ON $p$-HYPONORMAL OPERATORS

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Abstract. In this paper we show that $p$-hyponormal operators with $0 \notin \sigma(|T|^2)$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

1. Introduction

Let $H$ and $K$ be separable complex Hilbert spaces and let $\mathcal{L}(H,K)$ denote the space of all bounded linear operators from $H$ to $K$. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H,K)$.

An operator $T \in \mathcal{L}(H)$ is said to be $p$-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where $T^*$ is the adjoint of $T$. If $p = 1$, $T$ is called hyponormal and if $p = 1/2$, $T$ is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia (see [Xi]), and $p$-hyponormal operators for a general $p$, $0 < p < 1$, have been studied by Aluthge. Any $p$-hyponormal operators are $q$-hyponormal if $q \leq p$. But there are examples to show that the converse of the above statement is not true (see [Al]).

A bounded linear operator $S$ on $H$ is called scalar of order $m$ if it has a spectral distribution of order $m$, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C^m_0(\mathbb{C}) \rightarrow \mathcal{L}(H)$$

such that $\Phi(z) = S$, where as usual $z$ stands for the identity function on $\mathbb{C}$ and $C^m_0(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order $m$, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace. We now define the weaker form of a subscalar operator. An operator $T \in \mathcal{L}(H)$ is quasi-subscalar if there exists a one-to-one $V \in \mathcal{L}(H,K)$ such that $VT = SV$ where $S (= \Phi(z)$ in the above definition) is a scalar operator.

This paper has been divided into three sections. Section 2 deals with some preliminary facts. In section 3, we show that $p$-hyponormal operators with the property $0 \notin \sigma(|T|^2)$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

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2. Preliminaries

Let $du(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let $H$ be a complex separable Hilbert space, and let $D$ be a bounded open disc in $C$. We shall denote by $L^2(D, H)$ the Hilbert space of measurable functions $f : D \to H$, such that

$$\|f\|_{2,D} = \left(\int_D \|f(z)\|^2 d\mu(z)\right)^{1/2} < \infty.$$

The space of functions $f \in L^2(D, H)$ which are analytic functions in $D$ (i.e., $\partial f = 0$) is defined by

$$A^2(D, H) = L^2(D, H) \cap \mathcal{O}(D, H)$$

where $\mathcal{O}(D, H)$ denotes the Fréchet space of $H$-valued analytic functions on $D$ with respect to uniform topology. $A^2(D, H)$ is called the Bergman space for $D$. Note that $A^2(D, H)$ is a Hilbert space. The operator $T - z$ on the space $\mathcal{O}(D, H)$ has property $(\beta)$, which means by definition that $T - z$ is one-to-one and has closed range for every disc $D$.

Let us define now a Sobolev type space, called $W^2(D, H)$ where $D$ is a bounded disc in $C$. $W^2(D, H)$ will be the space of those functions $f \in L^2(D, H)$ whose derivatives $\partial f, \partial^2 f$ in the sense of distributions still belong to $L^2(D, H)$. Endowed with the norm $\|f\|_{W^2}^2 = \sum_{i=0}^2 \|\partial^i f\|_{2,D}^2$, $W^2(D, H)$ becomes a Hilbert space contained continuously in $L^2(D, H)$.

Now for $f \in C^2_0(C)$, let $M_f$ denote the operator on $W^2(D, H)$ given by multiplication by $f$. This has a spectral distribution of order 2, defined by the functional calculus

$$\Phi_M : C^2_0(C) \to \mathcal{L}(W^2(D, H)), \quad \Phi_M(f) = M_f.$$

Therefore $M_z$ is a scalar operator of order 2. In fact, it can be shown [Pu] that $M_z$ is subnormal.

3. Subscalarity

This section deals with the characterization for some $p$-hyponormal operators. Recall that an operator $T \in \mathcal{L}(H)$ is said to be $p$-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where $T^*$ is the adjoint of $T$.

We need the following lemmas to give a proof of the main theorem.

**Lemma 1** ([Xi], Lemma 2.1). Let $T = U|T|_r$ be the polar decomposition of $T$, $Q = |T|_r - |T|_l$, $z = pe^{i\theta}$, $0 < \rho$, and $|e^{i\theta}| = 1$ where $|T|_r = (T^*T)^{\frac{1}{2}}$ and $|T|_l = (TT^*)^{\frac{1}{2}}$. Then

$$\|(T - z)f\|_{2,D}^2 = \|(T|_r - \rho)f\|_{2,D}^2 + \rho\|T|^\frac{1}{2}(U - e^{i\theta})^*f\|_{2,D} + \rho(Qf, f)$$

for all $f \in L^2(D, H)$.

For reference, we quote Lemma 2 from [Pu].

**Lemma 2** ([Pu], Proposition 2.1). For every bounded disk $D$ in $C$ there is a constant $C_D$, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$ we have

$$\|(I - P)f\|_{2,D} \leq C_D(\|(T - z)^*\partial f\|_{2,D} + \|(T - z)^*\partial^2 f\|_{2,D})$$

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. 
For $p$-hyponormal operator $T = U|T|$, Aluthge ([Al]) introduced the operator $\tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2}$ and showed very interesting results on $\tilde{T}$.

**Lemma 3** ([Al]). Let $T = U|T|_r$ be a $p$-hyponormal operator, $0 < p < 1$, and $U$ unitary. Then the operator $\tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2}$ is hyponormal if $\frac{1}{2} \leq p < 1$, and $(p + \frac{1}{2})$-hyponormal if $0 < p < \frac{1}{2}$.

**Lemma 4.** Let $T = U|T|_r$ be semi-hyponormal and let $U$ be unitary. Let $D$ be a bounded disk which contains $\sigma(T)$. Then the map $V : H \to H(D)$ defined by $Vh = 1 \otimes h \mp (\tilde{T} - z) W^2(D, H)$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to $h$.

**Proof.** Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$
\lim_{n \to \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0.
$$

Then by the definition of the norm of Sobolev space (1) implies

$$
\lim_{n \to \infty} \|(T - z)\bar{\partial}^i f_n\|_{2,D} = 0
$$

for $i = 1, 2$. Since $T$ is a semi-hyponormal operator, Lemma 1 and equation (2) imply

$$
\begin{cases}
\lim_{n \to \infty} \||(T - \rho)\bar{\partial}^i f_n\|_{2,D} = 0, \\
\lim_{n \to \infty} \rho\|\|T\|^\frac{1}{2} (U - e^{i\theta})^* \bar{\partial}^i f_n\|_{2,D} = 0, \\
\lim_{n \to \infty} \rho\|Q\bar{\partial}^i f_n, \bar{\partial}^i f_n\| = 0.
\end{cases}
$$

We note that for $i = 1, 2$

$$
(T - z)^* \bar{\partial}^i f_n = |T|^\frac{1}{2} \left[ |T|^{\frac{1}{2}} (U - e^{i\theta})^* \bar{\partial}^i f_n \right] + e^{-i\theta}(\{(T) - \rho\) \bar{\partial}^i f_n].
$$

By equations (3) and (4), we get

$$
\lim_{n \to \infty} \|(T - z)^* \bar{\partial}^i f_n\|_{2,D} = 0.
$$

**Lemma 2** and equation (5) imply

$$
\lim_{n \to \infty} \|(I - P)f_n\|_{2,D} = 0,
$$

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto $A^2(D, H)$. Then by

$$
\lim_{n \to \infty} \|(T - z)Pf_n + 1 \otimes h_n\|_{2,D} = 0.
$$

Let $\Gamma$ be a curve in $D$ surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$
\lim_{n \to \infty} \|Pf_n(z) + (T - z)^{-1}(1 \otimes h_n)\| = 0
$$

uniformly. Hence

$$
\lim_{n \to \infty} \|\frac{1}{2\pi i} \int_{\Gamma} Pf_n(z)dz + h_n\| = 0.
$$

But by Cauchy’s theorem,

$$
\int_{\Gamma} Pf_n(z)dz = 0.
$$

Hence $\lim_{n \to \infty} h_n = 0$. Thus $V$ is one-to-one and has closed range. 

$\square$
Proposition 5. Let $T = U|T|_r$ be a $p$-hyponormal operator with the property $0 \notin \sigma(|T|^\frac{1}{p})$, $0 < p < 1$, and $U$ unitary. Let $D$ be a bounded disk which contains $\sigma(T)$. Then the map $V : H \to H(D)$ defined by $V h = 1 \otimes h (\equiv 1 \otimes h + (T - z)W^2(D, H))$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to $h$.

Proof. Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

\begin{equation}
\lim_{n \to \infty} \| (T - z)f_n + 1 \otimes h_n \|_{W^2} = 0.
\end{equation}

Then equation (7) implies

\begin{equation}
\lim_{n \to \infty} \| (T - z)\partial^i f_n \|_{2,D} = 0
\end{equation}

for $i = 1, 2$.

(a) If $\frac{1}{2} \leq p < 1$, then $T$ is semi-hyponormal. Therefore, Proposition 5 follows from Lemma 4.

(b) Let $0 < p < \frac{1}{2}$. Since $T = U|T|_r$,

\begin{equation}
\lim_{n \to \infty} \| |T|^\frac{1}{p} (U|T|_r - z)\partial^i f_n \|_{2,D} = 0.
\end{equation}

Since $\tilde{T} = |T|^\frac{1}{p} U|T|_r$, we have

\begin{equation}
\lim_{n \to \infty} \| (\tilde{T} - z)\partial^i (|T|^\frac{1}{p} f_n) \|_{2,D} = 0.
\end{equation}

Now we note that for $i = 1, 2$

\begin{equation}
(\tilde{T} - z)^* \partial^i (|T|^\frac{1}{p} f_n) = [\tilde{T}]^\frac{1}{p} [\tilde{T}]^\frac{1}{p} (W - e^{i\theta})^* \partial^i (|T|^\frac{1}{p} f_n)]
\end{equation}

\begin{equation}
+ e^{-i\theta} [\tilde{T}]^\frac{1}{p} (W - e^{i\theta})^* \partial^i (|T|^\frac{1}{p} f_n)].
\end{equation}

By (10) and (11), we get

\begin{equation}
\lim_{n \to \infty} \| (\tilde{T} - z)^* \partial^i (|T|^\frac{1}{p} f_n) \|_{2,D} = 0.
\end{equation}

Since $|T|^\frac{1}{p} (T - z) = (\tilde{T} - z)|T|^\frac{1}{p}$ and $0 \notin \sigma(|T|^\frac{1}{p})$, it follows from (7) that $\sigma(T) = \sigma(\tilde{T})$ and

\begin{equation}
\lim_{n \to \infty} \| (\tilde{T} - z)|T|^\frac{1}{p} f_n + |T|^\frac{1}{p} (1 \otimes h_n) \|_{2,D} = 0.
\end{equation}

By (13) and (14), we have

\begin{equation}
\lim_{n \to \infty} \| (\tilde{T} - z)P(|T|^\frac{1}{p} f_n) + |T|^\frac{1}{p} (1 \otimes h_n) \|_{2,D} = 0.
\end{equation}
Let $\Gamma$ be a curve in $D$ surrounding $\sigma(T) =: \sigma(T)$). Then for $z \in \Gamma$
\[
\lim_{n \to \infty} \| P([T]^\frac{1}{2} f_n(z)) + (\tilde{T} - z)^{-1}([T]^\frac{1}{2} (1 \otimes h_n)) \| = 0
\]
uniformly. Hence
\[
\lim_{n \to \infty} \| \frac{1}{2\pi i} \int_{\Gamma} P([T]^\frac{1}{2} f_n(z))dz \| = 0.
\]
But by Cauchy’s theorem,
\[
\frac{1}{2\pi i} \int_{\Gamma} P([T]^\frac{1}{2} f_n(z))dz = 0.
\]
Therefore $\lim_{n \to \infty} [T]^\frac{1}{2} h_n = 0$. Since $0 \notin \sigma([T]^\frac{1}{2})$, $[T]^\frac{1}{2}$ is bounded below. Hence $\lim_{n \to \infty} h_n = 0$. \[
\square
\]

**Theorem 6.** Let $T = U|T|_r$ be $p$-hyponormal, $0 < p < 1$, and $U$ unitary. If $0 \notin \sigma([T]^\frac{1}{2})$, then $T$ is subscalar of order 2.

**Proof.** Consider an arbitrary bounded open disk $D$ in the complex plane $\mathbb{C}$ and the quotient space
\[
H(D) = W^2(D, H)/\langle (T-z)W^2(D, H) \rangle
\]
endowed with the Hilbert space norm. The class of a vector $f$ or an operator on $H(D)$ will be denoted by $\mathbf{f}$, respectively $\mathbf{A}$. Let $M$ be the operator of multiplication by $z$ on $W^2(D, H)$. As noted at the end of section 2, $M$ is a scalar of order 2 and has a spectral distribution $\Phi$. Let $S \equiv M$. Since $\langle (T-z)W^2(D, H) \rangle$ is invariant under every operator $Mf$, $f \in C^2(D)$, we infer that $S$ is a scalar operator of order 2 with spectral distribution $\Phi$.

Consider the natural map $V : H \to H(D)$ defined by $Vh = \tilde{1} \otimes h$, for $h \in H$, where $1 \otimes h$ denotes the constant function identically equal to $h$. Note that $VT = SV$. In particular ran $V$ is an invariant subspace for $S$. Since $V$ is one-to-one and has closed range by Proposition 5, $T$ is subscalar of order 2. \[
\square
\]

**Corollary 7.** Every invertible $p$-hyponormal operator is subscalar of order 2.

**Proof.** Assume $T = U|T|_r$ is an invertible $p$-hyponormal operator where $U$ is unitary. Then $|T|_r$ is invertible. By [Ru, Theorem 12.33], $|T|^\frac{1}{2}$ is invertible. Therefore, $0 \notin \sigma([T]^\frac{1}{2})$. By Theorem 6, $T$ is subscalar of order 2. \[
\square
\]

**Corollary 8.** Let $T = U|T|_r$ be a $p$-hyponormal operator with the property $0 \notin \sigma([T]^\frac{1}{2})$, $0 < p < 1$, and $U$ unitary. If $\sigma(T)$ has interior in the plane, then $T$ has a non-trivial invariant subspace.

**Proof.** The corollary follows from Theorem 6 and $|Es|$. \[
\square
\]

**Corollary 9.** Let $T$ be as in Corollary 8. Then $T$ has the property $(\beta)$.

**Proof.** Since every subscalar operator has the property $(\beta)$, the corollary follows from Theorem 6. \[
\square
\]

Recall that an $X$ in $\mathcal{L}(H, K)$ is called a quasi-affinity if it has trivial kernel and dense range. An operator $A$ in $\mathcal{L}(H)$ is said to be a quasi-affine transform of an operator $T$ in $\mathcal{L}(K)$ if there is a quasi-affinity $X$ in $\mathcal{L}(H, K)$ such that $XA = TX$ (notation: $A \prec T$).
Corollary 10. Let $T$ be as in Corollary 8. If $A$ is any operator such that $A \prec T$, then $\sigma(T) \subseteq \sigma(A)$.

Proof. This is clear from [Ko, Theorem 3.2] and Corollary 9.

Corollary 11. Under the same hypothesis as Corollary 10, $A \in \mathcal{L}(H)$ is quasi-subscalar.

Proof. Let $X \in \mathcal{L}(H, K)$ be a quasi-affinity such that $XA = TX$. Since $V$ (in the construction of $V$ and $S$) and $X$ are one-to-one, $VX$ is one-to-one. Therefore $VX$ implements the quasi-subscalar properties. Thus $A$ is quasi-subscalar.

References


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