UNIVERSAL $\mathbb{Z}$-LATTICES OF MINIMAL RANK

BYEONG-KWEON OH

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Abstract. Let $U_{\mathbb{Z}}(n)$ be the minimal rank of $n$-universal $\mathbb{Z}$-lattices, by which we mean positive definite $\mathbb{Z}$-lattices which represent all positive $\mathbb{Z}$-lattices of rank $n$. It is a well known fact that $U_{\mathbb{Z}}(n) = n + 3$ for $1 \leq n \leq 5$. In this paper, we determine $U_{\mathbb{Z}}(n)$ and find all $n$-universal lattices of rank $U_{\mathbb{Z}}(n)$ for $6 \leq n \leq 8$.

1. Introduction

A positive definite $\mathbb{Z}$-lattice (or simply a lattice) is said to be $n$-universal if it represents all positive definite $\mathbb{Z}$-lattices of rank $n$. It is well known that the ranks of $n$-universal lattices should be greater than or equal to $n + 3$. In fact, for each $n$, $1 \leq n \leq 5$, the lattice $I_{n+3}$ is $n$-universal because $I_{n+3}$ has class number 1 and is universal over the $p$-adic integer ring $\mathbb{Z}_p$ for all $p$, where $I_n$ is the lattice $\mathbb{Z}^n$ equipped with the standard inner product (see [10], [12] and [15]). For $n \geq 6$, however, no diagonal lattice can be $n$-universal. Moreover, there does not exist a lattice of rank $n + 3$ which has class number 1 and represents all integral lattices of rank $n$ over $\mathbb{Z}_p$ for all $p$ (see [18], [20]). To be more precise, we define $U_{\mathbb{Z}}(n) = \min \{ \text{rank}(L) \mid L \text{ is } n\text{-universal} \}$.

Let $L_1, L_2, \ldots, L_k$ be all unimodular lattices of rank $n + 3$ up to isometry. Then the lattice $L_1 \perp L_2 \perp \cdots \perp L_k$ is $n$-universal and therefore $U_{\mathbb{Z}}(n)$ exist for all $n$. As was mentioned above, $U_{\mathbb{Z}}(n) = n + 3$ for $1 \leq n \leq 5$. In this paper, we investigate the minimal rank $U_{\mathbb{Z}}(n)$ of $n$-universal $\mathbb{Z}$-lattices for $6 \leq n \leq 10$. We prove that $U_{\mathbb{Z}}(n) = 13, 15, 16, 28, 30$ for $n = 6, 7, 8, 9, 10$ respectively, and find all $6, 7, 8$-universal $\mathbb{Z}$-lattices of rank $13, 15, 16$, respectively. For the complete list of $1, 2$-universal $\mathbb{Z}$-lattices of minimal rank, see [6], [7], [16] and [21].

In [1], Bannai proved that most unimodular lattices (even or odd) have trivial automorphism groups if the rank is sufficiently large, and that such lattices are indecomposable. If a lattice $L$ is $n$-universal, then $L$ must represent all indecomposable unimodular lattices of rank $n$ as direct summands. So from this we may guess that $U_{\mathbb{Z}}(n)$ grows very quickly.

Remark. Note that if we define $U_{\mathbb{Q}}(n)$ to be the minimal rank of $n$-universal positive definite quadratic space over $\mathbb{Q}$, then $U_{\mathbb{Q}}(n) = n + 3$ for all $n$. 

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We adopt terminologies and notations from [2], [3] and [14]. By \( l \to L \) we mean that the lattice \( L \) represents the lattice \( l \). For a sublattice \( l \) of \( L \) \( \perp M \) of the form \( l = \mathbb{Z}(x_1 + y_1) + \mathbb{Z}(x_2 + y_2) + \cdots + \mathbb{Z}(x_n + y_n) \) for \( x_i \in L \) and \( y_i \in M \), we define sublattices \( l(L) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n \) and \( l(M) = \mathbb{Z}y_1 + \mathbb{Z}y_2 + \cdots + \mathbb{Z}y_n \). A lattice \( l \) is said to be additively indecomposable if either \( l(L) = 0 \) or \( l(M) = 0 \) whenever \( l \to L \perp M \).

2. Determination of \( U_\mathbb{Z}(6) \)

We assume that \( L \) is a 6-universal \( \mathbb{Z} \)-lattice. Since \( L \) must represent the lattice \( I_6 \), it decomposes into \( I_6 \perp L' \). Furthermore, since the root lattice \( E_6 \) is additively indecomposable, it should be represented by \( L' \). Therefore \( U_\mathbb{Z}(6) \geq 12 \). Suppose rank \( L = 12 \); then \( L = I_6 \perp E_6 \). But this cannot represent the root lattice \( A_6 \). On the other hand, the lattice \( E_6 \perp I_{10} \) is 6-universal because \( E_6 \) is the unique additively indecomposable lattice of rank 6 and \( E_6^{(2)} = E_8 \) is represented by \( I_8 \), where \( E_6^{(2)} \) is the lattice obtained from scaling \( E_6 \) by 2 (see [8] and [11]). Therefore \( 13 \leq U_\mathbb{Z}(6) \leq 16 \). If \( L \) is 6-universal and rank \( L = 13 \), then \( L \) must be equal to \( E_6 \perp I_7 \) or \( E_7 \perp I_6 \) because the only lattice of rank 7 which represents both \( A_6 \) and \( E_6 \) is \( E_7 \). In this section, we prove that \( E_6 \perp I_7 \) and \( E_7 \perp I_6 \) are indeed 6-universal lattices of rank 13.

**Lemma 2.1.** If a lattice \( L \) of rank \( n + 3 \) has a square free determinant and its quadratic norm \( Q(L) \) is not contained in \( 2\mathbb{Z} \), then every lattice \( l \) of rank \( n \) is represented by a lattice in the genus of \( L \).

**Proof.** The local lattice \( L_p \) is \( n \)-universal over \( \mathbb{Z}_p \) by [13]. So the lemma follows directly from [14, 102:9]. (See also [4].)

Now, we prove the following technical lemma, which is useful in the sequel.

**Lemma 2.2.** Let \( l \) be a lattice of rank \( n \) which is represented by \( I_m, m \geq 7 \).

1. If \( 5 \leq n \leq m - 2 \), then \( l \) is represented by \( D_{n-i} \perp I_{m-n+i} \) for some \( i = 1, 2, \ldots, n-1 \), where \( D_k \) is the root lattice of type \( D \) for \( k \geq 4 \), \( D_3 = A_3 \), \( D_2 = A_1 \perp A_1 \), and \( D_1 = \{0\} \).

2. If \( n = m - 1 \), then \( l \) is represented by \( D_{n+1-i} \perp I_i \) for some \( i = 0, 1, 2, \ldots, n \).

Furthermore, if \( l \) is represented by \( D_{n+1-i} \perp I_i \) only for \( i = 0 \) or 1 and \( n \equiv i \) (mod 2), then \( dl \equiv n - i + 1 \) (mod 4).

**Proof.** We only prove (1). The proof of (2) is quite similar to that of (1). It suffices to show this when \( m \) is equal to \( n+2 \). We may assume that \( l = \bigoplus_{i=1}^{n} \mathbb{Z}(\sum_{k=1}^{m} a_{ik} c_k) \) is a sublattice of \( I_m \), where the \( e_i \)'s are the standard orthonormal basis of \( I_m \). By suitable base change, we may also assume that \( a_{ij} = 0 \) for all \( i, j \) satisfying \( i \geq 2 \) and \( j \geq m+2-i \), and that the \( a_{k(m+1-i)} \)'s are even for \( 1 \leq k \leq i-1 \) if \( a_{i(m+1-i)} \) is odd for some \( i \geq 2 \). For a subset \( J = \{j_1, j_2, \ldots, j_r \} \subseteq \{1, 2, \ldots, m\} \), we define the lattice \( M_J = \bigoplus_{i=1}^{n} \mathbb{Z}(\sum_{t=1}^{r} a_{ij_t} c_{j_t}) \). We let \( J = \{m\} \) if \( a_{1m} \) is even. Then \( M_J = \mathbb{Z}(a_{1m} c_m) \to D_1 \). Assume that \( a_{i(m+1-i)} \) is even for some \( i, 2 \leq i \leq n-1 \). Let \( i \) be the smallest such. Let \( J \) be the set containing \( (m+1-i) \) and all \( (m-k+1) \)'s, \( 1 \leq k \leq i-1 \), for which the \( a_{k(m+1-i)} \)'s are odd. Then \( M_J \to D_{J | J} \) and hence \( l \to D_{J | J} \perp I_{m-J | J} \). Therefore we may assume that the \( a_{i(m+1-i)} \)'s are odd for all \( i, 1 \leq i \leq n-1 \).

Now assume that \( a_{nj} \) is even for some \( j, 1 \leq j \leq 3 \). If not all \( a_{kj} \)'s are odd for \( 1 \leq k \leq n-1 \), then \( M_J \to D_{J | J} \) as above. Hence if one of \( a_{nj} \) is even for \( j = 1, 2, 3 \),
then we may assume that the $a_{kj}'$s are all odd for $k = 1, 2, \ldots, n - 1$. If two of
the $a_{n}$'s are even, then $l \rightarrow D_{2} \perp I_{m-2}$. Therefore, without loss of generality,
we may assume that $a_{n}$ is odd and $a_{k}$ are all even for $k = 1, 2, \ldots, n - 1$. For
a fixed $s$, $s = 1$ or 2, if the number of $a_{k}$'s which are odd is less than $n - 1$ for
$k = 1, 2, \ldots, n$, then $l \rightarrow D_{s|j|} \perp I_{m-|j|}$, where $J$ is the set containing $s$ and the
$(m - k + 1)$'s for which the $a_{k}'$s are odd. In the remaining case, it is easy to see
that $l \rightarrow D_{k} \perp I_{m-k}$, where $k$ is 2 or 3 or 4.

**Theorem 2.3.** The lattice $E_{7} \perp I_{6}$ is 6-universal. In particular, $U_{Z}(n) = 13$.

**Proof.** First observe that $\text{gen}(I_{8} \perp A_{1}) = \{ I_{8} \perp A_{1}, E_{7} \perp I_{2} \}$. Hence it suffices
to show that every sublattice $l$ of $A_{1} \perp I_{8}$ of rank 6 is represented by $E_{7} \perp I_{6}$ by
Lemma 2.1. By Lemma 2.2 (1), $l(l(I_{8}) \rightarrow D_{5-i} \perp I_{3+i}$, for some $i = 0, 1, \ldots, 4$.
If $i \neq 4$, then we have

$$
l \rightarrow A_{1} \perp I_{8} \rightarrow A_{1} \perp D_{5-i} \perp I_{i+3} \rightarrow E_{7} \perp I_{6}.
$$

If $i = 4$, then $l' = l(l(I_{8}) \rightarrow D_{1} \perp I_{7}$. We apply Lemma 2.2 (2) to $l'(I_{7})$. By similar
reasoning as above, we need only consider the case when $l'(I_{7}) \rightarrow D_{7}$. This indeed
implies $l \rightarrow A_{1} \perp D_{1} \perp D_{7}$ and $d(l(D_{7})) \equiv 1 \pmod{7}$. For all prime $p$ (including
$\infty$), since $l(D_{7})_{p}$ is represented by $(E_{7} \perp A_{1})_{p}$ and the class number of $E_{7} \perp A_{1}$ is
1 [19], we have $l(D_{7}) \rightarrow E_{7} \perp A_{1}$, which proves the theorem.

In order to prove that $E_{6} \perp I_{7}$ is the other 6-universal lattice of rank 13, we
need the following lemma.

**Lemma 2.4.** If a $Z$-lattice $l$ of rank 6 is not represented by a sum of squares, then
$l \rightarrow E_{6} \perp I_{5}$.

**Proof.** We may assume that $l \rightarrow E_{7} \perp I_{2}$. By [8], we may also assume that
d($l(E_{7})$) is an odd determinant. Since the class number of $E_{6} \perp A_{2}$ is 1, it can
easily be checked that $l(E_{7}) \rightarrow E_{6} \perp A_{2}$, and hence $l \rightarrow E_{6} \perp I_{5}$ if $d(l(E_{7})) \neq 1$
(mod 3). So we assume that $d(l(E_{7})) \equiv 1 \pmod{6}$. By considering local conditions
for representation, we can conclude that $l(E_{7}) \rightarrow \text{gen}(E_{6} \perp I_{2})$ and consequently
$l \rightarrow E_{6} \perp I_{5}$ from the fact that $\text{gen}(E_{6} \perp I_{2}) = \{ E_{6} \perp I_{5}, \langle 3 \rangle \perp I_{7} \}$.

**Remark.** Ko conjectured [11] that if $l$ is of rank 6 and represented by a sum of
squares, then $l \rightarrow I_{9}$, and if $l$ is of rank 6 and not represented by a sum of squares,
then $l \rightarrow E_{6} \perp I_{3}$ and $l(E_{6}) = E_{6}$. But both conjectures are false because $l = A_{2} \perp
A_{2} \perp A_{1}10[1, 2]$ is represented by $I_{10}$ but not by $I_{9}$ for the former conjecture (see
[8], [9] for further results) and $l = D_{5}124[1, 2]$, which is not represented by a sum of
squares, is represented by $E_{6} \perp I_{3}$ but does not satisfy $l(E_{6}) = E_{6}$.

**Theorem 2.5.** The lattice $E_{6} \perp I_{7}$ is 6-universal.

**Proof.** Let $l$ be a $Z$-lattice of rank 6. By the above lemma, we may assume that $l$
is represented by a sum of squares, and hence by [8] we may assume that $l \rightarrow I_{10}$.
This implies that $l \rightarrow D_{5-i} \perp I_{5+i}$ for some $i = 0, 1, \ldots, 4$ by Lemma 2.2 (1). If
$i \neq 3, 4$, then $l \rightarrow D_{5-i} \perp I_{5+i} \rightarrow E_{6} \perp I_{7}$. The desired conclusion for the case
when $i = 3, 4$ can be deduced by applying Lemma 2.2 again if necessary.
3. Determination of $U_Z(n)$ for $7 \leq n \leq 10$

**Theorem 3.1.** The lattice $E_8 \perp I_8$ is a unique 8-universal $Z$-lattice of rank 16, and $U_Z(8) = 16$.

**Proof.** Note that the lattice $E_8 \perp I_8$ is the unique candidate of 8-universal $Z$-lattice of rank 16, for $E_8$ is the unique additively indecomposable $Z$-lattice of rank 8. Let $l$ be a $Z$-lattice of rank 8. Since $l \rightarrow \text{gen}(E_8 \perp I_3) = \{ E_8 \perp I_3, I_{11} \}$, we may assume that $l \rightarrow I_{11}$. By Lemma 2.2, we may further assume that $l \rightarrow A_1 \perp A_1 \perp D_9$ and $d(l(D_9)) \equiv 1 \pmod{8}$. Clearly, $l(D_9)$ is contained in one of the sublattices of $I_9$ of rank 9 with determinant 9. The following are all such sublattices of $I_9$:

\[
\begin{align*}
9 \perp & \quad I_8, A_118[12] \perp I_7, A_2 \perp \langle 3 \rangle \perp I_6, A_336[14] \perp I_5, A_445[25] \perp I_4, \\
A_56[32] & \perp I_3, A_663[37] \perp I_2, A_772[38] \perp I_1, \text{ and } A_89[33].
\end{align*}
\]

One can easily check that if $l(D_9)$ is represented by one of these lattices except the first one, then $l \rightarrow E_8 \perp I_8$. So assume that $l(D_9) \rightarrow \langle 9 \rangle \perp I_8$. Then $l(D_9)$ is represented by $Z(e_1 - e_2) + Z(e_2 - e_3) + \cdots + Z(e_7 - e_8) + Z(e_8 - 3e_9) + Z(e_8 + 3e_9)$ and hence is represented by $A_8 \perp I_5$. Therefore $l$ is represented by $E_8 \perp I_8$.

**Remark.** In [5], Conway and Schneeberger proved the so-called 15-Theorem, i.e., every integral $Z$-lattice which represents 1, 2, 3, 5, 6, 7, 10, 14, 15 is 1-universal. An analogy for 8-universal $Z$-lattices can be deduced from Theorem 3.1: Every $Z$-lattice which represents both $I_8$ and $E_8$ is 8-universal.

**Corollary 3.2.** The lattice $E_8 \perp I_7$ is 7-universal and $U_Z(7) = 15$.

**Proof.** The 7-universality of $E_8 \perp I_7$ follows from the above theorem. Consider the only possible candidate for a 7-universal $Z$-lattice of rank 14; namely, $E_7 \perp I_7$. But this cannot represent $A_677[2\frac{1}{2}]$, and the result follows.

**Theorem 3.3.** There are exactly three 7-universal $Z$-lattices of rank 15. They are $E_8 \perp I_7, E_7 \perp I_8$, and $E_76[1\frac{1}{2}] \perp I_7$.

**Proof.** Suppose that $L$ is a 7-universal $Z$-lattice of rank 15. Then $L = I_7 \perp L'$ and rank $(L') = 8$. Clearly, $E_7 \rightarrow L'$. If the lattice $L'$ represents 1, then $L = I_8 \perp E_7$. So assume that $L'$ does not represent 1. Since $A_677[2\frac{1}{2}] \rightarrow L$, either $D_7 \rightarrow L'$ or $A_677[2\frac{1}{2}] \rightarrow L'$. In the first case, $L'$ must be $E_8$, for $E_8$ is the only lattice of rank 8 which represents $E_7$ and $D_7$ simultaneously. In the second case, since the minimum quadratic norm of the dual lattice $E_7^{\perp}$ of $E_7$ is $\frac{1}{2}$, it can be easily deduced that $L'$ must be $E_76[1\frac{1}{2}]$. Hence we have exactly three candidates $E_8 \perp I_7, E_7 \perp I_8$ and $E_76[1\frac{1}{2}] \perp I_7$ for 7-universal $Z$-lattices of minimal rank, 15.

It suffices to show the 7-universality for the latter two. First, we show that $E_7 \perp I_8$ is 7-universal. Let $l$ be any $Z$-lattice of rank 7. Note that

\[
l \rightarrow \text{gen}(E_8 \perp I_2) = \{ E_8 \perp I_2, I_{10} \}.
\]

If $l \rightarrow I_{10}$, it is easy to check that $l \rightarrow E_7 \perp I_8$ by Lemma 2.2(1). So assume that $l \rightarrow E_8 \perp I_2$. Note that $l(E_8)$ can be represented by one of the sublattices of $E_8$ with determinant 4; the only such sublattices are $E_7 \perp A_1$ and $D_8$. Therefore the 7-universality of $E_7 \perp I_8$ follows immediately.
Now we prove that $E_7[1 \frac{1}{2}] \perp I_7$ is 7-universal. Note that for every $\mathbb{Z}$-lattice $l$ of rank 7

$$l \rightarrow \text{gen}(E_7[1 \frac{1}{2}] \perp I_2) = \{E_7[1 \frac{1}{2}] \perp I_2, A_2 \perp I_8\}.$$ 

So we assume that $l \rightarrow A_2 \perp I_8$. Then $l(I_8)$ is contained in one such sublattice of $I_8$ of rank 8 with determinant 9. It is easy to check that $l \rightarrow E_7[1 \frac{1}{2}] \perp I_7$ if $l(I_8)$ is contained in one such sublattice except $A_772[3 \frac{1}{2}]$. Therefore, we may restrict ourselves to the case when

$$l(I_8) \rightarrow A_772[3 \frac{1}{8}] = \{\sum_{i=1}^{8} a_ie_i \mid \sum_{i=1}^{8} a_i \equiv 0 \pmod{3}\}.$$ 

Furthermore, we may assume that $d(l(I_8)) \equiv 2 \pmod{3}$, for we may assume that $l(I_8)$ is not contained in any sublattice of $I_8$ of rank 8 with determinant 9 other than $A_772[3 \frac{1}{8}]$. By Lemma 2.2, we obtain $l \rightarrow D_{8-i} \perp I_i \perp A_2$ for $i = 0, 1, \ldots, 7$. If $i \neq 0, 1$, then this implies $l \rightarrow E_7[1 \frac{1}{2}] \perp I_7$, as desired. If $i = 0$, then

$$l(I_8) \rightarrow A_772[2 \frac{1}{4}] = \{\sum_{i=1}^{8} a_ie_i \mid \sum_{i=1}^{8} a_i \equiv 0 \pmod{6}\} \rightarrow E_7[1 \frac{1}{2}] \perp I_3$$

and hence $l \rightarrow E_7[1 \frac{1}{2}] \perp I_7$. If $i = 1$, then we may assume that $d(l(I_8)) \equiv 11 \pmod{12}$ by Lemma 2.2(2). Therefore

$$l(I_8) \rightarrow \text{gen}(A_2 \perp I_7) = \{A_2 \perp I_7, E_7[1 \frac{1}{2}] \perp I_1\}.$$ 

Consequently, $l \rightarrow E_7[1 \frac{1}{2}] \perp I_7$ as desired. 

\[\square\]

**Theorem 3.4.** The lattice $E_8 \perp I_9 \perp D_{10}A_1[11]$ is a 9-universal $\mathbb{Z}$-lattice and $U_2(9) = 28$.

**Proof.** Suppose that $L$ is a 9-universal $\mathbb{Z}$-lattice. Then $L$ must decompose into $E_8 \perp I_9 \perp L'$. There exist exactly two additively indecomposable $\mathbb{Z}$-lattices of rank 9, namely, $A_663[4 \frac{1}{2}]$ and $A_4A_415[33 \frac{1}{2}]$ (see [17]). Since $L'$ must represent these lattices, the rank of $L'$ is greater than 9. Suppose that the rank of $L'$ is 10. Then $A_9 \rightarrow L'$, since $1 \notin Q(L')$. Furthermore, $L'$ has a vector of norm 3, since $A_863[4 \frac{1}{2}] \rightarrow L'$. The possible candidates for $L'$ satisfying these properties are the following:

$$A_9210[1 \frac{1}{10}], A_935[2 \frac{1}{5}], A_990[3 \frac{1}{10}], A_915[4 \frac{1}{5}], A_9A_1[5 \frac{1}{2}], \text{ and } A_9 \perp (3).$$

Among these lattices, only $A_915[4 \frac{1}{5}]$ and $A_9A_1[5 \frac{1}{2}]$ can represent $A_863[4 \frac{1}{2}]$ and $A_4A_415[33 \frac{1}{2}]$ simultaneously. But neither $E_8 \perp I_9 \perp A_915[4 \frac{1}{5}]$ nor $E_8 \perp I_9 \perp A_9A_1[5 \frac{1}{2}]$ can represent $A_9117[2 \frac{3}{2}]$. Therefore the rank of $L'$ is greater than 10. On the other hand, it can be easily checked by using Lemma 2.2 that every $\mathbb{Z}$-lattice $l$ of rank 9, which is represented by $A_1 \perp I_{11}$, is represented by $E_8 \perp D_{10}A_1[11] \perp I_9$. Hence from

$$\text{gen}(A_1 \perp I_{11}) = \{A_1 \perp I_{11}, E_7 \perp I_5, D_{10}A_1[11] \perp I_1, E_8 \perp I_3 \perp A_1\}$$

we may conclude that $E_8 \perp D_{10}A_1[11] \perp I_9$ is 9-universal, and the result follows. 

\[\square\]

**Theorem 3.5.** The lattice $E_8 \perp I_{10} \perp D_{12}[1]$ is 10-universal and $U_2(10) = 30$. 


Proof. Suppose that $L$ is a 10-universal $\mathbb{Z}$-lattice. The lattice $L$ must decompose into $E_8 \perp I_{10} \perp L'$. The lattices $D_9[1\frac{1}{2}]$, $A_9A_1[51]$ are additively indecomposable $\mathbb{Z}$-lattices of rank 10 (see [17]), so $L'$ must represent these lattices. Suppose that the rank of $L'$ is 11; then $A_{10} \rightarrow L'$, since $1 \notin Q(L')$. But there does not exist a lattice of rank 11 which represents the lattices $A_{10}$ and $D_9[1\frac{1}{2}]$ simultaneously. Hence the rank of $L'$ is greater than 11. On the other hand, since 

$$\text{gen}(I_{13}) = \{I_{13}, E_8 \perp I_5, D_{12}[1] \perp I_1\},$$

it can be easily checked by applying Lemma 2.2 that $E_8 \perp I_{10} \perp D_{12}[1]$ is 10-universal. Therefore the result follows. 

Remark. It seems to be a very difficult problem to find the exact value of $U_\mathbb{Z}(n)$ for large $n$. For example, one can easily obtain $U_\mathbb{Z}(24) \geq 6673$ from a simple counting of all indecomposable unimodular $\mathbb{Z}$-lattices of rank less than or equal to 24 (see [2]).

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References


DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL, 151-742, KOREA

E-mail address: oandhan@math.snu.ac.kr