LIPSCHITZ CONTINUITY OF OBLIQUE PROJECTIONS

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Abstract. Let $W$ and $L$ be complementary spaces of a finite dimensional unitary space $V$ and let $P(W, L)$ denote the projection of $V$ on $W$ parallel to $L$. Estimates for the norm of $P(W, L) - P(W, M)$ are derived which involve the norm of the restriction of $P(W, L)$ to $M$ or the gap between $L$ and $M$.

1. Introduction and preliminaries

Let $V = W \oplus L$ be a nontrivial direct sum decomposition of an $n$-dimensional unitary space $V$ and let $P(W, L)$ denote the oblique projection on $W$ along $L$. If the distance between subspaces is measured in the gap metric, then all subspaces $M$ contained in a sufficiently small neighbourhood $U(L)$ of $L$ are also complementary to $W$ (see e.g. [1, p. 390] or [5]). For $M \in U(L)$ set $\pi(M) = P(W, M)$. In this note we study the map $\pi(M)$. An estimate for $\|\pi(M) - \pi(L)\|$ will be obtained which involves the restriction of $P(W, L)$ to $M$. A Lipschitz constant for $\pi$ in [1] will be improved.

Notation. For a linear map $A: Y \to V$ the norm $\|A\|$ denotes the operator norm, i.e. $\|A\| = \sup\{\|Ay\|, y \in Y, \|y\| = 1\}$. Let $P_W$ denote the orthogonal projection of $V$ on $W$ and set

$$P(W, L; M) = P(W, L)|_M.$$  

We write $d(x, M)$ for the distance of $x \in V$ from $M$. The gap between two subspaces $L$ and $M$ is defined by

$$\theta(L, M) = \|P_L - P_M\|.$$  

We shall need the following facts on the gap, for which we refer to [2] and [1]. First of all $\theta$ is a metric on the set of subspaces of $V$, and $\theta(L, M) \leq 1$. If $\theta(L, M) < 1$, then

$$\dim M = \dim L. \tag{1.1}$$  

In the case of (1.1) we have

$$\theta(L, M) = \|P_L(I - P_M)\| = \|P_M(I - P_L)\|. \tag{1.2}$$
Lemma 1.1. Assume $V = W \oplus L$.

(i) For a subspace $M$ of $V$ we have

$$\|P(W, L; M)\| = \|P(W, L)P_M\|. \tag{1.3}$$

(ii) If $\dim M = \dim L$, then

$$\|P(W, L; M)\| \leq \|P(W, L)\|\theta(L, M). \tag{1.4}$$

Proof. (i) For $y \in M, M \neq 0$, we have $P(W, L; M)y = P(W, L)P_My$. Therefore

$$\|P(W, L; M)\| = \max \left\{ \frac{\|P(W, L)P_My\|}{\|y\|}, y \neq 0, y \in M \right\} \leq \|P(W, L)P_M\|$$

$$= \max \left\{ \frac{\|P(W, L)P_My\|}{\|y\|}, y \neq 0, y \in V \right\}$$

$$\leq \max \left\{ \frac{\|P(W, L)P_My\|}{\|P_My\|}, y \in V, P_My \neq 0 \right\} = \|P(W, L; M)\|.$$ 

(ii) From $P(W, L) = P(W, L)(I - P_L)$ and (1.3) follows

$$\|P(W, L; M)\| = \|P(W, L)(I - P_L)P_M\|.$$ 

Hence (1.2) yields (1.4). \qed

The following observations do not seem to be widely known.

Lemma 1.2. Assume $V = W \oplus L$.

(i) If $W \neq 0$, then

$$\max_{x \in W, \|x\| = 1} \frac{1}{d(x, L)} = \|P(W, L)\|. \tag{1.5}$$

(ii) If $W \neq 0$ and $L \neq 0$, then

$$\|P(W, L)\| = \|P(L, W)\|. \tag{1.6}$$

Proof. (i) We shall see that (1.5) is equivalent to the identity

$$\frac{1}{1 - \|P_LP_W\|^2} = \|P(W, L)\|^2, \tag{1.7}$$

which is due to Ljance [3] (see [4] or [6]). Set

$$\tau = \min_{x \in W, \|x\| = 1} d(x, L) = \min_{x \in W, \|x\| = 1} \|(I - P_L)P_Wx\|.$$ 

Then the left-hand side of (1.5) is equal to $1/\tau$. If $x \in W$ and $\|x\| = 1$, then

$$\|(I - P_L)P_Wx\|^2 + \|P_LP_Wx\|^2 = 1.$$ 

Hence

$$\tau^2 = 1 - \max_{x \in W, \|x\| = 1} \|P_LP_Wx\|^2 = 1 - \|P_LP_W\|^2.$$ 

Therefore

$$\frac{1}{\tau^2} = \frac{1}{1 - \|P_LP_W\|^2},$$

and (1.5) follows from (1.7).

(ii) Since $P_LP_W = 0$ implies $P_LP_WP_L = 0$ and thus $P_WP_L = 0$, we note that either $P_LP_W = P_WP_L = 0$ or both $P_LP_W \neq 0$ and $P_WP_L \neq 0$. In each case we have $\|P_LP_W\| = \|P_WP_L\|$. Hence (1.7) implies (1.6). \qed
2. Estimates for oblique projections

**Theorem 2.1.** Assume \(V = W \oplus L, W \neq 0, L \neq 0\).

(i) Let \(M\) be a subspace of \(V\) with \(\dim M = \dim L\) and

\[
\mu = \|P(W, L; M)\| < 1.
\]

Then

\[
V = W \oplus M
\]

and

\[
\|P(W, M) - P(W, L)\| \leq \frac{\mu}{1 - \mu} \|P(W, L)\|.
\]

(ii) If a subspace \(M\) satisfies

\[
\theta(L, M) \leq (1 - c)\|P(W, L)\|^{-1}, \quad 0 < c < 1,
\]

then we have (2.2) and

\[
\|P(W, M) - P(W, L)\| \leq \frac{1}{c} \|P(W, L)\|^2 \theta(L, M).
\]

**Proof.** (i) Suppose \(x \neq 0\) for some \(x \in W \cap M\). Then \(P(W, L; M)x = x\). Hence \(\|P(W, L; M)\| \geq 1\), which contradicts (2.1). Therefore we have \(W \cap M = 0\) and (2.2). Now put \(S = P(M, W)P_L\). Then

\[
P(M, W)[I - P_L P(L, W)] = P(M, W)P(L, W) = 0
\]

implies \(P(M, W) = SP(L, W)\). Using \(P(W, L) = I - P(L, W)\) and (1.6) we obtain

\[
\|P(W, M) - P(W, L)\| = \|P(M, W) - P(L, W)\|
\]

\[
= \|SP(L, W) - P_L P(L, W)\| \leq \|S - P_L\| \|P(W, L)\|.
\]

Thus our target inequality is

\[
\|S - P_L\| \leq \frac{\mu}{1 - \mu}.
\]

Since \(P(L, W)P(W, M)x = 0\) for all \(x \in V\), we have \(P(L, W)[I - P(M, W)] = 0\) or \(P(L, W)P(M, W) = P(L, W)\). Similarly \(P(L, W)P_L = P_L\). Hence

\[
P(L, W)P(M, W)P_L = P(L, W)S = P_L,
\]

and we obtain \(S - P_L = [I - P(L, W)]S = P(W, L)P_M S\). Then

\[
(S - P_L) (S - P_L) = S^* S - P_L S - S^* P_L + P_L
\]

(2.7)

\[
= S^* P_M P(W, L)^* P(W, L) P_M S,
\]

which implies

\[
\]

For the left-hand side of (2.8) we obtain

\[
\|P_L S + S^* P_L\| \leq 2\|P_L\| \|S\| \leq 2\|S\|.
\]

Put \(T = I - P_M P(W, L)^* P(W, L) P_M\) such that the right-hand side of (2.8) equals \(R = S^* T S + P_L\). Since \(P_L\) is the identity map on \(L\) and \(SP_L = S\), it is not difficult to show that \(\|R\| = \|S^* T S\| + \|P_L\| \|S\| \leq 2\|S\|\). Hence \(\|S^* T S\| \geq \|S\| \|S\| (1 - \mu^2)\) is the smallest eigenvalue of \(T\). Hence \(\|S^* T S\| \geq \|S\| \|S\| (1 - \mu^2)\). Thus \(\|S\|\) satisfies

\[
0 \leq \|S\|^2 (1 - \mu^2) + 1 \leq 2\|S\|,
\]

which completes the proof.
which is equivalent to

\[(2.9)\quad 0 < \frac{1}{1 + \mu} \leq \|S\| \leq \frac{1}{1 - \mu}\]

Then (2.7) yields

\[\|S - P_L\| \leq \|S\| \|P(W, L)P_M\| \leq \frac{\mu}{1 - \mu},\]

and we have (2.6), which completes the proof of (2.3).

(ii) From (2.4) and (1.4) we obtain

\[(2.10)\quad \mu \leq \|P(W, L)\|\theta(L, M) \leq 1 - c < 1.

Then \(\|P(W, L)\| \geq 1\) implies \(\theta(L, M) < 1\) and \(\dim M = \dim L\). Because of \(\mu < 1\) we can use (i) and conclude that \(P(W, M)\) exists. Since \(0 \leq \mu \leq 1 - c\) is equivalent to

\[0 \leq \frac{1}{1 - \mu} \leq \frac{1}{c},\]

the estimate (2.5) follows immediately from (2.3).

In the neighbourhood of \(L\) given by (2.4) the estimate (2.5) yields a Lipschitz constant for \(P(W, M)\) of the form

\[(2.11)\quad \frac{1}{c}\|P(W, L)\|^2.

In [1, p. 390] we find for sufficiently small \(\theta(L, M)\) an estimate

\[\|P(W, M) - P(W, L)\| \leq K\theta(L, M)\]

with

\[(2.12)\quad K = 2\|P(W, L)\| \max_{x \in W, \|x\|=1} \frac{1}{d(x, L)}.

According to Lemma 1.2 the Lipschitz constant \(K\) in (2.12) is equal to (2.11) with \(c = 1/2\).