A LOCAL VERSION OF WONG-ROSAY’S THEOREM FOR PROPER HOLOMORPHIC MAPPINGS

NABIL OURIMI

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Abstract. In the present paper, we generalize Wong-Rosay’s theorem for proper holomorphic mappings with bounded multiplicity. As an application, we prove the non-existence of a proper holomorphic mapping from a bounded, homogeneous domain in $\mathbb{C}^n$ onto a domain in $\mathbb{C}^n$ whose boundary contains strongly pseudoconvex points.

1. Introduction and results

The purpose of this paper is to prove a version of Wong-Rosay’s theorem [15],[10] for families of proper holomorphic mappings with bounded multiplicity. Our main result can be stated as follows:

Theorem 1. Let $D \subset \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^n$ be domains. Suppose there exist a point $p \in D$ and a sequence $\{f_k\}_k$ of proper holomorphic mappings $f_k : D \to G$ of multiplicity equal to $m$ such that $\{f_k(p)\}_k$ converges to a strongly pseudoconvex boundary point $q \in \partial G$. Then there exists a proper holomorphic mapping defined from $D$ onto the unit ball in $\mathbb{C}^n$ of multiplicity less than or equal to $m$.

This theorem implies that domain $D$ is necessarily pseudoconvex and furthermore, if $G$ is a strongly pseudoconvex, bounded, simply connected domain with $C^\infty$-boundary, then according to [2] $G$ is biholomorphic to the unit ball in $\mathbb{C}^n$.

The assumption about a uniform bound on the multiplicities on the mappings is necessary for our proof, but it is rather natural in view of a result of Bedford [1] which states that there is an absolute bound on the multiplicity of a proper holomorphic mapping between bounded pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries.

By using Theorem 1, we give a generalization of a result of Lin and Wong [7] for unbounded domains in $\mathbb{C}^n$.

Corollary 1. Let $D \subset \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^n$ be domains. Suppose there exist a point $p \in D$ and a sequence $\{f_k\}_k$ of unbranching proper holomorphic mappings $f_k : D \to G$ such that $\{f_k(p)\}_k$ converges to a strongly pseudoconvex boundary point $q \in \partial G$. Then both $D$ and $G$ are biholomorphic to the unit ball in $\mathbb{C}^n$. 

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The following example proves that Corollary 1 cannot be extended to sequences of branched proper holomorphic mappings.

Let \( D = \{(z, w) \in \mathbb{C}^2 : |z|^4 + |w|^2 < 1\} \), \( B = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\} \) be domains in \( \mathbb{C}^2 \) and let us consider the proper holomorphic mapping \( f : D \to B \),

\[
(z, w) \mapsto (z^2, w).
\]

Let \( q \in \partial B \) be a boundary point and \( (q_k) \) be a sequence in \( B \), which converges to \( q \). Since \( B \) is homogeneous, there exists a sequence \( (\varphi_k)_k \subset B \) of automorphisms such that \( q_k = \varphi_k(0) \). Let \( f_k = \varphi_k \circ f \). Then \( \{f_k\}_k \) is a sequence of proper holomorphic mappings with bounded multiplicity and \( \{f_k(0)\}_k \) converges to \( q \) which is a strongly pseudoconvex boundary point, but the domain \( D \) is not biholomorphic to the unit ball in \( \mathbb{C}^2 \).

For strongly pseudoconvex domains in \( \mathbb{C}^n \), we have the following result.

**Corollary 2.** Let \( D \subset \subset \mathbb{C}^n \) and \( G \subset \mathbb{C}^n \) be strongly pseudoconvex domains. Suppose there exist a point \( p \in D \) and a sequence \( \{f_k\}_k \) of proper holomorphic mappings \( f_k : D \to G \) such that \( \{f_k(p)\}_k \) converges to a strongly pseudoconvex boundary point \( q \in \partial G \). Then both \( D \) and \( G \) are biholomorphic to the unit ball in \( \mathbb{C}^n \).

In the case where \( D = G \), we obtain a local version of Wong-Rosay’s theorem for proper holomorphic mappings as follows:

**Theorem 2.** Let \( D \) be a bounded domain in \( \mathbb{C}^n \). Suppose there exist a point \( p \in D \) and a sequence \( \{f_k\}_k \) of proper holomorphic mappings \( f_k : D \to D \) of bounded multiplicity such that \( \{f_k(p)\}_k \) converges to a strongly pseudoconvex boundary point \( q \in \partial D \). Then \( D \) is biholomorphic to the unit ball in \( \mathbb{C}^n \).

As another application of Theorem 1, we establish the following result concerning bounded homogenous domains in \( \mathbb{C}^n \).

**Corollary 3.** Let \( D \) be a bounded homogenous domains in \( \mathbb{C}^n \) and \( G \) be a domain in \( \mathbb{C}^n \) whose boundary contains strongly pseudoconvex points. If there exists a proper holomorphic mapping from \( D \) onto \( G \), then \( D \) is biholomorphic to the unit ball in \( \mathbb{C}^n \).

2. **Notations and preliminary results**

For the proof of Theorem 1, we need to introduce the notion of proper holomorphic correspondences. Let \( D \) and \( G \) be two domains in \( \mathbb{C}^n \) and let \( \Gamma \) be a complex purely \( n \)-dimensional subvariety contained in \( D \times G \). We denote by \( \pi_1 : \Gamma \to D \) and \( \pi_2 : \Gamma \to G \) the natural projections. When \( \pi_1 \) is proper, then \( (\pi_2 \circ \pi_1^{-1})(z) \) is a non-empty finite subset of \( G \) for any \( z \in D \) and one may therefore consider the set-valued mapping \( f = \pi_2 \circ \pi_1^{-1} \). Such a map is called a holomorphic correspondence between \( D \) and \( G \); \( \Gamma \) is said to be the graph of \( f \) and it will be denoted by \( \text{graph} f \). Since \( \pi_1 \) is proper, there exist a complex subvariety \( V \subset \text{graph} f \) and an integer \( m \) such that \( f(z) = \{f^1(z), \ldots, f^m(z)\} \) for all \( z \in D \setminus \pi_1(V) \) and the \( f^j \)'s are distinct holomorphic functions in a neighborhood of \( z \in D \setminus \pi_1(V) \) (see for instance [5]). The integer \( m \) is called the multiplicity of \( f \). The correspondence \( f \) is proper if \( \pi_2 \) is proper and it is irreducible if its graph is irreducible. Furthermore, for bounded domains \( f \) is proper if and only if \( \partial \text{graph} f \subset \partial D \times \partial G \). Correspondences were introduced by Stein [12] in order to generalize meromorphic mappings.
between complex spaces. Properties of correspondences can be found in Stein’s papers [12, 13]. For example, it can be shown that \( f \) gives rise to a holomorphic mapping \( \tilde{f} : D \to G'_{\text{sym}} \) into the \( m \)-fold symmetric product of \( G \) ([3]).

Now let \( z_0 \) be a point in \( D \) and \( \{ z_1, z_2, \ldots, z_m \} \) be a set in \( G \). We say that \( f(z) = \{ f^1(z), \ldots, f^m(z) \} \) converges to \( \{ z_1, z_2, \ldots, z_m \} \) when \( z \) tends to \( z_0 \) if after a possible renumberation of \( f^j \), one has \( \lim_{z \to z_0} f^j(z) = z^j \). Equivalently \( f(z) \) tends to \( \{ z_1, z_2, \ldots, z_m \} \) in the sense of Hausdorff convergence of sets.

We denote by \( \text{Cor}(D, G, m) \) the set of all \( \nu \)-valued holomorphic mappings from \( D \) onto \( G \) for \( \nu = 1, \ldots, m \). Let \( f \in \text{Cor}(D, G, m) \) be irreducible, \( a \in A \subseteq D \), \( b \in f(a) \); then we define \( \tilde{f}_{\alpha}^{(a,b)} \in \text{Cor}(\hat{A}, G, m) \) to be the correspondence obtained by analytic continuation of the germ of \( f \) at \( (a, b) \) by paths which lie in \( A \). Equivalently, \( \text{graph} \tilde{f}_{\alpha}^{(a,b)} \) is the union of those irreducible components of \( \text{graph} f \cap \{ A \times G \} \), which contain \( (a, b) \).

Let \( \{ f_k \} \subseteq \text{Cor}(D, G, m) \). We say that \( \{ f_k \} \) is compactly divergent if \( \forall K_1 \subseteq D, K_2 \subseteq G, \exists j_0 \forall j \geq j_0 : \)

\[
f_k(K_1) \cap K_2 = \emptyset.
\]

If the \( f_k \) are irreducible, we say that \( f_k \) converge to \( f \in \text{Cor}(D, G, m) \) if \( \exists (a, b_k) \in \text{graph} f_k \) with \( b_k \to b \in G \) and for all \( K \subseteq D \) with \( a \in K \):

\[
\tilde{f}_{k, K}^{(a, b_k)} \to f_K \text{ for some } f_K \in \text{Cor}(\hat{K}, G, m)
\]

and

\[
\bigcup_{K \subseteq D} \text{graph} f_K = \text{graph} f.
\]

If \( D \) and \( G \) are bounded domains in \( \mathbb{C}^n \), the set of proper holomorphic correspondence \( \text{Cor}(D, G, m) \) is normal for any \( m \in \mathbb{N} \) ([6]), i.e. every sequence \( \{ f_k \} \) of proper holomorphic correspondence in \( \text{Cor}(D, G, m) \) is either compactly divergent or has a convergent subsequence.

3. Proofs of results

Proof of Theorem 1. Since \( q \) is strongly pseudoconvex boundary point, according to [4] the sequence \( \{ f_k \} \) converges to \( q \) uniformly on compact subsets of \( D \). We use scaling methods introduced by S.Pinchuk [8]. Let \( U \) be a neighborhood of \( q \) in \( \mathbb{C}^n \) which does not intersect the set of weakly pseudoconvex points of \( \partial G \). For all \( \xi \in \partial G \cap U \), we consider the change of variables \( \alpha^\xi \) defined by:

\[
\begin{align*}
z_j^* &= \frac{\partial \rho}{\partial z_n}(\xi)(z_j - \xi_j) - \frac{\partial \rho}{\partial \bar{z}_j}(\xi)(z_n - \xi_n), \quad 1 \leq j \leq n - 1, \\
z_n^* &= \sum_{1 \leq j \leq n} \frac{\partial \rho}{\partial \bar{z}_j}(\xi)(z_j - \xi_j)
\end{align*}
\]

where \( \rho \) is a defining function of \( G \). The mapping \( \alpha^\xi \) maps \( \xi \) to \( 0 \) and the real normal at \( 0 \) to \( \partial G \) to the line \( \{ z = 0, y_n = 0 \} \).

Let \( K \subseteq D \) be a compact. There exists an integer \( k_0 \) such that, for all \( k \geq k_0 \) and \( z \in K \), the point \( f_k(z) \in U \cap G \). We denote by \( w^k \) the projection of \( q^k \) on \( \partial G \cap U \) and \( \alpha^k = \alpha^{w^k} \) the mapping as above. We have \( \alpha^k(q^k) = 0, -\delta_k \) with \( q^k = f_k(p) \) and \( \delta_k = \text{dist}(\alpha^k(q^k), \partial \alpha^k(G^k)) \). We define now the inhomogenous dilatations \( \varphi^k \) by \( \varphi^k(z, z_n) = (\delta_k^2 z, \delta_k z_n) \) and let \( G^k = \varphi^k \circ \alpha^k(G) \). For all \( k \),
the mapping \( g_k = \varphi^k \circ \alpha^k \circ f_k : D \to G^k \) is a proper holomorphic mapping with multiplicity \( m \), which satisfies \( g_k(p) = s = (0, -1) \). The sequence \( \{g_k\}_k \) is a normal family, passing to subsequence, \( \{g_k\}_k \) converges uniformly on the compact subsets of \( D \) to a holomorphic mapping \( g : D \to \Sigma \), where

\[
\Sigma = \{ (z, z_n) \in \mathbb{C}^n : 2 \text{Re}(z_n) + |z|^2 < 0 \}.
\]

To finish the proof we shall prove that the mapping \( g \) is proper. We will need to study the convergence of the correspondence \( h_k = g_k^{-1} \). For this, we will use a similar method introduced by W. Klingenberg and S. Pinchuk in [6] to study the problem of normality of proper holomorphic correspondences between bounded domains in \( \mathbb{C}^n \).

The correspondence \( h_k : G^k \to D \) is a proper, holomorphic irreducible one which satisfies \( (s, p) \in \text{graph}(h_k) \) for all \( k \). Given \( K \subset \Sigma \), a compact, \( s \in K \), we have \( \hat{h}_{k, K}^{(s, p)} \in \text{Cor}(\hat{K}, D, m) \). Since \( D \) is bounded, there is a subsequence which converges to an element \( h \in \text{Cor}(K, \mathcal{D}, m) \). Since \( \Sigma \) is biholomorphic to the unit ball \( \mathbb{B} \), then by exhausting \( \mathbb{B} \) with compact and passing to diagonal subsequence, we obtain \( h \in \text{Cor}(\Sigma, \mathcal{D}, m) \). The following fact was proved in [6]. For completeness, we include a proof.

**Claim.** \( h \in \text{Cor}(\Sigma, D, m) \).

**Proof.** The branches \( \{h^1, \ldots, h^m\} \) of \( h \) are locally defined and holomorphic on \( D \setminus \pi_1(V) \). Now the jacobians of \( h^i \) induce in a natural manner a holomorphic function \( \text{Jac}(h) \) on \( \text{graph}(h) \setminus V \); then there exists only one \( i \in \{1, \ldots, m\} \) such that \( z \in \text{graph}(h^i) \). We define \( \text{Jac}(h)(z) = \text{Jac}(h^i)(\pi_1 z) \).

First we show that \( \text{Jac}(h) \neq 0 \). We need the following lemma.

We will write \( \hat{h}_A^{(a, b)} = h_A^{(a, b)}(A) \).

**Lemma 1** ([6]). Let \( D \) and \( G \) be bounded domains in \( \mathbb{C}^n \) and \((a, b) \in D \times G \). Then for all \( U(b) \subset G \) there exists \( U(a) \subset D \), such that for all \( h \in \text{Cor}(D, G, m) \) with \( b \in h(a) \) we have: \( \hat{h}_U^{(a, b)}(A) \subset U(b) \).

Let \( U(p) \subset \subset D \) be a neighborhood of \( p \in D \). By Lemma 1, there exists \( U(s) \) a neighborhood of \( s \in \Sigma \) with \( \hat{h}_{k, U(s)}^{(s, p)} \subset U(p) \) for all \( k \). Then we have \( z = g_k = g \circ \hat{h}_{k, U(s)}^{(s, p)}(z) \) for all \( z \in U(s) \). Passing to a convergent subsequence, we have \( z = g_k \circ \hat{h}_{U(s)}^{(s, p)}(z) \) for all \( z \in U(s) \), which implies that \( \text{Jac}(h_U(s)) \neq 0 \). Since \( \text{graph}(h) \setminus V \) is connected, we conclude that \( \text{Jac}(h) \neq 0 \). Let \( W \subset \text{graph}(h) \setminus V \) denote the variety \( \{ \text{Jac}(h) = 0 \} \). Now assume that the claim is false, i.e. there exist \((x, y) \in \Sigma \times \partial D \) with \( y \in h(x) \). Since the branches of \( h \) are locally open maps on \( D \setminus \pi_1(V \cup W) \), we must have \( x \in \pi_1(V \cup W) \). The variety \( V \cup W \) is a subvariety of \( D \) of dimension \( n - 1 \); thus there exists a holomorphic disc \( \hat{\Delta} \) in \( D \) such that \( \hat{\Delta} \cap \pi_1(V \cup W) = x \). Since \( h(\hat{\Delta}) \subset G \cup \{ y \} \) is a disc, by the theorem of Cartan-Thullen (see [14]), the maps \( g_k \) and \( g \) extend analytically to a fixed neighborhood of \( y \), say \( U(y) \). The domain \( \Sigma \) is biholomorphic to the unit ball which is a bounded domain; then there exists a subsequence of \( g_k \) which converges to \( g \) on the compact subsets of \( D \cup U(y) \). It follows from the assumption that there exists \( y_k \in \hat{h}_{k, K}(x) \) with \( y_k \to y \). But since \( h_k \) is the inverse of \( g_k \), this implies \( x = g_k(y_k) \), and we may pass to the limit, which gives \( x = g(y) \). Since \( g_k \) is proper, \( g_k(y) \in \partial G_k \) and then by passing
to a convergent subsequence, the limit implies that $g(y) \in \partial \Sigma$. This contradicts $x \in \Sigma$.

We continue now with the proof of Theorem 1. Let $z \in D$ and $U(g(z))$ be a neighborhood of $g(z)$ in $\Sigma$. Lemma 1 implies that there exists $U(z)$ a neighborhood of $z$ in $D$ such that for large $k$'s we have $g_k(U(z)) \subset U(g(z))$. One has $z \in h_k \circ U(g(z)) \circ g_k(z)$ for all $z \in U(z)$. Passing to a convergent subsequence and to limit, we get

$$z \in h \circ g(z), \forall z \in D.$$  

Suppose that there exists a sequence $\{z_j\} \subset D$, which converges to $z \in \partial D$ and $g(z_j)$ converges to $z' \in \Sigma$. According to (*), we have $z_j \in h \circ g(z_j)$ for all $j$. The limit implies that $z \in h(z')$, which contradicts $z \in \partial D$ and then $g$ is proper. This finishes the proof of Theorem 1.

Proof of Corollary 1. First we show that $D$ is simply connected. According to [4], the sequence $\{f_k(p)\}_k$ converges uniformly on compact subsets of $D$ to $q$. Suppose that $D$ is not simply connected; then there exists a nontrivial closed loop $\gamma$ in $\pi_1(D)$. The boundary of $G$ is smooth near $q$; then there exists a neighborhood $U$ of $q$ such that $G \cap U$ is simply connected. For large $k$'s, $f_k(\gamma)$ is a closed loop $\beta \subset G \cap U$. Nevertheless, $f_k : D \to G$ is a covering, and $(f_k)_* : \pi_1(D) \to \pi_1(G)$ is one to one. This is a contradiction to the fact that $f_k(\gamma)$ must be a nontrivial element in $(f_k)_*\pi_1(D)) \subset \pi_1(G)$.

The mappings $f_k$ are a covering and $D$ is simply connected. Then the order of $\pi_1(G)$ is equal to the multiplicity of $f_k$ for all $k$ and then the multiplicity of $f_k$ is bounded. According to Theorem 1, there exists a proper holomorphic mapping $f : D \to \mathbb{B}$. Hurwitz's theorem implies that $f$ is a covering. Since $\mathbb{B}$ is simply connected, $f$ is biholomorphic and then $D$ is biholomorphic to the unit ball.

For any $k$, the map $h = f_k \circ \mathbb{B} \to G$ is a holomorphic covering. The ball $\mathbb{B}$ is simply connected, and $h$ is factored by automorphisms, i.e., there exists a subgroup $\Gamma$ of automorphism groups of $\mathbb{B}$ such that for all $z \in \mathbb{B}$, $h^{-1}(h(z)) = \{\gamma(z), \gamma \in \Gamma\}$. According to [11], $\{\gamma(z) = z\}$ is non-empty. Since $\{\gamma(z) = z\} \subset V_h$ ($V_h$ is the branch locus of $h$) for all $\gamma \in \Gamma \setminus \{I_\mathbb{B}\}$ and $h$ is a covering, the group $\Gamma$ is reduced to $\{I_\mathbb{B}\}$ and then $h$ is biholomorphic.

Proof of Corollary 2. The domains $D$ and $G$ are strongly pseudoconvex, according to [8], and $f_k$ is a covering. The proof can be completed by using Corollary 1.

Proof of Theorem 2. Theorem 1 implies that there exists a proper holomorphic mapping $f : D \to \mathbb{B}$. The correspondence $f \circ f_k \circ f^{-1}$ is an irreducible self-proper one, according to [2]; $f \circ f_k \circ f^{-1}$ is an automorphism of the unit ball. There exists then $\phi \in Aut(\mathbb{B})$ such that $f \circ f_k = \phi \circ f$. From this, we conclude that the mapping $f_k$ is one to one. Otherwise the multiplicity of the mapping $f \circ f_k$ is greater than the multiplicity of the mapping $\phi \circ f$, but $f \circ f_k = \phi \circ f$. Then $f_k$ is biholomorphic for all $k$. Now Corollary 1 can be applied to finish the proof.

Proof of Corollary 3. Let $q$ be a strongly pseudoconvex boundary point of $G$ and $\{f(p_k)\}_k$ be a sequence in $G$ which converges to $q$, where $(p_k)_k$ is a sequence in $D$. Since $D$ is homogenous, there exists a sequence of automorphisms $\{g_k\}_k \subset Aut(D)$ such that $g_k(0) = p_k$. The sequence $\{f \circ g_k(0)\}$ converges to $q$. Theorem 1 implies that there exists a proper holomorphic mapping from $D$ onto the unit ball in $\mathbb{C}^n$. According to [9], $D$ is biholomorphic to the unit ball.
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C.M.I., 39 RUE JOLIOT-CURIE, 13453 MARSEILLE CEDEX 13, FRANCE
E-mail address: ourimi@gyptis.univ-mrs.fr
Current address: Faculte des Sciences de Monastir, Route de Kairouan, 5000 Monastir, Tunisia