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INJECTIVE MODULES AND LINEAR GROWTH OF PRIMARY DECOMPOSITIONS

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Abstract. The purposes of this paper are to generalize, and to provide a short proof of, I. Swanson's Theorem that each proper ideal $\mathfrak{a}$ in a commutative Noetherian ring R has linear growth of primary decompositions, that is, there exists a positive integer h such that, for every positive integer n, there exists a minimal primary decomposition $\mathfrak{a}^n = \mathfrak{q}_{n1} \cap \ldots \cap \mathfrak{q}_{nk_n}$ with $\sqrt{\mathfrak{q}_{ni}}^{hn} \subseteq \mathfrak{q}_{ni}$ for all $i = 1, \ldots, k_n$. The generalization involves a finitely generated R-module and several ideals; the short proof is based on the theory of injective R-modules.

0. Introduction

Throughout, let R denote a commutative Noetherian ring, let M be a finitely generated R-module, and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ be ideals of R such that, for all $i = 1, \ldots, t$, we have $M \neq \mathfrak{a}_i M$. We say that $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ *have linear growth of primary decompositions with respect to* M if there exists a positive integer h such that, for every $n := (n_1, \ldots, n_t) \in \mathbb{N}^t$ (we use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{N}_0$ to denote the set of non-negative integers), there exists a minimal primary decomposition of $\mathfrak{a}_1^{n_1} \ldots \mathfrak{a}_t^{n_t} M$ in M,

$$\mathfrak{a}_1^{n_1} \ldots \mathfrak{a}_t^{n_t} M = Q_{n1} \cap \ldots \cap Q_{nk_n},$$

with

$$\sqrt{(Q_{ni} : M)}^{h(n_1 + \cdots + n_t)} \subseteq (Q_{ni} : M) \quad \text{for all } i = 1, \ldots, k_n.$$

One purpose of this paper is to prove that it is always the case that $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ have linear growth of primary decompositions with respect to M; this generalizes the theorem of I. Swanson [S, Theorem 3.4] that every (single) proper ideal in R has linear growth of primary decompositions (with respect to R). In [He-S], W. Heinzer and Swanson gave an alternative proof of Swanson's Theorem in the special case where R is locally at each prime ideal formally equidimensional and analytically unramified. Motivation for this work comes from the so-called "localization problem" in the theory of tight closure: see K. E. Smith and Swanson [S-S, Corollary 1.3] (and its proof) and M. Hochster and C. Huneke [H-H, p. 43].

Swanson's proof in [S, Theorem 3.4] relies on a "linear uniform Artin-Rees Lemma", and several pages of her paper are devoted to a proof (based on ideas of

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C. Huneke [Hu]) of that lemma. A second purpose of this paper is to present a much shorter proof of Swanson's Theorem (and, indeed, of the above-mentioned generalization) based on the well-known theory of injective R-modules and D. Kirby's Artin-Rees Lemma for Artinian modules [K, Proposition 3]. This proof is offered in the hope that the ideas might lead to progress on the above-mentioned "localization problem".

1. Some preliminaries

1.1 Notation. In addition to the notation introduced in Section 0, the following further notation will be used in this paper.

The proof of the result described in the Introduction can be quickly reduced to the case where each $\mathfrak{a}_i$ $(i = 1, \dots, t)$ is principal and generated by an element which is a non-zerodivisor on R and a non-zerodivisor on M: we shall use $u_1, \dots, u_t$ to denote elements of R which are non-zerodivisors on both R and M. For $n := (n_1, \dots, n_t) \in {\mathbb{N}_0}^t$, we shall use $\mathfrak{a}^n$ as an abbreviation for the ideal $\mathfrak{a}_1^{n_1} \dots \mathfrak{a}_t^{n_t}$; likewise, u^n will be used as an abbreviation for $u_1^{n_1} \dots u_t^{n_t}$. For use with this notation, it will be convenient to set $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in {\mathbb{N}_0}^t$ (where the 1 is in the i-th position), and to use $+$ to denote componentwise addition in ${\mathbb{N}_0}^t$ and 0 to denote the t-tuple of zeros in ${\mathbb{N}_0}^t$.

For $n := (n_1, \dots, n_t) \in {\mathbb{N}_0}^t$, we define $|n| := \sum_{i=1}^t n_i$.

For each $i = 1, \dots, t$, set $\mathfrak{P}_i := \operatorname{Ass}(M/u_iM)$; since u_i is a non-zerodivisor on M, we have $\mathfrak{P}_i := \operatorname{Ass}(M/u_i^{n_i}M)$ for all $n_i \in \mathbb{N}$. Set $\mathfrak{P} := \bigcup_{i=1}^t \mathfrak{P}_i$, a finite set. An easy inductive argument based on the exact sequences

$$0 \longrightarrow u_iM/u^{n+e_i}M \longrightarrow M/u^{n+e_i}M \longrightarrow M/u_iM \longrightarrow 0$$

$(i = 1, \dots, t)$ shows that $\operatorname{Ass}(M/u^nM) \subseteq \mathfrak{P}$ for all $0 \neq n \in {\mathbb{N}_0}^t$.

We shall use $E(N)$ (or $E_R(N)$) to denote the injective envelope of the R-module N. We can deduce from the Matlis-Gabriel theory of injective R-modules (see [M, §18], for example) that, for each $i = 1, \dots, t$, there are a finite family $(\mathfrak{p}_\alpha)_{\alpha \in \Lambda_{e_i}}$ of prime ideals of $\mathfrak{P}$ (so that the indexing set Λ_{e_i} is finite) and an R-monomorphism

$$\eta_{e_i} : M/u_iM \longrightarrow \bigoplus_{\alpha \in \Lambda_{e_i}} E(R/\mathfrak{p}_\alpha);$$

for each $\beta \in \Lambda_{e_i}$, let $\pi_\beta : \bigoplus_{\alpha \in \Lambda_{e_i}} E(R/\mathfrak{p}_\alpha) \longrightarrow E(R/\mathfrak{p}_\beta)$ denote the canonical projection; then $\operatorname{Im}(\pi_\beta \circ \eta_{e_i})$ is annihilated by some power of $\mathfrak{p}_\beta$ (by [M, Theorem 18.4(v)]). Note that it is possible to choose $h_0 \in \mathbb{N}$ sufficiently large so that

$$\mathfrak{p}_\beta^{h_0} \operatorname{Im}(\pi_\beta \circ \eta_{e_i}) = 0 \quad \text{for all } \beta \in \Lambda_{e_i} \text{ and all } i = 1, \dots, t.$$

1.2 Lemma. *Let $\mathfrak{p} \in \operatorname{Spec}(R)$, and set $E := E(R/\mathfrak{p})$. Denote $\operatorname{Hom}_R(M, E)$ by H. Let $b \in R$.*

(i) *There exists $h_1 \in \mathbb{N}_0$ such that*

$$(0 :_H bR) + (0 :_H \mathfrak{p}^n) = \big(\big((0 :_H bR) + (0 :_H \mathfrak{p}^{h_1})\big) :_H \mathfrak{p}^{n-h_1}\big) \quad \textit{for all } n \geq h_1.$$

(ii) *Assume now that b is a non-zerodivisor on M, and let $f \in H$ and $m \in \mathbb{N}$ be such that $\mathfrak{p}^m f = 0$. Then there exists $f' \in H$ such that $f = bf'$ and $\mathfrak{p}^{m+h_1} f' = 0$, where h_1 is as in part* (i).

Proof. (i) By [M, 18.4(vi)], the R-module E has a natural structure as an $(R, R_{\mathfrak{p}})$-bimodule, and so H also inherits a structure as $R_{\mathfrak{p}}$-module; furthermore, it is easy to see that this $R_{\mathfrak{p}}$-module structure is such that, for each ideal $\mathfrak{c}$ of R, we have $(0 :_H \mathfrak{c}) = (0 :_H \mathfrak{c}R_{\mathfrak{p}})$; hence $(0 :_H bR) + (0 :_H \mathfrak{p}^k)$ (for $k \in \mathbb{N}$) is an $R_{\mathfrak{p}}$-submodule of H, and so

$$\big(((0 :_H bR) + (0 :_H \mathfrak{p}^k)) :_H \mathfrak{p}^l\big) = \big(((0 :_H bRR_{\mathfrak{p}}) + (0 :_H \mathfrak{p}^k R_{\mathfrak{p}})) :_H \mathfrak{p}^l R_{\mathfrak{p}}\big) \ \ \forall\, l \in \mathbb{N}.$$

Now $E \cong E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ as $R_{\mathfrak{p}}$-modules (by [M, 18.4(vi)]), and it is well known that $E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ is an Artinian $R_{\mathfrak{p}}$-module (see [M, 18.6(v)], for example). It therefore follows from the additivity and exactness of the functor $\mathrm{Hom}_R(\bullet, E)$ (from the category of R-modules to the category of $R_{\mathfrak{p}}$-modules) that H is $R_{\mathfrak{p}}$-isomorphic to a submodule of $\mathrm{Hom}_R(F, E)$ for some finitely generated free R-module F; hence H is an Artinian $R_{\mathfrak{p}}$-module. The claim in part (i) therefore follows from D. Kirby's Artin-Rees Lemma [K, Proposition 3].

(ii) Since b is a non-zerodivisor on M, it follows from the exactness of the functor $\mathrm{Hom}_R(\bullet, E)$ that $H = bH$. Hence there exists $g \in H$ such that $f = bg$. Since $\mathfrak{p}^m f = 0$, it follows that

$$\begin{aligned} g \in ((0 :_H bR) :_H \mathfrak{p}^m) &\subseteq \big(((0 :_H bR) + (0 :_H \mathfrak{p}^{h_1})) :_H \mathfrak{p}^m\big) \\ &= (0 :_H bR) + (0 :_H \mathfrak{p}^{m+h_1}), \end{aligned}$$

and so there exist $f_0 \in (0 :_H bR)$ and $f' \in (0 :_H \mathfrak{p}^{m+h_1})$ such that $g = f_0 + f'$. Since $f = bg = b(f_0 + f') = bf'$, the proof is complete. □

1.3 Remark. Let the situation be as in Lemma 1.2(i). Observe that, if h_1' is any integer such that $h_1' \geq h_1$, then we also have

$$(0 :_H bR) + (0 :_H \mathfrak{p}^n) = \Big(\Big((0 :_H bR) + (0 :_H \mathfrak{p}^{h_1'})\Big) :_H \mathfrak{p}^{n-h_1'}\Big) \quad \text{for all } n \geq h_1'.$$

In other (and less precise) words, the conclusion of 1.2(i) holds for any 'larger h_1'.

1.4 Notation. Bearing in mind (the last sentence of) 1.1, Lemma 1.2(i) and Remark 1.3, we choose h to be a positive integer such that

(i)

$$\mathfrak{p}_\beta^h \,\mathrm{Im}(\pi_\beta \circ \eta_{e_i}) = 0 \quad \text{for all } \beta \in \Lambda_{e_i} \text{ and all } i = 1, \ldots, t,$$

and

(ii) for all $\mathfrak{p} \in \mathfrak{P}$, we have, for $H(\mathfrak{p}) := \mathrm{Hom}_R(M, E(R/\mathfrak{p}))$,

$$(0 :_{H(\mathfrak{p})} u_i R) + (0 :_{H(\mathfrak{p})} \mathfrak{p}^n) = \big(((0 :_{H(\mathfrak{p})} u_i R) + (0 :_{H(\mathfrak{p})} \mathfrak{p}^h)) :_{H(\mathfrak{p})} \mathfrak{p}^{n-h}\big)$$

for all $n \geq h$ and all $i = 1, \ldots, t$.

1.5 Lemma. *Let $\mathfrak{p} \in \mathfrak{P}$, and set $E := E(R/\mathfrak{p})$. Let $0 \neq n := (n_1, \ldots, n_t) \in {\mathbb{N}_0}^t$, let $i \in \{1, \ldots, t\}$, and let $g : u_i M/u^{n+e_i}M \longrightarrow E$ be an R-homomorphism such that $\mathfrak{p}^n \,\mathrm{Im}\, g = 0$, where $n \in \mathbb{N}$. Then g can be extended to an R-homomorphism $g' : M/u^{n+e_i}M \longrightarrow E$ such that $\mathfrak{p}^{n+h} \,\mathrm{Im}\, g' = 0$ (where h is as in 1.4).*

Proof. Since u_i is a non-zerodivisor on M, there is an isomorphism

$$\theta : u_i M/u^{n+e_i}M \longrightarrow M/u^n M$$

which is such that $\theta(u_i m + u^{n+e_i}M) = m + u^n M$ for all $m \in M$. Let $\sigma : M \longrightarrow M/u^n M$ be the natural epimorphism. Then

$$f := g \circ \theta^{-1} \circ \sigma \in \mathrm{Hom}_R(M, E) \quad \text{and} \quad \mathfrak{p}^n f = 0.$$

Our choice of h means that we can use Lemma 1.2 to deduce that there exists $f' \in \mathrm{Hom}_R(M, E)$ such that $f = u_i f'$ and $\mathfrak{p}^{n+h} f' = 0$. Since f vanishes on $u^n M$, it follows that f' vanishes on $u^{n+e_i}M$, and so $f' = g' \circ \tau$ for some $g' \in \mathrm{Hom}_R(M/u^{n+e_i}M, E)$, where $\tau : M \longrightarrow M/u^{n+e_i}M$ is the canonical epimorphism. This g' has the desired properties. □

2. The result

2.1 Theorem. *Let M be a finitely generated R-module, and let $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ be ideals of R such that, for all $i = 1, \dots, t$, we have $M \neq \mathfrak{a}_i M$. Then $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ have linear growth of primary decompositions with respect to M.*

More precisely, there exists a positive integer h such that, for every $0 \neq n := (n_1, \dots, n_t) \in {\mathbb{N}_0}^t$, there exists a minimal primary decomposition of $\mathfrak{a}_1^{n_1} \dots \mathfrak{a}_t^{n_t} M$ in M,

$$\mathfrak{a}_1^{n_1} \dots \mathfrak{a}_t^{n_t} M = Q_{n1} \cap \dots \cap Q_{nk_n},$$

such that

$$\sqrt{(Q_{ni} : M)}^{h(n_1 + \dots + n_t)} \subseteq (Q_{ni} : M) \quad \textit{for all } i = 1, \dots, k_n.$$

Proof. We first show that it is enough to prove the result in the special case in which, for all $i = 1, \dots, t$, the ideal $\mathfrak{a}_i$ is principal and generated by an element which is both a non-zerodivisor on M and a non-zerodivisor on R.

Let $\mathbb{Z}$ denote the set of integers. Take independent indeterminates $T_1, \dots, T_t$, denote $R[T_1, \dots, T_t, T_1^{-1}, \dots, T_t^{-1}] \otimes_R M$ by $M[T_1, \dots, T_t, T_1^{-1}, \dots, T_t^{-1}]$, and denote its element $T_1^{r_1} \dots T_t^{r_t} \otimes m$ (for $r_1, \dots, r_t \in \mathbb{Z}$ and $m \in M$) by $mT_1^{r_1} \dots T_t^{r_t}$; the Rees module of M with respect to $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ is

$$\bigoplus_{(r_1, \dots, r_t) \in \mathbb{Z}^t} \mathfrak{a}_1^{r_1} \dots \mathfrak{a}_t^{r_t} M T_1^{r_1} \dots T_t^{r_t} =: \mathfrak{M},$$

considered as a module over the (extended) Rees ring

$$\mathfrak{R} := R[\mathfrak{a}_1 T_1, \dots, \mathfrak{a}_t T_t, T_1^{-1}, \dots, T_t^{-1}]$$

of R with respect to $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ (where, as usual, a negative power of an ideal of R is interpreted as R).

For each $i = 1, \dots, t$, set $u_i := T_i^{-1}$, which is a non-zerodivisor on $\mathfrak{R}$ and a non-zerodivisor on $\mathfrak{M}$. Now $\mathfrak{R}$ and $\mathfrak{M}$ are naturally $\mathbb{Z}^t$-graded, the 0-th component of $\mathfrak{R}$ is R, and, for every $n := (n_1, \dots, n_t) \in {\mathbb{N}_0}^t$, the 0-th component of $u_1^{n_1} \dots u_t^{n_t} \mathfrak{M}$ is $\mathfrak{a}_1^{n_1} \dots \mathfrak{a}_t^{n_t} M$. Therefore, if the result could be proved for the $\mathfrak{R}$-module $\mathfrak{M}$ and the ideals $\mathfrak{R}u_1, \dots, \mathfrak{R}u_t$ of $\mathfrak{R}$, then it could be proved for the R-module M and the ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ simply by contraction to 0-th components of appropriate primary decompositions of the $u_1^{n_1} \dots u_t^{n_t} \mathfrak{M}$ in $\mathfrak{M}$.

Consequently, we can, and do, assume for the remainder of this proof that, for each $i = 1, \dots, t$, the ideal $\mathfrak{a}_i$ is principal and generated by u_i, where u_i is a non-zerodivisor on M and a non-zerodivisor on R. We now re-employ the notations

$\mathfrak{P}_i, \Lambda_{e_i}, \eta_{e_i}$ $(i = 1, \dots, t)$ and $\mathfrak{P}$ introduced in 1.1; we also define h as in 1.4. Whenever $(\mathfrak{p}_\alpha)_{\alpha\in\Lambda}$ is a family of prime ideals of $\mathfrak{P}$, and $\beta \in \Lambda$, we shall use

$$\pi_\beta : \bigoplus_{\alpha\in\Lambda} E(R/\mathfrak{p}_\alpha) \longrightarrow E(R/\mathfrak{p}_\beta)$$

to denote the canonical projection.

The next stage in this proof is to show, by induction on $|n|$, that for all $0 \neq n \in {\mathbb{N}_0}^t$, there exist a finite family $(\mathfrak{p}_\alpha)_{\alpha\in\Lambda_n}$ (so that the indexing set Λ_n is finite) of prime ideals of $\mathfrak{P}$ and an R-monomorphism

$$\eta_n : M/u^nM \longrightarrow \bigoplus_{\alpha\in\Lambda_n} E(R/\mathfrak{p}_\alpha)$$

such that, for all $\beta \in \Lambda_n$, we have $\mathfrak{p}_\beta^{h|n|}\operatorname{Im}(\pi_\beta \circ \eta_n) = 0$ (that is, $\mathfrak{p}_\beta^{h|n|}(\pi_\beta \circ \eta_n) = 0$). (It should be noted that it is *not* claimed that the monomorphism η_n provides the injective envelope of M/u^nM.) By choice of h, this is certainly true when $|n| = 1$. We therefore suppose that we have proved this statement for $0 \neq n := (n_1, \dots, n_t) \in {\mathbb{N}_0}^t$, and we show now how to deduce that the statement is true for $n + e_i$, where $i \in \{1, \dots, t\}$.

Since u_i is a non-zerodivisor on M, there is an isomorphism

$$\theta : u_iM/u^{n+e_i}M \longrightarrow M/u^nM$$

which is such that $\theta(u_im+u^{n+e_i}M) = m+u^nM$ for all $m \in M$. For each $\alpha \in \Lambda_n$, let $g_\alpha = \pi_\alpha\circ\eta_n\circ\theta : u_iM/u^{n+e_i}M \longrightarrow E(R/\mathfrak{p}_\alpha)$; by the inductive hypothesis, $\mathfrak{p}_\alpha^{h|n|}g_\alpha = 0$, and so it follows from 1.5 that g_α can be extended to an R-homomorphism $g'_\alpha : M/u^{n+e_i}M \longrightarrow E(R/\mathfrak{p}_\alpha)$ such that $\mathfrak{p}_\alpha^{h|n|+h}g'_\alpha = 0$. Define

$$\mu : M/u^{n+e_i}M \longrightarrow E' := \bigoplus_{\alpha\in\Lambda_n} E(R/\mathfrak{p}_\alpha)$$

by $\mu(x) = (g'_\alpha(x))_{\alpha\in\Lambda_n}$ for all $x \in M/u^{n+e_i}M$. Observe that the R-homomorphism μ has the property that $\mathfrak{p}_\alpha^{h|n|+h}(\pi_\alpha \circ \mu) = 0$ for all $\alpha \in \Lambda_n$.

Now set $E := \bigoplus_{\alpha\in\Lambda_{e_i}} E(R/\mathfrak{p}_\alpha)$, and consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & u_iM/u^{n+e_i}M & \longrightarrow & M/u^{n+e_i}M & \longrightarrow & M/u_iM & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & E' & \longrightarrow & E' \oplus E & \longrightarrow & E & \longrightarrow & 0,
\end{array}$$

in which the top row is the natural exact sequence, the bottom row is the canonical split exact sequence, the left-hand vertical homomorphism is

$$\eta_n \circ \theta : u_iM/u^{n+e_i}M \longrightarrow E',$$

and the right-hand vertical homomorphism is $\eta_{e_i} : M/u_iM \longrightarrow E$.

Define $\eta_{n+e_i} : M/u^{n+e_i}M \longrightarrow E' \oplus E$ by

$$\eta_{n+e_i}(m + u^{n+e_i}M) = \big(\mu(m + u^{n+e_i}M), \eta_{e_i}(m + u_iM)\big) \quad \text{for all } m \in M.$$

It is routine to check that when one inserts the R-homomorphism η_{n+e_i} as a central vertical map in the above diagram, then the resulting extended diagram is commutative; consequently, η_{n+e_i} is a monomorphism, and since $E' \oplus E$ is isomorphic to a direct sum of finitely many indecomposable injective R-modules of the form $E(R/\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{P}$, and since $\mathfrak{p}_\beta^h(\pi_\beta \circ \eta_{e_i}) = 0$ for all $\beta \in \Lambda_{e_i}$, the inductive step is complete.

Let $0 \neq n \in {\mathbb{N}_0}^t$, and let $\tau_n : M \longrightarrow M/u^n M$ be the natural epimorphism. The inductive argument above has shown that there exist a finite family $(\mathfrak{p}_\alpha)_{\alpha \in \Lambda_n}$ of prime ideals of $\mathfrak{P}$ and an R-monomorphism

$$\eta_n : M/u^n M \longrightarrow \bigoplus_{\alpha \in \Lambda_n} E(R/\mathfrak{p}_\alpha)$$

such that, for all $\beta \in \Lambda_n$, we have $\mathfrak{p}_\beta^{h|n|} \operatorname{Im}(\pi_\beta \circ \eta_n \circ \tau_n) = 0$. For $\alpha \in \Lambda_n$, let $Q_\alpha := \operatorname{Ker}(\pi_\alpha \circ \eta_n \circ \tau_n)$; since $M/Q_\alpha \cong \operatorname{Im}(\pi_\alpha \circ \eta_n \circ \tau_n)$, it follows from well-known properties of indecomposable injective R-modules (see [M, 18.4(iii), (v)], for example) that Q_α, if different from M, is a $\mathfrak{p}_\alpha$-primary submodule of M such that $\mathfrak{p}_\alpha^{h|n|} M \subseteq Q_\alpha$. Therefore we can refine the expression

$$u^n M = \bigcap_{\alpha \in \Lambda_n} Q_\alpha$$

to a minimal primary decomposition of $u^n M$ in M with the desired properties. $\square$

References

[He-S] W. Heinzer and I. Swanson, *Ideals contracted from* 1*-dimensional overrings with an application to the primary decomposition of ideals*, Proc. Amer. Math. Soc. **125** (1997), 387–392. MR **97d:**13008

[H-H] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda Theorem*, J. Amer. Math. Soc. **3** (1990), 31–116. MR **91g:**13010

[Hu] C. Huneke, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992), 203–223. MR **93b:**13027

[K] D. Kirby, *Artinian modules and Hilbert polynomials*, Quart. J. Math. Oxford (2) **24** (1973), 47–57. MR **47:**4993

[M] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986. MR **88h:**13001

[S-S] K. E. Smith and I. Swanson, *Linear bounds on growth of associated primes*, Communications in Algebra **25** (1997), 3071–3079. MR **98k:**13003

[S] I. Swanson, *Powers of ideals: primary decompositions, Artin-Rees lemma and regularity*, Math. Annalen **307** (1997), 299–313. MR **97j:**13005

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