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CONJUGATE $SU(r)$-CONNECTIONS AND HOLONOMY GROUPS

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Abstract. In this article we show that when the structure group of the reducible principal bundle P is $SU(r)$ and $Q \subset P$ is an $SO(r)$-subbundle of P, the rank of the holonomy group of a connection which is gauge equivalent to its conjugate connection is less than or equal to $\left[\frac{r}{2}\right]$, and use the estimate to show that for all odd prime r, if the holonomy group of the *irreducible* connection as above is simple and is not isomorphic to E_8, F_4, or G_2, then it is isomorphic to $SO(r)$.

1. Introduction

In their paper [7], S. Kobayashi and E. Shinozaki introduced the concept of conjugate connection in a reducible principal G-bundle P. By looking at principal bundles they had a better understanding of conjugate connections whose notion in affine differential geometry had been known for a long time. Moreover, they showed that the group consisting of automorphisms of a Lie group G fixing a Lie subgroup H, the automorphism group, induces a compatible action on the quotient space of connections modulo gauge group on a principal G-bundle P that is reducible to a principal H-subbundle Q. It was also observed in [4] that the action does not depend on the reduction of a principal G-bundle to a subbundle. When we fix a Riemannian metric on the base manifold and G is compact, the automorphism group also acts on the moduli space of Yang-Mills connections and on the moduli space of anti-self-dual (ASD) connections, when the base manifold is 4-dimensional.

However, it turns out that the inner automorphism group which consists of inner automorphisms in the automorphism group trivially acts on the quotient space of connections. Thus, it is natural to consider the action on the quotient space of connections for the outer automorphism group. It is well known that for compact simple Lie groups only Lie groups of type A_r $(r > 1)$, D_r $(r \geq 4)$, and E_6 have a nontrivial outer automorphism group (see [10]). In case of $G = SU(3)$, we showed a localization theorem of the moduli space of irreducible ASD connections in a reducible principal $SU(3)$-bundle P along the moduli spaces of irreducible ASD connections in the $SO(3)$-subbundles Q of P (see [3]). One of the key ingredients of the proof was to know what is the holonomy group of an irreducible connection which is gauge equivalent to its conjugate connection. This is relatively easy since the rank of the structure group is just 2. But as the rank of the structure group

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becomes higher, it does not seem to be a trivial problem to know what are the holonomy groups at all.

In this paper, as a continuation of the paper [3] we give a sharp rank estimate of the holonomy group of an irreducible connection which is gauge equivalent to its conjugate connection, and as an easy consequence we prove that when the rank of the structure group plus one is odd prime, if the holonomy group is simple and not isomorphic to E_8, F_4, or G_2, then it is isomorphic to $SO(r)$.

We organize this paper as follows. In Section 2, we briefly recall the definition of conjugate connection and an important theorem of Kobayashi and Shinozaki. We will show a sharp rank estimate of the holonomy group in Section 3.

2. Preliminaries

In this section, we will set up some notations and recall briefly the definition of conjugate connection in reducible principal bundles (see [7] for more details) and a theorem from [8], due to Kobayashi and Shinozaki, which motivates this paper.

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let H be a closed subgroup with Lie algebra $\mathfrak{h}$. Let P be a principal G-bundle over a manifold M with projection π, and Q a principal H-subbundle of P. In general, such a subbundle does not necessarily exist. We cover M by open sets U_i with local sections $\xi_i : U_i \to Q$. Then, the transition functions $a_{ij} : U_i \cap U_j \to H$ are defined by

$$\xi_j(x) = \xi_i(x) a_{ij}(x).$$

A **connection form** $\{A_i\}$ on P is defined as a family of $\mathfrak{g}$-valued 1-forms A_i on U_i which satisfies the following transformation rule from A_i to A_j:

$$A_j = a_{ij}^{-1} A_i a_{ij} + a_{ij}^{-1} da_{ij} \quad \text{on } U_i \cap U_j. \tag{2.1}$$

Since A_i is not defined on all of M, we define a $\mathfrak{g}$-valued 1-form on P from A_i as follows:

$$A = g^{-1} A_i g + g^{-1} dg, \quad g \in G,$$

on $\pi^{-1}(U_i) = U_i \times G$. Then it is easy to see that A is a $\mathfrak{g}$-valued 1-form on P, and it follows that $\xi_i^*(A) = A_i$.

Given $\sigma \in \mathrm{Aut}(G, H)$, we set $A_i^\sigma = \sigma(A_i)$ and apply σ to (2.1). Since we took local sections $s_i : U_i \to Q$ and thus the transition functions a_{ij}'s are H-valued, we have

$$A_j^\sigma = a_{ij}^{-1} A_i^\sigma a_{ij} + a_{ij}^{-1} da_{ij}.$$

Thus, $\{A_i^\sigma\}$ defines a connection on P. We call it the σ-**conjugate connection** of A relative to Q, denoted A^σ. Note that A^σ is not $\sigma(A)$. In fact, $\sigma(A)$ may not be a connection 1-form.

Now, let us set up some notations which will used later in this paper. Let $\mathrm{Aut}(G, H)$ be the group of automorphisms of G leaving all elements of H fixed, $\mathrm{Inn}(G, H)$ be the group of all inner automorphisms in $\mathrm{Aut}(G, H)$, and let

$$\mathrm{Out}(G, H) = \mathrm{Aut}(G, H)/\,\mathrm{Inn}(G, H).$$

$\mathrm{Out}(G, H)$ is called the **outer automorphism group**. Given $\sigma \in \mathrm{Aut}(G, H)$, the induced automorphism of $\mathfrak{g}$ is denoted also σ. For $G = SU(r)$, we fix an automorphism σ given by $a \mapsto \bar{a}$. Note that the automorphism σ of $SU(r)$ $(r \geq 3)$ is outer and a generator of the outer automorphism group (see [10]).

We fix a point $u_0 \in Q$. We denote by $H_{u_0}(A)$ the holonomy group of a connection A with reference to u_0. We call a connection in P **generic** if its holonomy group coincides with G, and call a connection **irreducible** if its isotropy group, considered as a closed Lie subgroup of G, coincides with the center of G (see [1] for more details).

Finally, we state an important theorem of Kobayashi and Shinozaki [8] in this paper.

Theorem 2.1. *Let $\sigma \in \mathrm{Aut}\,(G, H)$ and A be a connection in P. Assume that A^σ is gauge equivalent to A under a gauge transformation φ. If we define an element $a \in G$ by $\varphi(u_0) = u_0 a$, then*

$$\sigma(g) = a^{-1} g a$$

for $g \in H_{u_0}(A)$. In particular, if the holonomy group is G, then σ is the inner automorphism defined by a^{-1} above.

As a consequence, $\mathrm{Out}\,(G, H)$ acts freely on the generic part of the quotient space of connections, and $\mathrm{Aut}\,(G, H)$ acts freely on the generic part of the framed quotient space of connections.

3. Main results

In this section we prove our main results in this paper. First, we begin with the following

Theorem 3.1. *Let P be a principal $SU(r)$-bundle over a simply connected manifold M that is reducible to an $SO(r)$-subbundle Q. Let A be an irreducible connection in P. Assume that the σ-conjugate connection A^σ of A is gauge equivalent to A under a gauge transformation φ. Then, the holonomy group $H_{u_0}(A)$ is a compact, connected, semisimple Lie subgroup of $SU(r)$ of rank less than or equal to $\left[\frac{r}{2}\right]$.*

Remark 3.2. The upper bound in the theorem is sharp because there is an irreducible connection A which defines one in the principal $SO(r)$-subbundle Q and whose holonomy group is $SO(r)$, and $SO(r)$ has rank $\left[\frac{r}{2}\right]$. Note that we need the irreducibility of the connection only to say that the holonomy group is semisimple.

To prove the theorem, we first prove the following

Lemma 3.3. *Let P be a principal $SU(r)$-bundle whose base manifold is simply connected and paracompact. Then, every holonomy group of an irreducible connection is a compact, connected, semisimple Lie subgroup of $SU(r)$.*

Remark 3.4. Since it is well known that the holonomy group is a connected Lie group ([6], Theorem 4.2), it remains to prove that the holonomy group is closed and semisimple. In general, the statement in the lemma may not be true. Thus, we would like to point out that the statement by S.K. Donaldson and P.B. Kronheimer, line 4 from the top on page 132 in [1], that it can be shown that the holonomy group is a closed Lie subgroup of G is false, in general. In fact, S. Kobayashi [5] gave a very simple counterexample to their claim: Let H be an open Lie subgroup of a Lie group G, and let Q be a principal H-bundle. Let P be the principal G-bundle extending Q. Since $H \subset G$, we have $Q \subset P$. That is, Q is a principal H-subbundle of P. Now, taking a connection in P which defines in the principal H-subbundle Q and whose holonomy is H completes the counterexample. So, the point of this

lemma is that if the structure group is $SU(r)$, then the holonomy group is always a *closed* Lie subgroup of $SU(r)$.

Proof. To prove the lemma, we first note that every non-semisimple subalgebra of type A_r is reducible by E. Cartan and E. B. Dynkin (e.g., see [2]), when we use the ordinary (or standard) representation. So, if the Lie algebra of the holonomy group $H_{u_0}(A)$, in short holonomy algebra, is not semisimple, then its centralizer would be larger than the center of $SU(r)$, $\mathbb{Z}_r$. Thus, the holonomy group is semisimple. Now, the proof follows immediately from the result of K. Yosida [11] (see also [6]) that every connected semisimple Lie subgroup of $GL(r;\mathbb{C})$ is closed and the fact that $SU(r)$ is closed in $GL(r;\mathbb{C})$. □

Proof (Theorem 3.1). We denote by $[x_1, \dots, x_r]$ the matrices of the form

$$\begin{pmatrix} \sqrt{-1}x_1 & & \\ & \ddots & \\ & & \sqrt{-1}x_r \end{pmatrix} \quad (x_i \text{ real}),$$

for the sake of simplicity.

We next define polynomial functions $g_1, g_2, \dots$ on the Lie algebra $\mathfrak{u}(r)$ by

$$\begin{aligned} \operatorname{tr}(\exp(\frac{\lambda}{\sqrt{-1}}X)) &= \operatorname{tr}(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!(\sqrt{-1})^k} X^k) \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!(\sqrt{-1})^k} \operatorname{tr}(X^k) \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} g_k(X). \end{aligned}$$

Then, it can be shown (see Theorem XII.2.5 in [6]) that $g_1, \dots, g_r$ are algebraically independent and generate the algebra of polynomial functions on $\mathfrak{u}(r)$ invariant by $\operatorname{ad}(U(r))$.

Let T be the subgroup of $U(r)$ consisting of diagonal elements. Then, its Lie algebra $\mathfrak{t}$ consists of the matrices of the form

$$[x_1, \dots, x_r] \quad (x_i \text{ real}).$$

Note that when restricted to $\mathfrak{t}$, the polynomial functions have the following special form:

$$g_k([x_1, \dots, x_r]) = \sum_{j=1}^{r} x_j^k$$

for all k.

By Lemma 3.3, the holonomy group $H_{u_0}(A)$ is a compact, connected, semisimple Lie subgroup of $SU(r)$. Thus, it remains to prove that the rank of the holonomy group $H_{u_0}(A)$ is less than or equal to $\left[\frac{r}{2}\right]$. To do this, let $\mathfrak{l}$ be the Lie algebra of the holonomy group $H_{u_0}(A)$. Then, if we define an element a in $SU(r)$ by $\varphi(u_0) = u_0 a$ as in Theorem 2.1, we have

$$\sigma(X) = a^{-1}Xa, \quad X \in \mathfrak{l}. \tag{3.1}$$

Thus, when k is an odd integer ($1 \le k \le r$) it follows from (3.1)

$$\begin{aligned} g_k(X) &= g_k(a^{-1}Xa) = g_k(\sigma(X)) \\ &= g_k(\bar{X}) = g_k(-X^T) = g_k(-X) \\ &= (-1)^k g_k(X) = -g_k(X), \quad X \in \mathfrak{l}. \end{aligned}$$

Hence, we obtain

$$g_k(X) = 0 \tag{3.2}$$

for all $X \in \mathfrak{l}$ and all odd integers k ($1 \le k \le r$).

Set

$$Z_k = \{[x_1, \dots, x_r] \mid g_k([x_1, \dots, x_r]) = 0\}$$

for all integers k ($1 \le k \le r$). We will first estimate the dimension of the complete intersection

$$Z = \bigcap_{\substack{k=\text{odd} \\ 1 \le k \le r}} Z_k$$

of the algebraic varieties Z_k. To do this, assuming that $r+1$ is even, it suffices to compute generically the maximal rank of the following $\left(\frac{r+1}{2}\right) \times r$-matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_r} \\ \frac{\partial g_3}{\partial x_1} & \dots & \frac{\partial g_3}{\partial x_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_r}{\partial x_1} & \dots & \frac{\partial g_r}{\partial x_r} \end{pmatrix} = \begin{pmatrix} 1 & \dots & 1 \\ 3x_1^2 & \dots & 3x_r^2 \\ \vdots & \ddots & \vdots \\ rx_1^r & \dots & rx_r^r \end{pmatrix}. \tag{3.3}$$

It is easy to see from the Vandermonde formula that the determinant of the $\left(\frac{r+1}{2}\right) \times \left(\frac{r+1}{2}\right)$-submatrix

$$\begin{pmatrix} 1 & \dots & 1 \\ 3x_1^2 & \dots & 3x_{\frac{r+1}{2}}^2 \\ \vdots & \ddots & \vdots \\ rx_1^r & \dots & rx_{\frac{r+1}{2}}^r \end{pmatrix} \tag{3.4}$$

of the matrix (3.3) is

$$3 \cdot 5 \cdots r \cdot (-1)^{\frac{(r-1)(r+1)}{8}} \prod_{\substack{i<j \\ 1 \le i,j \le \frac{r+1}{2}}} (x_i^2 - x_j^2),$$

which is not zero generically, i.e., the matrix (3.4) has rank $\left(\frac{r+1}{2}\right)$ generically. Thus, the dimension of Z is less than or equal to

$$r - \left(\frac{r+1}{2}\right) = \frac{r-1}{2} = \left[\frac{r}{2}\right].$$

The case that r is odd can be shown similarly.

On the other hand, since Z always contains a vector space of the form

$$\{[x_1, -x_1, \dots, x_{\left[\frac{r}{2}\right]}, -x_{\left[\frac{r}{2}\right]}] \mid x_i \text{ real}\} \quad (r \text{ even}),$$
$$\text{or } \{[x_1, -x_1, \dots, x_{\left[\frac{r}{2}\right]}, -x_{\left[\frac{r}{2}\right]}, 0] \mid x_i \text{ real}\} \quad (r \text{ odd}),$$

whose dimension is $\left[\frac{r}{2}\right]$, we see that the dimension of Z is actually equal to $\left[\frac{r}{2}\right]$.

Now, let $\tilde{\mathfrak{t}}$ be a maximal torus of the Lie algebra $\mathfrak{l}$. Since the maximal torus $\tilde{\mathfrak{t}}$ of $\mathfrak{l}$ can be assumed to be contained in a maximal torus of $\mathfrak{su}(r)$ of the form

$$\{[x_1, \ldots, x_r] \,|\, x_1 + x_2 + \ldots + x_r = 0\},$$

the above argument together with (3.2) shows immediately that $\tilde{\mathfrak{t}}$ is contained as a vector space in the complete intersection Z, which implies that the rank of the Lie algebra $\mathfrak{l}$ is less than or equal to $\left[\frac{r}{2}\right]$. This completes the proof. □

Using Theorem 3.1 and the classification of compact, connected, simple Lie groups of E. Cartan, we get the following

Theorem 3.5. *Let P be a principal $SU(r)$-bundle over a simply connected manifold M that is reducible to an $SO(r)$-subbundle Q. Let A be an irreducible connection in P. Assume that the σ-conjugate connection A^σ of A is gauge equivalent to A and that r is a positive odd prime integer. If the holonomy group $H_{u_0}(A)$ is simple and is not isomorphic to E_8, F_4, or G_2, then $H_{u_0}(A)$ is isomorphic to $SO(r)$.*

Remark 3.6. The theorem extends the result for $r = 3$ in [3] to all positive odd prime integers. It is not clear to the author whether or not there is an irreducible connection which is gauge equivalent to its conjugate connection and whose holonomy group is isomorphic to E_8, F_4, or G_2.

Proof. Since r is odd prime, the center of the holonomy group is isomorphic either to the center of $SU(r)$, $\mathbb{Z}_r$ or to $\{0\}$. From the classification of compact, simple Lie groups by E. Cartan, we have only Lie groups of type $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4$ and G_2. It is also well known (see Theorem II.4.10 and V.6.38 in [9]) that the center of Lie groups $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4$, and G_2 is isomorphic to

$$\mathbb{Z}_l, 0, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, 0, 0, \text{ and } 0,$$

respectively. Since there is no compact simple Lie group whose center is $\mathbb{Z}_r$ and whose rank is less than or equal to $\frac{r-1}{2}$, the holonomy group must be isomorphic to B_k for k $(1 \leq k \leq r)$, unless $H_{u_0}(A)$ is isomorphic to E_8, F_4,or G_2. However, if $H_{u_0}(A)$ is isomorphic to B_k for $1 \leq k \leq r-1$, it is easy to see that the centralizer of $H_{u_0}(A)$ is strictly larger than $\mathbb{Z}_r$. Thus, the holonomy group $H_{u_0}(A)$ must be isomorphic to $B_{\left[\frac{r}{2}\right]}$, unless $H_{u_0}(A)$ is isomorphic to E_8, F_4, or G_2. This completes the proof. □

Now, a few concluding remarks are in order. Our ultimate goal of this program is to use the above results to deduce information about the geometry of 4-manifolds or the set of stable holomorphic structures, and so on, as we did in [3]. To do this we first need to extend above results to the semisimple case, and we hope that we can deal with these problems in the future.

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