THETA FUNCTIONS OF INDEFINITE QUADRATIC FORMS
OVER REAL NUMBER FIELDS

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Abstract. We define theta functions attached to indefinite quadratic forms
over real number fields and prove that these theta functions are Hilbert mod-
ular forms by regarding them as specializations of symplectic theta functions.
The eighth root of unity which arises under modular transformations is deter-
mined explicitly.

1. Introduction

We construct a theta function attached to a quadratic form over a totally real
number field and show that this theta function is a modular form. If the quadratic
form is not totally positive, the usual sum over some ideal in the ring of integers
will not converge. Stopple [8] solves this problem by introducing the analogue of
a spherical harmonic for the theta function to ensure convergence. Furthermore,
he follows Eichler [2] to show that his theta function is a modular form on some
Γ₀ subgroups. Following Siegel [5] and [6], Friedberg [3] defines a theta function of
indefinite quadratic forms over C by using the majorants of the quadratic forms to
guarantee that the theta function will converge. By converting this theta function
into a symplectic theta function, Friedberg proves that his theta function is indeed
a modular form on some special subgroups of SL₂(ℝ). The advantage of his method
is that one can compute the theta multiplier explicitly without too much effort using
the main result of Stark [7]. We will follow his method and obtain results similar
to his over totally real number fields, i.e. we will prove that the theta function we
will construct is a Hilbert modular form on some Γ₀ subgroups and also compute
the theta multiplier explicitly using the main result of [7].

2. Symplectic theta functions

The symplectic group, Spₙ(ℝ), consists of those 2n × 2n real matrices

\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

(each entry is n × n) such that

\[ {}^t M J M = J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \].

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The corresponding symmetric space is the Siegel upper half plane \( \mathcal{H}^{(n)} \) which consists of \( n \times n \) symmetric complex matrices \( Z \) with \( \text{Im}(Z) > 0 \) (positive definite). The action of \( M \) on \( Z \) is given by

\[
M \circ Z = (AZ + B)(CZ + D)^{-1}.
\]

Let \( \Gamma^{(n)} = \text{Sp}_n(\mathbb{Z}) \). The theta subgroup \( \Gamma^{(n)}_0 \) of \( \Gamma^{(n)} \) is the set of all \( (A, B, C, D) \) in \( \Gamma^{(n)} \) such that both \( A^tB \) and \( C^tD \) have even diagonal entries. The subgroup acts on the symplectic theta function,

\[
\vartheta \left( Z, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \sum_{m \in \mathbb{Z}^n} \exp \left\{ \pi i \left[ \frac{1}{4} (m + v)^t Z (m + v) - 2^t m u - 4^t v u \right] \right\},
\]

where \( u \) and \( v \) are column vectors in \( \mathbb{C}^n \). It is well known (see Eichler [1], for example) that for

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ in } \Gamma^{(n)}_0,
\]

\[(1) \quad \vartheta \left( M \circ Z, M \begin{pmatrix} u \\ v \end{pmatrix} \right) = \chi(M) [\det(CZ + D)]^{1/2} \vartheta \left( Z, \begin{pmatrix} u \\ v \end{pmatrix} \right),
\]

where \( \chi(M) \) is an eighth root of unity which depends upon the chosen square root of \( \det(CZ + D) \), but which is otherwise independent of \( Z, u, \) and \( v \). It is also known that \( \chi(M) \) can be expressed in terms of Gaussian sums. Stark [7] determined \( \chi(M) \) in the important special case that \( pD^{-1} \) is integral for some odd prime \( p \). The main result in [7] is

**Theorem 1.** Suppose \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is in \( \Gamma^{(n)}_0 \) where \( C^{-1} \) and \( D^{-1} \) exist. Suppose further that for some odd prime \( p, pD^{-1} \) is integral. Then \((\text{mod } p)\), the symmetric matrix \( pD^{-1}C \) has rank \( h \) where \( \det(D) = \pm p^h \). Let \((pD^{-1}C)^{(h)} \) be a nonsingular \((\text{mod } p)\) \( h \times h \) principal submatrix of \( pD^{-1}C \) and let \( s \) be the signature (the number of positive eigenvalues minus the number of negative eigenvalues) of \( C^{-1}D \). Then \( \chi(M) [\det(CZ + D)]^{1/2} \)

\[
\begin{align*}
&= \varepsilon_p^{-h} \left( \frac{2^h \det \left( (pD^{-1}C)^{(h)} \right)}{p} \right) \varepsilon_p^{\frac{s^2}{2}} \left| \det(C) \right|^{1/2} \left( \det[-iC^{-1}(CZ + D)] \right)^{1/2},
\end{align*}
\]

where \( \varepsilon_p = 1 \) for \( p \equiv 1 \text{ mod } 4, \varepsilon_p = i \) for \( p \equiv 3 \text{ mod } 4, \left( \frac{2}{p} \right) \) is the Legendre symbol, \( \left| \det(C) \right|^{1/2} \) is positive and \( \{ \det[-iC^{-1}(CZ + D)] \}^{1/2} \) is given by analytic continuation from the principal value when \( Z = -C^{-1}D + iY \). Alternatively, if just \( C^{-1} \) exists and \( pC^{-1} \) is integral, \( \det(C) = \pm p^h \), then \( pC^{-1}D \) (mod \( p \)) has rank \( h \) and \( \chi(M) [\det(CZ + D)]^{1/2} \)

\[
\begin{align*}
&= \varepsilon_p^{-h} \left( \frac{-2}{p} \right)^h \left( \frac{\det \left( (pC^{-1}D)^{(h)} \right)}{p} \right) \left| \det(C) \right|^{1/2} \left( \det[-iC^{-1}(CZ + D)] \right)^{1/2}.
\end{align*}
\]
3. Theta functions attached to indefinite quadratic forms

Let $K$ be a totally real number field of degree $r_1$. Let $\Delta_K$ be the discriminant of $K$, let $\delta_K$ be the different of $K$, and let $\mathcal{O}_K$ be the ring of integers of $K$. The algebraic conjugates of an algebraic number $\alpha$ in $K$ are given by $\alpha^{(1)}, \ldots, \alpha^{(r_1)}$. Furthermore, let $\Gamma = SL_2(\mathcal{O}_K)$ and as usual, for an integral ideal $\mathfrak{N}$, let

$$\Gamma_0(\mathfrak{N}) = \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ M \in \Gamma \text{ and } \gamma \in \mathfrak{N} \right\}.$$ 

We define the upper half plane $\mathfrak{H} = \mathbb{H}^{r_1}$, where $\mathbb{H} = \{ z \in \mathbb{C}, \text{Im}z > 0 \}$ is the usual upper half plane. The matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$$

acts on $z = (z_1, \ldots, z_{r_1}) \in \mathfrak{H}$ by

$$M \circ z = \left( M^{(1)} \circ z_1, \ldots, M^{(r_1)} \circ z_{r_1} \right),$$

where

$$M^{(j)} = \begin{pmatrix} \alpha^{(j)} & \beta^{(j)} \\ \gamma^{(j)} & \delta^{(j)} \end{pmatrix}$$

and

$$M^{(j)} \circ z_j = \left( \alpha^{(j)} z_j + \beta^{(j)} \right) \left( \gamma^{(j)} z_j + \delta^{(j)} \right)^{-1}.$$ 

For $\gamma$ and $\delta$ in $K$ and $z$ in $\mathfrak{H}$, we define

$$\mathcal{N}(\gamma z + \delta) = \prod_{j=1}^{r_1} \left( \gamma^{(j)} z_j + \delta^{(j)} \right)$$

and

$$\mathcal{N} \left[ (\gamma z + \delta)^{1/2} \right] = \prod_{j=1}^{r_1} \left( \gamma^{(j)} z_j + \delta^{(j)} \right)^{1/2},$$

where each of the $r_1$ square roots is given by the principal value.

Let $Q$ be a symmetric $n \times n$ matrix defining the quadratic form $Q[x] = x^T Q x$, where $x \in \mathbb{R}^n$. If $Q$ has entries in $\mathcal{O}_K$ and diagonal entries which are divisible by 2, we say that $Q$ is of level $N$ ($N \in \mathcal{O}_K$) whenever the following two conditions are satisfied:

a) The matrix $NQ^{-1}$ has entries in $\mathcal{O}_K$, and 2 divides the diagonal entries of $NQ^{-1}$.

b) For any $M \in \mathcal{O}_K$, $N$ divides $M$ whenever $MQ^{-1}$ has entries in $\mathcal{O}_K$ and 2 divides the diagonal entries of $MQ^{-1}$.

If $Q^{(j)}$ has signature $(p, q)$ for $j = 1, \ldots, r_1$, then there exist matrices $L_j$ in $GL_n(\mathbb{R})$ such that $Q^{(j)} = L_j^T E_{p,q} L_j$, where

$$E_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix},$$

and $I_p$ and $I_q$ are the $p \times p$ and $q \times q$ identity matrices, respectively.

Set $R_j = L_j^T L_j$. Then $R_j$ is a majorant of $Q^{(j)}$, i.e.

$$R_j Q^{(j)-1} R_j = Q^{(j)} \text{ and } R_j = R_j > 0.$$
For the vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \), we set \( \lambda^{(j)} = (\lambda_1^{(j)}, \ldots, \lambda_n^{(j)}) \) where \( \lambda_1, \ldots, \lambda_n \) are in \( K \). We define the theta function \( \Theta_Q \) of an indefinite quadratic form by

**Definition 1.** Let \( Q \) be a symmetric \( n \times n \) matrix with entries in \( \mathcal{O}_K \) such that 2 divides the diagonal entries of \( Q \) and such that \( Q \) is of level \( N \). Furthermore, assume that each \( Q^{(j)} \) has the same signature \((p,q)\) for \( j = 1, \ldots, r_1 \). Let \( u_1, \ldots, u_{r_1} \) and \( v_1, \ldots, v_{r_1} \) be vectors in \( \mathbb{C}^n \). For an ideal \( \mathfrak{I} \subset \mathcal{O}_K \) and \( z = (z_1, \ldots, z_{r_1}) \in \mathfrak{I} \) set

\[
\Theta_Q \left( z, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \left( \prod_{j=1}^{r_1} y_j \right)^{q/2} \sum_{\lambda \in \mathfrak{I}^n} \exp \left\{ \pi i \left[ \sum_{j=1}^{r_1} Q^{(j)} \lambda^{(j)} + v_j x_j \right] + i R_j [\lambda^{(j)} + v_j y_j - 2 \lambda^{(j)} Q^{(j)} u_j - v_j Q^{(j)} u_j] \right\},
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \), \( u = (\lambda u_1, \ldots, \lambda u_{r_1}) \) and \( v = (\lambda v_1, \ldots, \lambda v_{r_1}) \).

Note that for any algebraic integer \( t \in K \), \( \sum_{j=1}^{r_1} Q^{(j)} [\lambda^{(j)}] t^{(j)} = \text{tr} (Q[\lambda] t) \) is an even rational integer, and thus \( \Theta_Q(z) \) is invariant under linear transformations, i.e.

\[
\Theta_Q \left( z + t, \begin{pmatrix} u + tv \\ v \end{pmatrix} \right) = \Theta_Q \left( z, \begin{pmatrix} u \\ v \end{pmatrix} \right).
\]

The first task is to convert \( \Theta_Q \) into a symplectic theta function. Let \( \omega_1, \ldots, \omega_{r_1} \) be an integral basis of the ideal \( \mathfrak{I} \subset \mathcal{O}_K \) and define the vector \( \omega^{(j)} = (\omega_1^{(j)}, \ldots, \omega_{r_1}^{(j)}) \).

We define the \( n \times nr_1 \) matrix

\[
W_j = \begin{pmatrix} \omega^{(j)} & \cdots & \omega^{(j)} \\ \vdots & \ddots & \vdots \\ \omega^{(j)} & \cdots & \omega^{(j)} \end{pmatrix}
\]

and the \( nr_1 \times nr_1 \) matrix \( W = (W_1, \ldots, W_{r_1}) \). Note that \( W^{-1} \) has entries in \( \mathcal{O}_K^{-1} \).

For \( z = (z_1, \ldots, z_{r_1}) \in \mathfrak{I} \), set

\[
Z^*_j = \begin{pmatrix} z_j I_p \\ -\overline{z}_j I_q \end{pmatrix}
\]

and

\[
Z^* = \begin{pmatrix} Z_1^* & \cdots & Z_{r_1}^* \\ & \ddots & \\ & & Z_{r_1}^* \end{pmatrix},
\]

where \( \overline{z}_j \) is the complex conjugate of \( z_j \). Define

\[
L = \begin{pmatrix} L_1 & \cdots & L_{r_1} \\ & \ddots & \\ & & L_{r_1} \end{pmatrix}
\]

and set

\[
T = LW \text{ and } Z = t^T Z^* T = \begin{pmatrix} t^T & 0 \\ 0 & T^{-1} \end{pmatrix} \circ Z^*.
\]
Observe that \((z T_{0 \ 0} - 1)\) is in \(\Sp_{nr}(\mathbb{R})\). Furthermore, \(Z^*\) is in the Siegel upper half plane \(\mathcal{H}(nr_1)\) and therefore \(Z\) is in \(\mathcal{H}(nr_1)\) as well. We have

\[
\left( \prod_{j=1}^{r_1} y_j \right)^{q/2} \Theta_{Q}(z, \begin{pmatrix} u & v \end{pmatrix}) = \vartheta \left( Z, \begin{pmatrix} tW \tilde{Q}u & W^{-1}v \end{pmatrix} \right),
\]

where

\[
\tilde{Q} = \begin{pmatrix} Q^{(1)} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ Q^{(r_1)} \end{pmatrix}.
\]

In order to apply Theorem 1, we need a symplectic matrix which expresses the action of \((\alpha \beta \gamma \delta) \in \Gamma = SL_2(\mathcal{O}_k)\) on our new variables \(u, v\) and \(Z\). For \((\alpha \beta \gamma \delta) \in \Gamma\), set

\[
\begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} I_n & \beta^{(1)} E_{p,q} \\ \cdots & \cdots \\ \alpha^{(r_1)} I_n & \beta^{(r_1)} E_{p,q} \\ \cdots & \cdots \end{pmatrix}.
\]

It is easy to check that the diagrams

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\]

commute, where

\[
(7) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left( \begin{pmatrix} tT & 0 \\ 0 & T^{-1} \end{pmatrix} \right) \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \left( \begin{pmatrix} tT & 0 \\ 0 & T^{-1} \end{pmatrix} \right)^{-1}
\]

\[
= \begin{pmatrix} tTA^* & tT^1 \\ T^{-1}C^* & tT^1 \end{pmatrix} \begin{pmatrix} tTB^*T \\ T^{-1}D^*T \end{pmatrix}.
\]

Hence

\[
z \mapsto \left( \begin{pmatrix} \alpha \beta \\ \gamma \delta \end{pmatrix} \right) \circ z
\]
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in \( \mathfrak{H} \) corresponds to

\[
Z \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z
\]

in \( \mathfrak{H}^{(n_{r_1})} \).

Let us introduce some more notation to show conditions under which the matrix in (7) is in the theta subgroup. Assume that \( S = (s_{il})_{i,l=1,\ldots,r_1} \) and \( R = (r_{km})_{k,m=1,\ldots,n} \) are matrices with entries in \( K \). We define the matrix \( R \circ S = ((\text{tr}(r_{km}s_{il}))_{i,l=1,\ldots,r_1})_{k,m=1,\ldots,n} \). Note that the entries of \( R \circ S \) are rational numbers.

Computation shows that \( A = I_n \circ A', B = Q \circ B', C = Q^{-1} \circ C' \) and \( D = I_n \circ D' \), where \( A', B', C' \) and \( D' \) are given by

\[
A' = \begin{pmatrix} \omega_1 \nu_1 \alpha & \omega_1 \nu_{r_1} \alpha \\ \omega_1 \nu_1 \alpha & \omega_{r_1} \nu_{r_1} \alpha \\ \vdots & \vdots \\ \omega_1 \nu_{r_1} \alpha & \omega_{r_1} \nu_{r_1} \alpha \end{pmatrix},
\]

\[
B' = \begin{pmatrix} \omega_1 \omega_1 \beta & \omega_1 \omega_{r_1} \beta \\ \omega_1 \omega_1 \beta & \omega_{r_1} \omega_{r_1} \beta \\ \vdots & \vdots \\ \omega_1 \omega_{r_1} \beta & \omega_{r_1} \omega_{r_1} \beta \end{pmatrix},
\]

\[
C' = \begin{pmatrix} \nu_1 \nu_1 \gamma & \nu_1 \nu_{r_1} \gamma \\ \nu_1 \nu_1 \gamma & \nu_{r_1} \nu_{r_1} \gamma \\ \vdots & \vdots \\ \nu_1 \nu_{r_1} \gamma & \nu_{r_1} \nu_{r_1} \gamma \end{pmatrix},
\]

\[
D' = \begin{pmatrix} \omega_1 \nu_1 \delta & \omega_{r_1} \nu_1 \delta \\ \omega_1 \nu_1 \delta & \omega_{r_1} \nu_{r_1} \delta \\ \vdots & \vdots \\ \omega_{r_1} \nu_1 \delta & \omega_{r_1} \nu_{r_1} \delta \end{pmatrix}.
\]

Clearly, \( ^tAC = ^tCA \), \( ^tBD = ^tDB \), and \( ^tDA - ^tBC = I_{n_{r_1}} \). Hence \( (\frac{A}{B}) \in \text{Sp}_{n_{r_1}}(\mathbb{R}) \). Furthermore, \( A'B = ^tTA' B'T = Q \circ (\alpha B') \) and \( C'D = T^{-1} C^* D^* T^{-1} = Q^{-1} \circ (\delta C') \). It follows that the entries of \( A, B, C \) and \( D \) are rational integers and that \( A'B \) and \( C'D \) have even diagonal entries if \( \gamma \) is in the ideal \( I_2 K N \). Hence for \( \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \Gamma_0 (\mathfrak{I}^2 K N) \),

we have

\[
\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \Gamma_{\delta}^{(n_{r_1})}.
\]

It is easy to verify that

\[
\det(CZ + D) = \det(C^* Z^* + D^*) = \mathcal{N}(\gamma z + \delta)^p \mathcal{N}(\gamma z + \delta)^q,
\]

and therefore by equations (1) and (5),

\[
\Theta_Q \left( \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \circ z, \left( \begin{array}{c} u \\ v \end{array} \right) \right)
= \chi \left( \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right), Q \right) \mathcal{N}(\gamma z + \delta)^{(p-q)/2} \Theta_Q \left( z, \left( \begin{array}{c} u \\ v \end{array} \right) \right),
\]

(8)
where $\chi((\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}), Q)$ is an eighth root of unity depending on $(\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix})$ and $Q$. Thus, $\Theta_Q(z, (\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}))$ is a (nonanalytic) Hilbert modular form on $\Gamma_0(3^2\delta_K N)$ of weight $(p-q)/2$.

4. THE EIGHTH ROOT OF UNITY

Now we determine $\chi((\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}), Q)$. Let us assume that $\delta \gg 0$ is a first degree prime in $\mathfrak{O}_k$ of norm $d$, where $d$ is a positive odd prime in $\mathbb{Z}$. In this case, $dD^{-1}$ is integral. Note that $\det(D) = \det(D^*) = d^n$, and thus by Theorem 1, $dD^{-1}$ has rank $n$ (mod $d$). Hence for $Q^{-1} = (r_d)_{i,l=1,\ldots,n}$, we see that

$$(dD^{-1}C)^{(n)} = \begin{pmatrix} \text{tr} (r_{11} \nu_1 \nu_1 d\delta^{-1} \gamma) & \cdots & \text{tr} (r_{1n} \nu_1 \nu_1 d\delta^{-1} \gamma) \\ \vdots & \ddots & \vdots \\ \text{tr} (r_{n1} \nu_1 \nu_1 d\delta^{-1} \gamma) & \cdots & \text{tr} (r_{nn} \nu_1 \nu_1 d\delta^{-1} \gamma) \end{pmatrix}$$

and

$$\det (dD^{-1}C)^{(n)} \equiv (d\delta^{-1} \gamma)^n (\nu_1 \nu_1)^n \det(Q)^{-1} \pmod{\delta}.$$

Some computation shows that

$$|\det(C)|^{1/2} \{\det[-iC^{-1}(CZ + D)]\}^{1/2} e^{2\pi i \frac{\gamma z}{\delta}} = N(\gamma z + \delta)^{(p-q)/2} |N(\gamma z + \delta)|^q.$$

Hence

$$(9) \quad \chi((\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}), Q) = \varepsilon_d^{-n} \left(\frac{(d\delta^{-1} \gamma)^n \det(Q)}{\delta}\right).$$

We have proved

**Theorem 2.** Suppose that $(\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}) \in \Gamma_0(3^2\delta_K N)$, where $\delta$ is a first degree prime in $\mathfrak{O}_K$ of norm $d$ (d is a positive odd prime in $\mathbb{Z}$). For $z \in \mathfrak{H}$, we have

$$\Theta_Q((\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}) \circ z, (\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}) (\begin{smallmatrix} u \\ v \end{smallmatrix})) = \varepsilon_d^{-n} \left(\frac{(d\delta^{-1} \gamma)^n \det(Q)}{\delta}\right) .$$

(10)

$$N(\gamma z + \delta)^{(p-q)/2} \Theta_Q(z, (\begin{smallmatrix} u \\ v \end{smallmatrix})),$$

where $\varepsilon_d = 1$ for $d \equiv 1 \pmod{4}$ and $\varepsilon_d = i$ for $d \equiv 3 \pmod{4}$.

Actually, we have determined the eighth root of unity more explicitly than it seems. In (4), we showed that for all algebraic integers $t$,

$$\Theta_Q((\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}) \circ z, (\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}) (\begin{smallmatrix} u \\ v \end{smallmatrix})) = \Theta_Q(z, (\begin{smallmatrix} u \\ v \end{smallmatrix})).$$

Together with (8) this implies that for $(\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}) \in \Gamma_0(3^2\delta_K N)$ and for all algebraic integers $t$,.

$$(11) \quad \chi((\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}) (\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}), Q) = \chi((\begin{smallmatrix} \alpha \\ \gamma \\ \delta \end{smallmatrix}), Q).$$

Furthermore, Hecke [4] gives a proof of Dirichlet’s primes in progression theorem for number fields. Hence for algebraic integers $\gamma$ and $\delta$ with $(\gamma, \delta) = 1$, the arithmetic progression $\{t + \delta\}_{t \in \mathfrak{O}_k}$ contains infinitely many primes $\pi \gg 0$ such that $N(\pi)$ is a positive odd prime in $\mathbb{Z}$. Hence the theta multiplier is determined explicitly after locating a totally positive first degree prime in the arithmetic progression $\{\gamma t + \delta\}_{t \in \mathfrak{O}_k}$.
There is a special case which also should be mentioned. Let $\delta \gg 0$ be a prime in $O_K$ with $N(\delta) = d^{r_1}$, where $d$ is a positive odd prime in $\mathbb{Z}$. As before, we observe that $dD^{-1}$ has rational integers as entries and hence $D$ is of level $d$. We see that $\det(D) = \det(D^*) = d^{nr_1}$ and by Theorem 1, $dD^{-1}C$ has rank $nr_1 \pmod{d}$. Thus,

$$\det(dD^{-1}C) = (N(\det(Q)))^{-1} \left( \Delta_K (N(I^2))^\frac{n}{2} (N(\gamma^n)) \right).$$

Hence

$$\chi \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} , Q \right) = \varepsilon_d^{-nr_1} \left( \frac{(\Delta_K)^n N((2\gamma)^n \det(Q))}{d} \right).$$

In the case that $n$ is even, we should see that (9) and (12) yield the same result.

For an odd rational prime $d$, an element $a$ is a square in $F_{d^{r_1}}$ (the field of $d^{r_1}$ elements) iff $N_{F_{d^{r_1}}/F_d}(a)$ is a square in $F_d$ (the field of $d$ elements). This can be seen by observing that the mapping $N : F_{d^{r_1}}^* \to F_d^*$ given by $a \to N(a) := N_{F_{d^{r_1}}/F_d}(a)$ is an epimorphism. Hence in the special case that $n$ is even, (9) becomes

$$\chi \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} , Q \right) = \left( \frac{(-1)^{n/2} \det(Q)}{\delta} \right)$$

and (12) becomes

$$\chi \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} , Q \right) = \left( \frac{N((-1)^{n/2} \det(Q))}{d} \right),$$

and the result from (9) coincides with the result from (12).

References


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