PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 755–765
S 0002-9939(99)05485-4
Article electronically published on September 9, 1999

RESONANCE PROBLEMS FOR THE ONE-DIMENSIONAL p-LAPLACIAN

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(Communicated by Hal L. Smith)

ABSTRACT. We consider resonance problems for the one dimensional p-Laplacian, and prove the existence of solutions assuming a standard Landesman-Lazer condition. Our proofs use variational techniques to characterize the eigenvalues, and then to establish the solvability of the given boundary value problem.

1. INTRODUCTION

Consider the boundary value problem

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda_n |u|^{p-2}u - f(u) + h = 0 \text{ in } (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that the limits $\lim\limits_{t\to\pm\infty} f(t) = f(\pm\infty)$ exist, $p > 1$, $h \in L^q(0,1)$ such that $p + q = pq$, and λ_n is an eigenvalue of the associated homogeneous problem

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda |u|^{p-2}u = 0 \text{ in } (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{2}$$

It is known that the eigenvalues of (2) are simple, positive, and form an unbounded increasing sequence, $\{\lambda_n\}$, whose eigenspaces are spanned by functions $\{\phi_n(x)\} \subset W_0^{1,p}(0,1) \cap C^1[0,1]$ such that ϕ_n has $n-1$ evenly spaced zeros in $(0,1)$, $||\phi_n||_{L^p} = 1$, and $\phi_n'(0) > 0$. See [6], pages 174-183, for further details and references.

We will show that (1) is solvable if either

$$f(+\infty)\int_0^1 \phi_n^+ + f(-\infty)\int_0^1 \phi_n^- > \int_0^1 \phi_n h > f(+\infty)\int_0^1 \phi_n^- + f(-\infty)\int_0^1 \phi_n^+ \tag{3}$$

Received by the editors April 21, 1998.
2000 *Mathematics Subject Classification.* Primary 34B15.
The first author's research was sponsored by the Grant Agency of the Czech Republic, Project no. 201/97/0395, and partly by the Ministry of Education of the Czech Republic, Project no. VS97156.

or

(4)

$$f(+\infty)\int_0^1 \phi_n^+ + f(-\infty)\int_0^1 \phi_n^- < \int_0^1 \phi_n h < f(+\infty)\int_0^1 \phi_n^- + f(-\infty)\int_0^1 \phi_n^+ ,$$

where $\phi_n^+ := \max\{0, \phi_n\}$ and $\phi_n^- := \min\{0, \phi_n\}$. Conditions (3) and (4) are known as Landesman-Lazer conditions after the pioneering work in [8] where the case $p = 2$ was studied. These conditions can be thought of as an adaptation of the orthogonality conditions in the Fredholm Alternative for compact self adjoint linear operators. As in [8], an illustrative example to consider is when $f(u) = \tan^{-1}(u)$, h is a constant, and $n = 1$. It is straightforward to check that (3) is satisfied if $\frac{-\pi}{2} < h < \frac{\pi}{2}$, a simple explicit criterion for solvability. Many authors have contributed to the generalization of Landesman and Lazer's work, but the focus has remained primarily upon the case $p = 2$. Of course, this is because when $p = 2$ the differential operator is linear and self adjoint with compact inverse. For further discussion, examples, and references, see [9]. The case $p \neq 2$ is not yet well understood, but is the subject of much current research. For example, an existence theorem for resonance problems associated with the principal eigenvalue, λ_1, has recently been proved in [2]. In this paper we prove an existence result allowing any $p > 1$ and allowing resonance at an arbitrary eigenvalue. It is interesting to note that not all theorems for the case $p = 2$ will carry over to the case $p \neq 2$. In fact it has been proved that in some respects these problems are quite different. (See [3], [4] and [5].)

Our proof relies on a saddle point theorem for *linked* sets, which in turn relies on a variational characterization of $\{\lambda_n\}$. The variational characterization of the spectrum and the geometry of the linked sets are of some independent interest, since they might have analogs in the PDE case. An important difficulty to overcome in the analogous PDE case is the lack of information about the spectrum. The properties of the principal eigenvalue are well established and the second eigenvalue has recently been characterized in [1], but many questions remain regarding the spectrum beyond the second eigenvalue.

2. Preliminaries

Let $\mathcal{X} = W_0^{1,p}(0,1)$ with norm $||u|| = \left(\int_0^1 |u'|^p\right)^{\frac{1}{p}}$, and let $\mathcal{X}^*$ be the dual space with the usual norm, $||\cdot||_*$, and duality pairing, $\langle \cdot, \cdot \rangle$, on $\mathcal{X}^* \times \mathcal{X}$.

In the following sections we will study several combinations of the functionals

$$A(u) := \frac{1}{p}\int_0^1 |u'|^p,$$

$$B(u) := \frac{1}{p}\int_0^1 |u|^p, \text{ and}$$

$$C(u) := \int_0^1 (F(u) - hu),$$

where $F(t) := \int_0^t f(s)ds$. These functionals are continuously differentiable on $\mathcal{X}$ with

$$\langle A'(u), v\rangle = \int_0^1 |u'|^{p-2}u'v',$$

$$\langle B'(u), v\rangle = \int_0^1 |u|^{p-2}uv, \text{ and}$$

$$\langle C'(u), v\rangle = \int_0^1 (f(u) - h)v, \ \forall v \in \mathcal{X}.$$

Lemma 1. *The operators $A', B', C' : \mathcal{X} \to \mathcal{X}^*$ have the following properties:*

(A') *A' is $(p-1)$-homogeneous, odd, continuously invertible, and $||A'(u)||_* = ||u||^{p-1}$ for any $u \in \mathcal{X}$.*

(B') *B' is $(p-1)$-homogeneous, odd and compact.*

(C') *C' is bounded and compact.*

Proof. See [6], Lemma 10.3, page 120.

It is not hard to see that eigenfunctions and eigenvalues of (2) are equivalent to critical points and critical values of the functional

$$E(u) := \frac{A(u)}{B(u)},$$

and that solutions of (1) are critical points of the functional

$$J(u) := A(u) - \lambda_n B(u) - C(u).$$

Studying E and J, respectively, will be the subject of the following two sections. For convenience we adopt the notation $\mathcal{E}_c := \{u \in \mathcal{X} : A(u) \geq cB(u)\} = \{u \in \mathcal{X} \setminus \{0\} : E(u) \geq c\} \cup \{0\}$, a super-level set, and $\mathcal{K}_c := \{u \in \mathcal{X} \setminus \{0\} : E(u) = c, E'(u) = 0\}$.

3. A variational characterization of λ_n

In this section we study the functional E and its critical values $\{\lambda_n\}$. By homogeneity it suffices to study $E|_{\mathcal{S}}$, where $\mathcal{S} := \{u \in \mathcal{X} : B(u) = 1\}$, an L^p sphere. Clearly, $E(u) = A(u)$ for $u \in \mathcal{S}$, and a direct computation gives $E'(u) = A'(u) - A(u)B'(u)$ for $u \in \mathcal{S}$. Moreover, it is straightforward to check that $\mathcal{K}_{\lambda_n} \cap \mathcal{S} = \{\pm p^{\frac{1}{p}}\phi_n\}$. We will exploit both the compactness and the symmetry of E.

Lemma 2. *$E|_{\mathcal{S}}$ satisfies the Palais-Smale condition, i.e., if $\{u_k\} \subset \mathcal{S}$ is a sequence with the properties*

(i) *$\exists\, c > 0$ such that $|E(u_k)| \leq c\ \forall k \in \mathbb{N}$, and*

(ii) *$E'(u_k) \to 0$ in $\mathcal{X}^*$,*

then $\{u_k\}$ contains a convergent subsequence.

Proof. Since $E(u_k) = A(u_k) = \dfrac{1}{p}||u_k||^p$, it is clear from (i) that $\{u_k\}$ is bounded in $\mathcal{X}$, so we may assume, without loss of generality, that $u_k \rightharpoonup u_0$ in $\mathcal{X}$, and that $A(u_k) \to A_0 \in \mathbb{R}$. By compactness $B'(u_k) \to B'(u_0)$ in $\mathcal{X}^*$. By (ii) we have that

$A'(u_k) - A(u_k)B'(u_k) \to 0$ in $\mathcal{X}^*$, and thus $A'(u_k) \to A_0 B'(u_0)$. Applying Lemma 1, $u_k \to u_0 = (A')^{-1}(A_0 B'(u_0))$ in $\mathcal{X}$. The proof is complete.

Since E is an even functional, we can apply a standard variational result to obtain a nondecreasing sequence of critical values with a minimax characterization. This requires the following definition.

Definition 1. Let $\mathcal{F} := \{\mathcal{A} \subset \mathcal{X} : \mathcal{A} \text{ closed}, \mathcal{A} = -\mathcal{A}\}$. Given nontrivial $\mathcal{A} \in \mathcal{F}$ we define the Krasnoselskii genus of $\mathcal{A}$ as follows. Let $\mathcal{M} := \{m \in \mathbb{N} : \exists h \in C^0(\mathcal{A}; \mathbb{R}^m \setminus \{0\}) \text{ such that } h(-u) = -h(u)\}$. Then

$$\gamma(\mathcal{A}) := \begin{cases} \inf \mathcal{M}, & \text{if } \mathcal{M} \neq \emptyset, \\ \infty, & \text{if } \mathcal{M} = \emptyset. \end{cases}$$

Intuitively, γ provides a measure of the dimension of a symmetric set. For example, if Ω is a bounded symmetric neighborhood of the origin in $\mathbb{R}^m$, then $\gamma(\partial\Omega) = m$. (See [10], Proposition 5.2, page 87.)

A nondecreasing sequence of critical values for E is characterized by

$$\beta_k := \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} E(u),$$

where $\mathcal{F}_k := \{\mathcal{A} \in \mathcal{F} : 0 \notin \mathcal{A}, \gamma(\mathcal{A}) \geq k\}$. (See [10], page 89.) The next lemma sharpens the given characterization and verifies that $\lambda_k = \beta_k \, \forall k$, i.e. all of the eigenvalues have the given variational characterization.

Lemma 3. *For any $k \in \mathbb{N}$ we have $\lambda_k = \beta_k = \min_{\mathcal{A} \in \tilde{\mathcal{F}}_k} \max_{u \in \mathcal{A}} E(u)$, where $\tilde{\mathcal{F}}_k := \{\mathcal{A} \in \mathcal{F}_k : \mathcal{A} \subset \mathcal{S}$ and $\mathcal{A}$ is compact $\}$.*

Proof. Let k be fixed. Clearly $\beta_k = \lambda_n$ for some n. Thus $\mathcal{K}_{\beta_k} \cap \mathcal{S} = \mathcal{K}_{\lambda_n} \cap \mathcal{S} = \{\pm p^{\frac{1}{p}} \phi_n\}$, and so $\gamma(\mathcal{K}_{\beta_k} \cap \mathcal{S}) = 1$. Moreover, it is known that if $\beta_j = \beta_{j+1} = \cdots = \beta_{j+m}$, then $\gamma(\mathcal{K}_{\beta_j} \cap \mathcal{S}) \geq m + 1$. (See [10], Lemma 5.6, page 89.) Thus $m = 0$ and $\{\beta_n\}$ must be an increasing sequence. It follows that $\beta_k \geq \lambda_k$.

Now consider the functions $\phi_{k,i} = \chi_{[\frac{i-1}{k}, \frac{i}{k}]} \phi_k$ for $i = 1, ..., k$, where $\chi_{[\frac{i-1}{k}, \frac{i}{k}]}$ is a characteristic function, and let

$$\Lambda_k := \{\alpha_1 \phi_{k,1} + \cdots + \alpha_k \phi_{k,k} : \alpha_i \in \mathbb{R} \text{ and } |\alpha_1|^p B(\phi_{k,1}) + \cdots + |\alpha_k|^p B(\phi_{k,k}) = 1\}.$$

Each $\phi_{k,i}$ is in $\mathcal{X}$ and is a principal eigenfunction, with eigenvalue λ_k, for the differential equation restricted to the appropriate subinterval. Observe that Λ_k is symmetric and is homeomorphic to the unit sphere in $\mathbb{R}^k$. Thus Λ_k is compact with $\gamma(\Lambda_k) = k$. Moreover, observe that for $u \in \Lambda_k$

$$\begin{aligned} B(u) &= B(\alpha_1 \phi_{k,1} + \cdots + \alpha_k \phi_{k,k}) \\ &= B(\alpha_1 \phi_{k,1}) + \cdots + B(\alpha_k \phi_{k,k}), \\ &\qquad \text{since } \{x : \phi_{k,i}(x) \neq 0\} \cap \{x : \phi_{k,j}(x) \neq 0\} = \emptyset \text{ for } i \neq j \\ &= |\alpha_1|^p B(\phi_{k,1}) + |\alpha_k|^p B(\phi_{k,k}), \text{ by the } p\text{-homogeneity of } B \\ &= 1. \end{aligned}$$

Thus $\Lambda_k \subset S$, and so $\Lambda_k \in \tilde{\mathcal{F}}_k$. A similar computation shows that $E(u) = A(u) = \lambda_k$ for all $u \in \Lambda_k$. This implies that

$$\beta_k = \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} E(u) \leq \lambda_k,$$

so $\lambda_k = \beta_k$. Moreover, the infsup has been achieved on a set in $\tilde{\mathcal{F}}_k$. Using the compactness of sets in $\tilde{\mathcal{F}}_k$ we replace sup by max, and since the inf is achieved we replace inf by min. The lemma is proved.

4. Existence of saddle point solutions

Now we prove the existence of at least one weak solution for the boundary value problem (1) assuming either (3) or (4). This is equivalent to proving the existence of critical points for the functional J. We will apply the following definition and saddle point theorem.

Definition 2. Let $\mathcal{E}$ be a closed subset of $\mathcal{X}$ and let $\mathcal{Q}$ be a submanifold of $\mathcal{X}$ with relative boundary $\partial\mathcal{Q}$. We say that $\mathcal{E}$ and $\partial\mathcal{Q}$ link if

(i) $\mathcal{E} \cap \partial\mathcal{Q} = \emptyset$, and
(ii) for any continuous map $h : \mathcal{X} \to \mathcal{X}$ such that $h|_{\partial\mathcal{Q}} = id$, there holds $h(\mathcal{Q}) \cap \mathcal{E} \neq \emptyset$.

(See [10], Definition 8.1, page 116.)

We note that the choice of notation, $\mathcal{E}$, is deliberate, since we will soon choose this set to be the super-level set $\mathcal{E}_{\lambda_{n+1}}$. We also note that, throughout the discussion, ∂ refers to the relative boundary.

Theorem 1. *Suppose $J \in C^1(\mathcal{X})$ satisfies the Palais-Smale condition. Consider a closed subset $\mathcal{E} \subset \mathcal{X}$ and a submanifold $\mathcal{Q} \subset \mathcal{X}$ with relative boundary $\partial\mathcal{Q}$, and let $\Gamma := \{h \in C^0(\mathcal{X}, \mathcal{X}) : h|_{\partial\mathcal{Q}} = id\}$. Suppose that*

(i) *$\mathcal{E}$ and $\partial\mathcal{Q}$ link, and*
(ii) $\inf\limits_{\mathcal{E}} J(u) > \sup\limits_{\partial\mathcal{Q}} J(u)$.

Then $\beta = \inf\limits_{h\in\Gamma} \sup\limits_{\mathcal{Q}} J(h(u))$ is a critical value. (See [10]*, Theorem 8.4, page 118.)*

The purpose of the following sequence of lemmas is to show that the hypotheses of Theorem 1 are satisfied provided that either (3) or (4) holds.

Lemma 4. *If either* (3) *or* (4) *is satisfied, then J satisfies the Palais-Smale condition.*

Proof. Suppose $\{u_k\} \subset \mathcal{X}$ such that $|J(u_k)| \leq c$ and $J'(u_k) \to 0$ in $\mathcal{X}^*$. We must show that $\{u_k\}$ has a subsequence that converges in $\mathcal{X}$. It is a helpful first step to show that $\{u_k\}$ is bounded.

Suppose that $||u_k|| \to \infty$ and consider $v_k := \dfrac{u_k}{||u_k||}$. Then $\{v_k\}$ is bounded and, without loss of generality, is weakly convergent to some v_0. We assume that

$$A'(u_k) - \lambda_n B'(u_k) - C'(u_k) \to 0,$$

so, dividing through by $||u_k||^{p-1}$, we have

$$A'(v_k) - \lambda_n B'(v_k) - \frac{C'(u_k)}{||u_k||^{p-1}} \to 0.$$

By the boundedness of C' we know that $\dfrac{C'(u_k)}{||u_k||^{p-1}} \to 0$, and by the compactness of B' we know that $B'(v_k) \to B'(v_0)$. Thus $v_k \to v_0 = (A')^{-1}(\lambda_n(B'(v_0)))$ in $\mathcal{X}$. It follows that $v_0 = \pm\phi_n$. We assume that $v_0 = \phi_n$ and remark that a similar argument follows if $v_0 = -\phi_n$.

Now we add the inequalities

$$-cp \leq pJ(u_k) \leq cp$$

and

$$-||J'(u_k)||_*||u_k|| \le -\langle J'(u_k), u_k\rangle \le ||J'(u_k)||_*||u_k||$$

to get

$$\begin{aligned} -cp - ||J'(u_k)||_*||u_k|| &\le -p\int_0^1 F(u_k) + \int_0^1 f(u_k)u_k + (p-1)\int_0^1 hu_k \\ &\le cp + ||J'(u_k)||_*||u_k||. \end{aligned}$$

Dividing by $||u_k||$ and writing $\dfrac{F(u_k)}{||u_k||} = g(u_k)v_k$, where

$$g(s) := \begin{cases} \dfrac{F(s)}{s} \text{ for } s \neq 0, \\ 0 \text{ for } s = 0 \end{cases}$$

we get

$$\left|\int_0^1 (f(u_k) - pg(u_k))v_k + (p-1)\int_0^1 hv_k\right| \le \frac{cp}{||u_k||} + ||J'(u_k)||_*.$$

The right hand side of the given inequality approaches 0 and $\displaystyle\int_0^1 hv_k \to \int_0^1 h\phi_n$, so

$$\lim_{k\to\infty}\int_0^1 (f(u_k) - pg(u_k))v_k = (1-p)\int_0^1 h\phi_n.$$

Recall that $\mathcal{X}$ embeds compactly in $C[0,1]$, so without loss of generality, $v_k = \dfrac{u_k}{||u_k||} \to \phi_n$ uniformly, and so $u_k(x) \to \infty$ on $\{x : \phi_n(x) > 0\}$ and $u_k(x) \to -\infty$ on $\{x : \phi_n(x) < 0\}$. But $u_k(x) \to \pm\infty$ implies $f(u_k(x)) \to f(\pm\infty)$ as well as $g(u_k(x)) \to f(\pm\infty)$, by an application of l'Hospital's rule to $\dfrac{F(s)}{s}$. Thus, by the Dominated Convergence Theorem,

$$\lim_{k\to\infty}\int_0^1 (f(u_k) - pg(u_k))v_k = (1-p)\left[f(+\infty)\int_0^1 \phi_n^+ + f(-\infty)\int_0^1 \phi_n^-\right],$$

and so

$$f(+\infty)\int_0^1 \phi_n^+ + f(-\infty)\int_0^1 \phi_n^- = \int_0^1 h\phi_n,$$

which contradicts (3) or (4). Hence $\{u_k\}$ is bounded.

By compactness there is a subsequence such that $B'(u_k)$ and $C'(u_k)$ converge in $\mathcal{X}^*$. Since $J'(u_k) \to 0$ we also have that $A'(u_k)$ converges in $\mathcal{X}^*$. Finally, $u_k = (A')^{-1}(A'(u_k))$ converges in $\mathcal{X}$. The proof is complete.

Now that the Palais-Smale condition is established we turn our attention to the linking properties required in the saddle point theorem. In particular the next lemmas prove condition (ii) of Definition 2 for appropriate sets. It will be helpful in the following arguments to allow the more general assumption that $h|_{\partial\mathcal{Q}}$ is odd.

Let $\mathcal{Q}_{n,T} := \{tu : 0 \le t \le T, u \in \Lambda_n\}$ for $T > 0$, where Λ_n was defined in the proof of Lemma 3.

Lemma 5. *If $h : \mathcal{Q}_{n,T} \to \mathcal{X}$ is a continuous map such that $h|_{\partial\mathcal{Q}_{n,T}}$ is odd, then $h(\mathcal{Q}_{n,T}) \cap \mathcal{E}_{\lambda_{n+1}} \neq \emptyset$.*

Proof. Suppose not, so $h(\mathcal{Q}_{n,T}) \subset (\mathcal{E}_{\lambda_{n+1}})^c$. Since $0 \in \mathcal{E}_{\lambda_{n+1}}$ we have $(\mathcal{E}_{\lambda_{n+1}})^c \subset \mathcal{X} \setminus \{0\}$ and we can compose with the radial projection onto $\mathcal{S}$ to get, without loss of generality, $h(\mathcal{Q}_{n,T}) \subset \mathcal{S} \cap (\mathcal{E}_{\lambda_{n+1}})^c$. Since $E(h(u)) < \lambda_{n+1}$ for $u \in \mathcal{Q}_{n,T}$, a compact set, we may assume that there is an $\epsilon > 0$ such that $E(h(u)) \leq \lambda_{n+1} - \epsilon \; \forall u \in \mathcal{Q}_{n,T}$. Lemma 3 implies that $\gamma(\{u \in \mathcal{S} : E(u) \leq \lambda_{n+1} - \epsilon\}) \leq n$, so there is a continuous odd $\tilde{h} : \{u \in \mathcal{S} : E(u) \leq \lambda_{n+1} - \epsilon\} \to \mathbb{R}^n \setminus \{0\}$. Hence the composed map $\tilde{h} \circ h : \mathcal{Q}_{n,T} \to \mathbb{R}^n \setminus \{0\}$ is continuous such that $\tilde{h} \circ h(-x) = -\tilde{h} \circ h(x)$ for $x \in \partial \mathcal{Q}_{n,T}$. But $\mathcal{Q}_{n,T}$ is homeomorphic to the closed unit ball in $\mathbb{R}^n$, so the previous statement contradicts the Borsuk-Ulam Theorem. (See [7], page 21.) The proof is complete.

For technical purposes in upcoming proofs we use a pseudo-gradient flow to *lower* $\mathcal{Q}_{n,T}$ and to *raise* $\mathcal{E}_{\lambda_n}$.

Lemma 6. *Given $\epsilon < \min\{|\lambda_{n+1} - \lambda_n|, |\lambda_n - \lambda_{n-1}|\}$, there is an $\tilde{\epsilon} \in (0, \epsilon)$ and a one-parameter family of homeomorphisms $\eta : [-1, 1] \times \mathcal{S} \to \mathcal{S}$ such that*

(i) *$\eta(t, u) = u$ if $E(u) \in (-\infty, \lambda_n - \epsilon] \cup [\lambda_n + \epsilon, \infty)$ or if $u \in \mathcal{K}_{\lambda_n}$,*
(ii) *$E(\eta(t, u))$ is strictly decreasing in t if $E(u) \in (\lambda_n - \tilde{\epsilon}, \lambda_n + \tilde{\epsilon})$ and $u \notin \mathcal{K}_{\lambda_n}$,*
(iii) *$\eta(t, -u) = -\eta(t, u)$.*

Proof. We present a variation of a well-known scheme. Let $v(u)$ denote a locally Lipschitz continuous symmetric pseudo-gradient vector field associated with E on $\tilde{\mathcal{S}} := \{u \in \mathcal{S} : E'(u) \neq 0\}$. More specifically, let $T_u\mathcal{S}$ and $T\mathcal{S}$ denote the tangent space at a point $u \in \mathcal{S}$ and the tangent bundle associated with $\mathcal{S}$, respectively; then $v : \tilde{\mathcal{S}} \to T\mathcal{S}$ such that

$$\begin{aligned}
&v(u) \in T_u\mathcal{S}, \\
&||v(u)|| < 2\min\{||E'(u)||, 1\}, \\
&\langle E'(u), v(u)\rangle > \min\{||E'(u)||, 1\}||E'(u)||, \text{ and} \\
&v(-u) = -v(u).
\end{aligned}$$

The existence of such a vector field is well known. (See [10], pages 77-79, for more details regarding the existence of, and properties of, pseudogradient vector fields and their related flows.)

Now let $\psi : \mathcal{S} \to [0, 1]$ be a smooth function such that $\psi(u) = 1$ for u satisfying $\lambda_n - \tilde{\epsilon} \leq E(u) \leq \lambda_n + \tilde{\epsilon}$ and $\psi(u) = 0$ for u satisfying $\lambda_n - \epsilon \geq E(u)$ or $\lambda_n + \epsilon \leq E(u)$. Consider the modified vector field

$$\tilde{v}(u) := \begin{cases} \psi(u) dist(u, \mathcal{K}_{\lambda_n}) v(u) \text{ for } u \in \tilde{\mathcal{S}}, \\ 0 \text{ for } u \in \mathcal{S} \setminus \tilde{\mathcal{S}}. \end{cases}$$

This extends v to a symmetric locally Lipschitz continuous vector field on all of $\mathcal{S}$. Let η be the solution of the initial value problem

$$\frac{\partial}{\partial t}\eta(t, u) = -\tilde{v}(\eta(t, u)), \;\; \eta(0, \cdot) = id.$$

It is straightforward to verify statements (i) and (iii). Statement (ii) follows from the computation

$$\frac{d}{dt}E(\eta(t, u)) = -\psi(\eta(t, u)) dist(\eta(t, u), \mathcal{K}_{\lambda_n}) \langle E'(\eta(t, u)), v(\eta(t, u))\rangle < 0,$$

and the proof is complete.

Define $\tilde{\mathcal{E}}_{\lambda_n} = \{tu : t \in \mathbb{R}, u \in \eta(-1, \mathcal{E}_{\lambda_n} \cap \mathcal{S})\}$ and $\tilde{\mathcal{Q}}_{n,T} = \{tu : 0 \leq t \leq T, u \in \eta(1, \Lambda_n)\}$. By Lemma 6, $A(u) - \lambda_n B(u) \geq 0$ for $u \in \tilde{\mathcal{E}}_{\lambda_n}$, with equality iff

$u = c\phi_n$ for some $c \in \mathbb{R}$. Similarly, $A(u) - \lambda_n B(u) \leq 0$ for $u \in \tilde{\mathcal{Q}}_{n,T}$, with equality iff $u = c\phi_n$ for some $c \in \mathbb{R}$. Using Lemma 5 and the fact that $\eta(t, \cdot)$ is an odd homeomorphism we can prove

Lemma 7. *If $h : \tilde{\mathcal{Q}}_{n,T} \to \mathcal{X}$ is continuous such that $h|_{\partial\tilde{\mathcal{Q}}_{n,T}}$ is odd, then $h(\tilde{\mathcal{Q}}_{n,T}) \cap \mathcal{E}_{\lambda_{n+1}} \neq \emptyset$.*

Proof. Suppose $h : \tilde{\mathcal{Q}}_{n,T} \to \mathcal{X}$ is continuous such that $h|_{\partial\tilde{\mathcal{Q}}_{n,T}}$ is odd with $h(\tilde{\mathcal{Q}}_{n,T}) \cap \mathcal{E}_{\lambda_{n+1}} = \emptyset$. Then the function $\tilde{h} : \mathcal{Q}_{n,T} \to \mathcal{X} : \tilde{h}(tu) := h(t\eta(1, u))$ for $u \in \Lambda_n$ and $0 \leq t \leq T$ is a function that contradicts Lemma 5. The proof is complete.

Lemma 8. *If $h : \mathcal{Q}_{n-1,T} \to \mathcal{X}$ is continuous such that $h|_{\partial\mathcal{Q}_{n-1,T}}$ is odd, then $h(\mathcal{Q}_{n-1,T}) \cap \tilde{\mathcal{E}}_{\lambda_n} \neq \emptyset$.*

Proof. Suppose not. Then, as in the proof of Lemma 5, we may assume that $h(\mathcal{Q}_{n-1,T}) \subset (\tilde{\mathcal{E}}_{\lambda_n})^c \cap \mathcal{S}$. Then $\tilde{h} : \mathcal{Q}_{n-1,T} \to \mathcal{X} : \tilde{h}(u) = \eta(1, h(u))$ is a continuous function which is odd on its boundary and with image in $(\mathcal{E}_{\lambda_n})^c \cap \mathcal{S}$, a contradiction of Lemma 5. The proof is complete.

The next two lemmas assume condition (3) and prove hypothesis (ii) of Theorem 1 with $\mathcal{E} = \mathcal{E}_{\lambda_{n+1}}$ and $\mathcal{Q} = \tilde{\mathcal{Q}}_{n,T}$. It follows from (ii) that $\partial\mathcal{Q} \cap \mathcal{E} = \emptyset$. This fact and Lemma 7 imply that $\mathcal{E}$ and $\partial\mathcal{Q}$ link. Hence the existence proof assuming (3) will be finished.

Lemma 9. *If (3) is satisfied, then $\exists R > 0$ and $\delta > 0$ such that $\langle J'(tu), u\rangle \leq -\delta\ \forall t, u$ with $t \geq R$ and $u \in \eta(1, \Lambda_n)$.*

Proof. Suppose $t_k \to \infty$ and $u_k \in \eta(1, \Lambda_n)$ such that

$$\limsup_{k\to\infty} \langle J'(t_k u_k), u_k\rangle \geq 0.$$

Since $\eta(1, \Lambda_n)$ is compact, we may assume that $u_k \to u_0$ in $\eta(1, \Lambda_n)$.

If $u_0 \neq \pm p^{\frac{1}{p}}\phi_n$, then

$$\int_0^1 |u_0'|^p - \lambda_n \int_0^1 |u_0|^p \leq -\epsilon$$

for some $\epsilon > 0$. (Note that this is the stage in the proof where it is technically important to use $\eta(1, \Lambda_n)$ rather than Λ_n.) Thus

$$\int_0^1 |u_k'|^p - \lambda_n \int_0^1 |u_k|^p \leq -\frac{\epsilon}{2}$$

for k large enough. Hence

$$\langle J'(t_k u_k), u_k\rangle \leq -\frac{\epsilon}{2} t_k^{p-1} - \int_0^1 (f(t_k u_k) - h)u_k, \text{ for large } k,$$

which leads to a contradiction of the limsup assumption.

Suppose $u_0 = p^{\frac{1}{p}}\phi_n$. We still have

$$\int_0^1 |u_k'|^p - \lambda_n \int_0^1 |u_k|^p \leq 0,$$

so

$$\langle J'(t_k u_k), u_k\rangle \leq -\int_0^1 (f(t_k u_k) - h)u_k\ \forall k.$$

The fact that f is bounded and that $u_k \to p^{\frac{1}{p}}\phi_n$ allows us to apply the Dominated Convergence Theorem to get

$$\lim_{k\to\infty}\langle J'(t_k u_k), u_k\rangle \leq -p^{\frac{1}{p}}\left(f(+\infty)\int_0^1 \phi_n^+ + f(-\infty)\int_0^1 \phi_n^- - \int_0^1 h\phi_n\right) < 0,$$

by (3). Once again a contradiction is reached. The case $u_0 = -p^{\frac{1}{p}}\phi_n$ is similar, so the proof is complete.

Lemma 10. *If* (3) *is satisfied, then* $\exists T > 0$ *such that* $\inf\limits_{\mathcal{E}_{\lambda_{n+1}}} J(u) > \sup\limits_{\partial\tilde{\mathcal{Q}}_{n,T}} J(u)$.

Proof. For $u \in \mathcal{E}_{\lambda_{n+1}}$ notice that

$$J(u) \geq \frac{1}{p}(\lambda_{n+1} - \lambda_n)||u||^p_{L^p} - \int_0^1 (F(u) - hu),$$

which is clearly bounded below by some value α. By Lemma 9 we have

$$\begin{aligned} J(tu) &= J(Ru) + J(tu) - J(Ru) \\ &= J(Ru) + \int_R^t \langle J'(su), u\rangle\, ds \\ &\leq c - \delta(t-R), \end{aligned}$$

for all $u \in \eta(1,\Lambda_n)$, all $t > R$, and some $c \in \mathbb{R}$. Thus there is a $T > R$ such that $J(tu) \leq c - \delta(T-R) < \alpha$ for all $t \geq T$ and $u \in \eta(1,\Lambda_n)$. The proof is complete.

Thus we have proven the following.

Theorem 2. *If* (3) *is satisfied, then* (1) *has at least one solution.*

If we assume condition (4), then similar arguments will show that, for some $T > 0$, $\mathcal{E} := \tilde{\mathcal{E}}_{\lambda_n}$ and $\mathcal{Q} := \mathcal{Q}_{n-1,T}$ satisfy the hypotheses of Theorem 1. The main difficulty in this case is to prove an estimate similar to the one in Lemma 9 for $u \in \eta(-1, \mathcal{E}_{\lambda_n} \cap \mathcal{S})$. This set does not enjoy the same compactness as $\eta(1,\Lambda_n)$, so the argument is more delicate. In the following proof the symbols ϵ and $\tilde{\epsilon}$ are the numbers associated with the flow η described in Lemma 6.

Lemma 11. *If* (4) *is satisfied, then* $\exists R > 0$ *and* $\delta > 0$ *such that* $\langle J'(tu), u\rangle \geq \delta$ $\forall t, u$ *with* $t \geq R$ *and* $u \in \eta(-1, \mathcal{E}_{\lambda_n} \cap \mathcal{S})$.

Proof. We begin by showing that given any $\epsilon' > 0\,\exists\,\delta > 0$ such that $E(u) \geq \lambda_n + \delta$ for $u \in \eta(-1, \mathcal{E}_{\lambda_n} \cap \mathcal{S}) \setminus B_{\epsilon'}(\pm\phi_n)$. Let $u \in \eta(-1, \mathcal{E}_{\lambda_n} \cap \mathcal{S}) \setminus B_{\epsilon'}(\pm\phi_n)$ and assume, without loss of generality, that $E(u) \leq \lambda_n + \tilde{\epsilon}$. Let $u_0 \in \mathcal{E}_{\lambda_n} \cap \mathcal{S}$ such that $\eta(-1, u_0) = u$. Observe that

$$||\frac{d}{dt}\eta(t,u)|| \leq ||\tilde{v}(\eta(t,u_0))|| \leq dist(\eta(t,u_0), \mathcal{K}_{\lambda_n})||v(\eta(t,u_0))|| < M$$

for some constant $M > 0$. It follows that $\eta(t,u_0) \notin B_{\frac{\epsilon'}{2}}(\pm\phi_n)\ \forall t \in [-1, -1 + \frac{\epsilon'}{2M}]$. Moreover, By Lemma 2, we can show that $\exists\rho > 0$ such that $||E'(u)||_* \geq \rho$ for all

$u \in \{u \in \mathcal{S} : \lambda_n \le E(u) \le \lambda_n + \tilde{\epsilon}\} \setminus B_{\frac{\epsilon'}{2}}(\pm\phi_n)$, so

$$\begin{aligned} ||\frac{d}{dt}E(\eta(t,u_0))|| \quad &= ||\psi(\eta(t,u_0))dist(\eta(t,u_0),\mathcal{K}_{\lambda_n})\,\langle E'(\eta(t,u_0)),v(\eta(t,u_0))\rangle\,|| \\ &\ge 1\cdot\tfrac{\epsilon'}{2}\cdot\min\{||E'(\eta(t,u_0))||,1\}||E'(\eta(t,u_0))|| \\ &\ge \tfrac{\epsilon'}{2}\rho^2, \forall t\in[-1,-1+\tfrac{\epsilon'}{2M}], \end{aligned}$$

where we have used the fact that $\psi(\eta(t,u_0)) \equiv 1$ for $-1 \le t \le 1$. Hence, $E(u) = E(\eta(-1,u_0)) \ge E(\eta(-1+\frac{\epsilon'}{2M},u_0)) + \frac{\epsilon'}{2}\rho^2 \cdot \frac{\epsilon'}{2M} \ge \lambda_n + \delta$, with $\delta = \frac{(\epsilon'\rho)^2}{4M}$.

Now we proceed as in the proof of Lemma 9. Suppose that $\{u_k\} \subset \eta(-1, \mathcal{E}_{\lambda_n} \cap \mathcal{S})$ and $\{t_k\} \subset \mathbb{R}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\limsup_{k\to\infty} \langle J'(t_k u_k), u_k\rangle \le 0$. If there is an ϵ' such that $u_k \in \eta(-1, \mathcal{E}_{\lambda_n} \cap \mathcal{S}) \setminus B_{\epsilon'}(\pm\phi_n)\,\forall k$, then the paragraph above leads to the estimate

$$\langle J'(t_k)u_k, u_k\rangle \ge \delta t_k^{p-1} - \int_0^1 (f(t_k u_k) - h)u_k,$$

contradicting the lim sup assumption. Thus it must be that $u_k \to \pm p^{\frac{1}{p}}\phi_n$. Finally we apply condition (4), exactly as condition (3) is applied in Lemma 9, to reach a contradiction, and the lemma is proved.

Thus J is bounded below on $\tilde{\mathcal{E}}_{\lambda_n}$, and is in fact coercive on this set. Estimates for J restricted to $\partial\mathcal{Q}_{n-1,T}$ are easily obtained, as were the estimates for J restricted to $\tilde{\mathcal{E}}_{\lambda_{n+1}}$ in Lemma 10. Hence we can prove the following lemma.

Lemma 12. *If* (4) *is satisfied, then* $\exists\, T > 0$ *such that* $\inf_{\tilde{\mathcal{E}}_{\lambda_n}} J(u) > \sup_{\partial\mathcal{Q}_{n-1,T}} J(u)$.

Proof. Similar to Lemma 10.

As a consequence of Lemmas 8 and 12 we now know that, for some $T > 0$, $\tilde{\mathcal{E}}_{\lambda_n}$ and $\mathcal{Q}_{n-1,T}$ satisfy the hypotheses of Theorem 1. Thus we have proven the following.

Theorem 3. *If 4 is satisfied, then* (1) *has at least one solution.*

Acknowledgement

The authors would like to thank the referee for many helpful comments that clarified and corrected an earlier version of this paper.

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