separable Banach space. Then there exists a linear operator $T : \ell_1 \to X$, $\|T\| \leq 1$, such that $\frac{1}{2} B_X \subseteq T(K_r)$.*

Proof. Choose a dense sequence $\{x_i\}$ in B_X. Put $T(e_i) = x_i$. Let $x \in X$, $\|x\| = 1$. Given $C > 0$, find inductively a sequence i_n in $\mathbb{N}$ such that

$$\|Te_{i_1} - x\| < 2^{-\frac{C}{r}},$$
$$\|2^{-\frac{C}{r}} Te_{i_2} + Te_{i_1} - x\| < 2^{-2\frac{C}{r}},$$
$$\left\| \sum_{n=1}^{k} 2^{-(n-1)\frac{C}{r}} Te_{i_n} - x \right\| < 2^{-\frac{kC}{r}}.$$

Put $y = \sum_{n=1}^{\infty} 2^{-(n-1)\frac{C}{r}} e_{i_n}$. Then $Ty = x$, $\|y\|_{\ell_1} \leq \sum_{i=1}^{\infty} |y_i|^r = \frac{1}{1-2^{-C}}$. Thus, given any $x' \in \frac{1}{2} B_X$, we can find (for C large enough) $y \in K_r$ such that $Ty = x'$. □

Given $p \in \mathbb{R}$, denote by $\bar{p}$ the largest integer *strictly smaller* than p. For $n \geq \bar{p}+1$ we define $I_{p,n} : \ell_p \to \ell_1$, $I_{p,n}(\{x_i\}) = \{x_i^n\}$. Clearly, $I_{p,n}$ is an n-homogeneous polynomial, $I_{p,n}(B_{\ell_p}) \supset K_{\frac{p}{n}}$.

Lemma 3. *Let X be a separable Banach space. Suppose $X \cong (X_1 \oplus X_2)_\infty$ where $X_1 \cong \ell_p$, $1 \leq p < \infty$, and X_2 does not admit a separating polynomial. Then for every $n \geq \bar{p} + 2$ there exists a non-P-continuous n-homogeneous polynomial on X.*

Proof. Let $n \geq \bar{p} + 2$. Denote by $i : \ell_p \to X_1$ an isomorphism, $\|i\| \leq 1$. Since X is separable, there exists a separable subspace Y of X^* which is 1-norming for X_2. Choose, using Lemma 2, an operator $T : \ell_1 \to Y$ such that $T(K_{\frac{p}{n-1}}) \supset 2B_Y$. We claim that the n-homogeneous scalar polynomial $P(x,y) : (X_1 \oplus X_2)_\infty \cong X \to \mathbb{R}$,

$$P(x,y) = \langle T \circ I_{p,n-1} \circ i^{-1}(x), y \rangle,$$

is not P-continuous (on $B_{(X_1 \oplus X_2)_\infty}$).

Indeed, let $\delta > 0$ and Q be any homogeneous polynomial on X. Choose $y \in S_{X_2}$ such that $|Q(0,y)| < \delta$. Since $T \circ I_{p,n-1} \circ i^{-1}(B_{X_1}) \supset B_Y$, there exists $x \in B_{X_1}$ such that $P(x,y) > \frac{1}{2}$ and $P(x,0) = 0$. The proof is completed. □

Before passing to the next theorem, let us recall the well-known fact that every polynomial of degree less than p on ℓ_p is weakly uniformly continuous when restricted to B_{ℓ_p}. Thus the only possible values n for which there might exist a non-P-continuous homogeneous polynomial of degree n are $n \geq \bar{p} + 1$. Similarly, if $p > 2$ every 2-homogeneous polynomial on $X = L_p[0,1]$ or $X = C(K)$ is P-continuous. This follows from the fact [DJT] that every linear operator from X into X^* factors through a Hilbert space (and thus is P-continuous) combined with Proposition 1 from [GGL].

The case $L_1[0,1]$ will be treated later together with $C(K)$.

Theorem 4. *Let $X = \ell_p$, p-noneven, $p \geq 1$, $n \geq \bar{p} + 2$. Then there exists a non-P-continuous n-homogeneous polynomial on X.*

Let $X = L_p[0,1]$, p-noneven, $p > 1$, $n \geq 3$. Then there exists a non-P-continuous n-homogeneous polynomial on X.

Proof. The ℓ_p case follows immediately by Lemma 3. To resolve the L_p case, it is enough to recall that L_p, $p > 1$, contains a complemented copy of ℓ_2 (see [D]).

We do not know the solution in case $L_p[0,1]$, $1 < p < 2$, and $n = 2$. Theorem 4 also leaves a gap for ℓ_p, p-noneven, $n = \bar{p}+1$. We offer some partial solutions below, but the general case remains open and seems difficult to us. Before we proceed, let us recall the well-known fact that the space ℓ_p, p-even, does not admit a separating polynomial. □

Proposition 5. *Let $1 < p$ be an odd integer. Then $Q(x) = \sum_{i=1}^{\infty} x_i^p$ is non-P-continuous on ℓ_p.*

Proof. Given any $\delta > 0$, $P \in \mathcal{P}(\ell_p)$, choose $x \in \frac{1}{2} S_{\ell_p}$ such that $|P(x) < \delta|$. Write $x = x^+ - x^-$, where $x_i^+ \geq 0$, $x_i^- \geq 0$, $\operatorname{supp}(x^+) \cap \operatorname{supp}(x^-) = \varnothing$. Assume e.g. $\|x^+\|_{\ell_p} \geq \|x^-\|_{\ell_p}$, in particular $\frac{1}{2^p} \geq \sum_{i=1}^{\infty} (x_i^+)^p \geq \frac{1}{2^{p+1}}$. We have

$$|Q(x^+) - Q(x^+ + x)| \geq |Q(x^+ + x)| - |Q(x^+)| \geq \frac{1}{2} - \frac{1}{2^{p+1}} - \frac{1}{2^p} \geq \frac{1}{4}.$$

□

Proposition 6. *Assume $\frac{1}{p} + \frac{1}{2m} = 1$ for $m \in \mathbb{N}$, $m > 1$. Then every 2-homogeneous polynomial on ℓ_p is P-continuous.*

Proof. Let $Q \in \mathcal{P}(^2\ell_p)$ be given. Consider the associated operator $T_Q : \ell_p \to \ell_p^* = \ell_{2m}$ such that

$$Q(x) = \langle T_Q x, x \rangle.$$

By the result of [GGL], it is enough to show that T_Q is P-continuous. Consider $R \in \mathcal{P}(^{2m}\ell_p)$ defined as

$$R(x) = \sum_{i=1}^{\infty} (T_Q x)_i^{2m}.$$

Whenever $|R(x - y)| < \varepsilon$, we have

$$\|T_Q(x) - T_Q(y)\|_{\ell_{2m}} < \varepsilon^{\frac{1}{2m}}.$$

□

We now pass to a slight modification of the above method in case ℓ_1 imbeds into a Banach space X. As we will see, in this case the complementability assumption can be dropped.

Proposition 7. *Let X be a Banach space, $\ell_1 \hookrightarrow X$, $n \geq 3$. Then there exists $P \in \mathcal{P}(^nX)$ which is not P-continuous.*

Proof. Let $Y \hookrightarrow X$, $Y \cong \ell_1$. It is convenient to split Y into $Y \cong (Y_1 \oplus Y_2)_1$, $Y_1, Y_2 \cong \ell_1$. There exists a linear operator $L : Y \to \ell_2$ such that $L(B_{Y_2}) \supseteq B_{\ell_2}$ and $L(B_{Y_1}) = 0$. As L is 2-summing, it can be extended to $\widetilde{L} : X \to \ell_2$ ([DJT]). Then $I_{2,n-1} \circ \widetilde{L}(B_{Y_2}) \supseteq K_{\frac{2}{n-1}}$. There exists a separable subspace Z of X^*, which is 1-norming for Y_1. By Lemma 2 there exists a linear operator $S : \ell_1 \to Z$, such that $S(K_{\frac{2}{n-1}}) \supseteq 2B_Z$.

Finally, consider the $(n-1)$-homogeneous polynomial $T = S \circ I_{2,n-1} \circ \widetilde{L} : X \to Z$. By construction, $T(B_{Y_2}) \supseteq 2B_Z$. We claim that the n-homogeneous scalar polynomial

$$P(x) := \langle T(x), x \rangle$$

is non-P-continuous.

Indeed, given any scalar homogeneous polynomial R on X, $\varepsilon > 0$, there exists $x_0 \in B_{Y_1}$ such that $|R(x_0)| < \varepsilon$. However, for some $y_0 \in B_{Y_2}$, $\langle T(y_0), x_0 \rangle \geq 1$. Thus

$$|P(x_0 + y_0) - P(y_0)| = |\langle T(y_0), x_0 + y_0 \rangle - \langle T(y_0), y_0 \rangle| \geq 1.$$

The proof is finished. □

Proposition 7 together with the remarks before Theorem 4 and in the introduction implies the following.

Corollary 8. *Let $X = C(K)$, $\ell_1 \hookrightarrow X$. Then there exists a non-P-continuous n-homogeneous polynomial of degree n if and only if $n \geq 3$.*

Let $X = L_1[0,1]$. Then there exists a non-P-continuous n-homogeneous polynomial if and only if $n \geq 2$.

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Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail address, J. G. Llavona: `llavona@sunam1.mat.ucm.es`

Current address, P. Hájek: Department of Mathematics, Texas A&M University, College Station, Texas 77843

E-mail address: `petr.hajek@math.tamu.edu`