EIGENVALUE COMPLETIONS BY AFFINE VARIETIES

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Abstract. In this paper we provide new necessary and sufficient conditions for a general class of eigenvalue completion problems.

1. Preliminaries

Let \( F \) be an algebraically closed field of characteristic zero. Let \( \text{Mat}_{n \times n} \) be the space of all \( n \times n \) matrices defined over the field \( F \). We will identify \( \text{Mat}_{n \times n} \) with the vector space \( F^{n^2} \). Let \( X \subset \text{Mat}_{n \times n} \) be an affine variety. If \( M \in \text{Mat}_{n \times n} \) is a particular matrix we will denote by \( \sigma_i(M) \) the \( i \)-th elementary symmetric function in the eigenvalues of \( M \), i.e. \( \sigma_i(M) \) denotes up to sign the \( i \)-th coefficient of the characteristic polynomial of \( M \).

In this note we will be interested in conditions on the variety \( X \) which guarantee that the morphism

\[
\chi: X \longrightarrow F^n, \quad X \longmapsto (\sigma_1(A + X), \ldots, \sigma_n(A + X))
\]

is dominant for a particular matrix \( A \). In other words we are interested under what conditions the image misses at most a proper algebraic subset, i.e. the image forms a generic subset of \( F^n \). This problem was treated in [HRW97] when \( X \) is a linear subspace of \( \text{Mat}_{n \times n} \) and the base field \( F \) consists of the complex numbers. In this paper we generalize those results to the situation when \( X \) represents a general affine (irreducible) variety defined over \( F \).

Our study is motivated in part by an extensive literature on matrix completion problems and by several applications arising in the control literature. We refer to the research monograph [GKS95], which provides a good overview on the large linear algebra literature on matrix completions and to the survey articles [Byr89, RW97] for the connections to the control literature and further references.

2. Main result

Theorem 2.1. The characteristic map \( \chi \) introduced in (1.1) is dominant for a generic set of matrices \( A \in \text{Mat}_{n \times n} \cong F^{n^2} \) if and only if \( \dim X \geq n \) and the trace function \( \text{tr} \in \mathcal{O}(X) \) is not a constant.
The stated conditions are obviously necessary. Our proof is mainly based on two propositions. The first one is a strong version of the Dominant Morphism Theorem. Our formulation is immediately deduced from [Bor91, Chapter AG, §17, Theorem 17.3].

**Proposition 2.2** (Dominant Morphism Theorem). Let $\phi : X \to Y$ be a morphism of affine varieties. Then $\phi$ is dominant if and only if there is a smooth point $P \in X$ having the property that $\phi(P)$ is smooth and the Jacobian $d\phi_P : T_P(X) \to T_{\phi(P)}(Y)$ is surjective.

The second proposition which we will need in the proof of Theorem 2.1 is:

**Proposition 2.3** ([HRW97]). Let $L \subset \text{Mat}_{n \times n}$ be a linear subspace of dimension $\geq n$, $L \not\subset \text{sl}_n$ (i.e. $L$ contains an element with nonzero trace). Let

$$\pi(L) = (l_{11}, l_{22}, \ldots, l_{nn})$$

be the projection onto the diagonal entries. Then there exists an $S \in \text{Gl}_n$ such that

$$\pi(SLS^{-1}) = F^n.$$

This proposition was formulated in [HRW97, Lemma 2.8] when the base field $F$ consists of the complex numbers. The proof presented in [HRW97] only requires linearizations of rational functions and it is therefore valid mutatis mutandis for an arbitrary base field $F$.

**Proof of Theorem 2.1.** As mentioned earlier it is enough to show the sufficiency of the stated conditions. It has been pointed out in [HRW97] that the characteristic map $\chi$ is dominant, respectively surjective, if and only if the trace map

\begin{equation}
\psi : X \longrightarrow F^n, \quad X \longmapsto (\text{tr}(A + X), \ldots, \text{tr}(A + X)^n)
\end{equation}

is dominant, respectively surjective. This follows from the so-called Newton identities which express the elementary symmetric functions $\sigma_i(M)$ in terms of the power sum symmetric functions $\{\text{tr}(M^j) | 1 \leq j \leq n\}$.

It is the strategy of our proof to show the existence of a smooth point $P \in X$ which has the property that the Jacobian $d\psi_P$ is surjective for a generic set of matrices $A \in \text{Mat}_{n \times n}$. Since the range of the trace map $\psi$ is a smooth variety, the proof would be complete.

Let $Q \in X$ be a smooth point and consider the polynomial function

$$f(M) := \text{tr}(M) - \text{tr}(Q) \in O(\text{Mat}_{n \times n}) = F[x_{11}, \ldots, x_{nn}].$$

Let $H$ be the linear hypersurface

$$H := \{U \in \text{Mat}_{n \times n} | f(U) = 0\}.$$

Since $Q \in H \cap X$, it follows by the affine dimension theorem that the dimension of $H \cap X$ is at least $\dim X - 1$. Since by assumption $X$ is irreducible and $\text{tr} \in O(X)$ is not a constant, it follows that

$$\dim(H \cap X) = \dim X - 1.$$

Let $S \subset X$ be the singular locus and let $I \subset H \cap X$ be the irreducible component of $H \cap X$ which contains the smooth point $Q$. It follows that $S \cap I$ is a proper algebraic subset of $I$. Because of this there exists a point $P \in I \subset X$ which is both smooth inside $I$ as well as inside $X$. 

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By construction the tangent space $T_P(\mathcal{I})$ is properly contained inside the tangent space $T_P(\mathcal{X})$ and one has the relation

$$T_P(\mathcal{I}) = T_P(\mathcal{X}) \cap sl_n.$$ 

By Proposition 2.3 there exists an $S \in GL_n$ such that

$$\pi(S(T_P(\mathcal{X})))S^{-1} = \mathbb{F}^n.$$ 

Consider the trace map $\psi$ introduced in (2.1). A direct computation shows that the Jacobian at the point $P$ is given through:

$$d\psi : T_P(\mathcal{X}) \rightarrow \mathbb{F}^n, \quad L \mapsto (\text{tr}(L), 2\text{tr}((A + P)L), \ldots, n \cdot \text{tr}((A + P)^{n-1}L)).$$

Since the characteristic of $\mathbb{F}$ is zero, $\mathbb{F}$ contains as a prime field the rational numbers $\mathbb{Q}$. The matrices

$$D := \begin{pmatrix} 1 & \cdots & 2 \\ & \ddots & \\ & & \ddots & \cdots & \cdots \\ & & & 1 & \cdots & n \\ & & & & \cdots & \cdots \\ & & & & & \cdots \end{pmatrix}, \quad V := \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 2^{n-1} \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ 1 & \cdots & n^{n-1} \end{pmatrix}$$

are therefore invertible. Define $A := S^{-1}DS - P$. With our choice of the matrix $A$ the Jacobian is given through:

$$d\psi_P(L) = (\text{tr}(SLS^{-1}), 2\text{tr}(SLS^{-1}D), \ldots, n\text{tr}(SLS^{-1}D^{n-1})) = \pi(SLS^{-1})VD.$$ 

Since both the matrices $V$ and $D$ describe invertible transformations on $\mathbb{F}^n$, it follows that $d\psi_P$ is surjective for the particular choice of the matrix $A$. By the Dominant Morphism Theorem 2.2, $\psi$ and therefore $\chi$ is dominant.

Since the set of matrices $A$ whose associated Jacobian $d\psi_P$ forms a Zariski open set, and since we just showed that it is nonempty, it follows that for a generic set of matrices the map $\chi$ is dominant.

In the remainder of the paper we assume that $\mathcal{X} \subseteq \mathbb{F}^{n^2}$ is a fixed affine variety of dimension $\dim \mathcal{X} = m \geq n$, with coordinate ring $\mathcal{O}(\mathcal{X})$ and vanishing ideal $I(\mathcal{X})$. We conclude the paper with an algebraic description of all matrices $A$ whose characteristic map is dominant.

Let $(f_1(X), \ldots, f_k(X))$ be generators of $I(\mathcal{X})$ and define

$$T(X) = \begin{pmatrix} \frac{\partial f_1(X)}{\partial x_{11}} & \cdots & \frac{\partial f_k(X)}{\partial x_{11}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1(X)}{\partial x_{kn}} & \cdots & \frac{\partial f_k(X)}{\partial x_{kn}} \\ \end{pmatrix}.$$ 

Let $t_{ij}(M)$ denote the $ij$th entry of the matrix $M$ and let

$$J(X) := \begin{pmatrix} t_{11}(I) & t_{11}(A^t + X^t) & \cdots & t_{11}((A^t + X^t)^{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ t_{1n}(I) & t_{1n}(A^t + X^t) & \cdots & t_{1n}((A^t + X^t)^{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ t_{nn}(I) & t_{nn}(A^t + X^t) & \cdots & t_{nn}((A^t + X^t)^{n-1}) \end{pmatrix}.$$
Theorem 2.4. Let $\Delta \subset \mathbb{F}[x_{11}, \ldots, x_{nn}]$ be the ideal generated by the $(m + n) \times (m + n)$ minors of the matrix $[J(X), T(X)]$. Then $\chi$ is not dominant for a particular matrix $A$ if and only if $\Delta \subset I(X)$.

Proof. With respect to the standard basis of $\text{Mat}_{m \times n}$ the matrix $T(X)$ defines a linear transformation $T(X) : \mathbb{F}^{n^2} \to \mathbb{F}^k$, $x \mapsto xT(X)$ and $\ker(T(X)) = T_X(\mathcal{X})$. The rank of $T(X)$ is by assumption at most $m$, the dimension of $\mathcal{X}$. Similarly the matrix $J(X)$ defines a linear transformation $J(X) : \mathbb{F}^{n^2} \to \mathbb{F}^n$, $x \mapsto xJ(X)$ and $J(X)$ restricted to $T_X(\mathcal{X})$ is exactly $d\psi_X$. The concatenated matrix $[J(X), T(X)]$ induces a linear map $\tau : \mathbb{F}^{n^2} \to \mathbb{F}^n \oplus \mathbb{F}^k$.

If $\Delta \subset I(\mathcal{X})$, then $\mathcal{X} \subset V(\Delta)$, the algebraic set defined by $\Delta$. It follows that the rank of $\tau$ is strictly less than $m + n$ for all matrices $X \in \mathcal{X}$. It is therefore not possible to find a smooth point $P$ whose associated map $d\psi_P$ has full rank $n$. By the dominant morphism theorem $\psi$ and therefore $\chi$ is not dominant.

On the other hand if $\Delta \not\subset I(\mathcal{X})$, then there is a smooth point $P$ such that $[J(P), T(P)]$ has rank $m + n$. Since $T(P)$ has rank $m$ it follows that for every $y \in \mathbb{F}^n$ the point $(y, 0) \in \mathbb{F}^n \oplus \mathbb{F}^k$ is in the image of $\tau$. It follows that $d\psi_P$ is surjective and once more by the dominant morphism theorem $\psi$ and $\chi$ are dominant.

The following statement is a reformulation:

Corollary 2.5. $\chi$ is dominant for a particular matrix $A$ if and only if the matrix $[J(X), T(X)]$ has rank $m + n$ over the ring $\mathcal{O}(\mathcal{X})$.

Remark 2.6. Theorem 2.1 did assume that the characteristic of $\mathbb{F}$ is zero. If the characteristic of $\mathbb{F}$ is $p$ and $p > n$, then our proof of Theorem 2.1 is still valid. If $0 < p \leq n$, then the Newton identities expressing the elementary symmetric functions $\sigma_i(M)$ in terms of the power sum symmetric functions $\{\text{tr}(M^j) \mid 1 \leq j \leq n\}$ do not exist and our proof method does not go through. It therefore remains an open question if Theorem 2.1 is also true in characteristic $p$, where $0 < p \leq n$.

References


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