

## ON AUTOMATIC CONTINUITY OF HOMOMORPHISMS

A. BEDDAA, S. J. BHATT, AND M. OUDADESS

(Communicated by Dale Alspach)

**ABSTRACT.** Introducing a weaker notion of regularity in a topological algebra, we examine and improve an automatic continuity theorem given by the second author. Examples and applications are given.

All topological algebras considered are commutative and Hausdorff, having a unit element. A topological algebra  $A$  is a  $Q$ -algebra if the set  $G(A)$  of invertible elements is open. Let  $B$  be a subalgebra of an algebra  $A$ . Then  $B$  is *inverse closed* in  $A$  if  $G(B) = B \cap G(A)$ .  $A$  is *strongly semisimple* if for every  $x \in A$ ,  $x \neq 0$ , there exists a nonzero continuous multiplicative linear functional  $\chi$  such that  $\chi(x) \neq 0$ .  $A$  is *advertibly complete* if a Cauchy net  $x_\alpha$  in  $A$  converges in  $A$  whenever for some  $y$  in  $A$ ,  $x_\alpha + y - x_\alpha y$  converges to 0. A  $Q$ -algebra is advertibly complete [Ma, p. 45]. A *uniform seminorm* on an algebra  $A$  is a seminorm  $p$  such that  $p(x^2) = p(x)^2$  for all  $x$  in  $A$ . Such a  $p$  is submultiplicative [BK]. A *uniform topological algebra*  $A$  is a topological algebra whose topology is defined by a family of uniform seminorms. Such an  $A$  is semisimple. The abbreviation *lmca* will stand for locally  $m$ -convex algebra.

In [B], the following is given.

**Theorem** ([B, Theorem 2.2]). *Let  $A$  be a spectrally bounded, regular, complete, uniform topological algebra. If  $B$  is an lmca and  $\phi : A \rightarrow B$  is a one-to-one homomorphism such that  $(\text{Im } \phi)^-$  (the closure of  $\text{Im } \phi$ ) is a semisimple  $Q$ -algebra, then  $\phi^{-1}/\text{Im } \phi$  is continuous.*

In the proof, the author considers the map  $\phi^* : \sigma(C) \rightarrow \sigma(A)$ , with  $\phi^*(f) = f \circ \phi$ ,  $\sigma(C)$  and  $\sigma(A)$  denoting respectively the spaces of nonzero continuous multiplicative functionals on  $C = (\text{Im } \phi)^-$  and  $A$ . In Math. Reviews, the reviewer R. J. Loy [L] asserted that the continuity of  $\phi$  has been implicitly used in [B]. Indeed,  $\phi^*$  is not always well defined when  $\phi$  is not continuous as the following example shows.

**Example 1.** Let  $\Omega$  denote the first uncountable ordinal and  $[0, \Omega)$  the set of all ordinals smaller than  $\Omega$ . Consider the algebra  $C[0, \Omega)$  of complex continuous functions on  $[0, \Omega)$  with compact open topology  $\tau$ . Every  $f \in C[0, \Omega)$  is bounded. It is a regular uniform lmca. The identity map  $\phi : (C[0, \Omega), \tau) \rightarrow (C[0, \Omega), \|\cdot\|_\infty)$ ,  $\phi(f) = f$ , satisfies the hypotheses of the above statement. It is well known that  $(C[0, \Omega), \tau)$  has discontinuous multiplicative linear functionals (see, for example, [Mi], [Z]); let  $\chi$

---

Received by the editors September 10, 1997 and, in revised form, May 18, 1998.

1991 *Mathematics Subject Classification.* Primary 46H40; Secondary 46H05.

*Key words and phrases.* Uniform topological algebra, locally  $m$ -convex algebra, weakly regular algebra, advertibly complete algebra,  $Q$ -algebra, automatic continuity of homomorphism.

be such a functional. It is in  $\sigma(C[0, \Omega], \|\cdot\|_\infty)$ , but  $\phi^*(\chi) = \chi$  is not in  $\sigma([0, \Omega], \tau)$ . One may give another example in a more general situation. Let  $X$  be a non-compact, locally compact pseudocompact space. Take  $A = C(X)$  the algebra of all continuous functions on  $X$  with compact open topology,  $B$  the algebra  $C(X)$  endowed with the sup norm  $\|\cdot\|_\infty$  and  $\phi : A \rightarrow B$  be  $\phi(f) = f$ . Since  $X$  is pseudocompact, non-compact, it is not realcompact [S, p. 44]. Let  $y$  be an element of the real compactification of  $X$  with  $y$  not in  $X$ . The evaluation  $\delta_y$  at  $y$  is in  $\sigma(B)$  but not in  $\sigma(A)$ . Hence  $\phi^*$  is not defined on  $\delta_y$ .

On the other hand,  $\phi^*$  can be well defined even when  $\phi$  is not continuous; in this case, the proof given in [B] works. Here is an example of such a situation.

**Example 2.** Let  $X$  be a compact Hausdorff space. Consider the algebra  $C(X)$  of continuous complex functions on  $X$ . Take  $A = (C(X), \tau_d)$ ,  $\tau_d$  being the topology of uniform convergence on all countable compact subsets of  $X$ ,  $B$  the algebra  $(C(X), \|\cdot\|_\infty)$  and  $\phi$  the identity map from  $A$  to  $B$ . It is of course discontinuous, only if  $X$  is uncountable. But in this case,  $\sigma(A)$  and  $\sigma(B)$  are both homeomorphic to  $X$ . So  $\phi^*$  is well defined.

The following theorem repairs the above result. On one hand, it provides a positive result in a context more general than above; on the other hand, it shows that if one assumes the continuity of  $\phi$  in the above result, then the algebra  $A$  is necessarily a Banach algebra.

**Theorem 1.** *Let  $A$  be weakly regular, advertibly complete, uniform topological algebra, let  $B$  be an lmca, and let  $\phi : A \rightarrow B$  be a one-to-one homomorphism such that  $(\text{Im } \phi)^-$  is a semisimple  $Q$ -algebra.*

- (1) *If  $A$  is functionally continuous, then  $\phi^{-1}/\text{Im } \phi$  is continuous.*
- (2) *If  $\phi$  is continuous, then the topology of  $A$  is normable.*

Following [Mi, p. 51],  $A$  is *functionally continuous (FC)* if every multiplicative functional on  $A$  is continuous. Note that  $(C(X), \tau_d)$  in Example 2 is FC, but not  $Q$ . A major unsolved problem in topological algebras is the Michael problem: Is every multiplicative linear functional on a Frechet lmc algebra continuous? This has led to several sufficient conditions for  $A$  to be FC. This makes FC a reasonable assumption.

Let  $A$  be a commutative topological algebra.  $A$  is *weakly regular* if given a closed set  $F \subset \sigma(A)$ ,  $F \neq \sigma(A)$ , there exists  $x \neq 0$  in  $A$  such that  $f(x) = 0$  for all  $f \in F$ . In the context of Banach algebras, weak regularity arises naturally in the study of uniqueness of the uniform norm [BD]; and it is weaker than regularity. This is exhibited in an example due to Barnes [Me, Example 1]. Let  $D = \{z \in \mathbb{C} : |z| < 1\}$ ,  $X = \bar{D} \times [0, 1]$ . Let  $A = \{f \in C(X) : f \text{ is holomorphic on } D \times \{0\}\}$ , a uniform Banach algebra. Then  $A$  is weakly regular, but not regular. Since regularity in a uniform algebra is a stringent property, the validity of Theorem 1 under weak regularity is interesting.

*Proof of Theorem 1.* (1) Let  $C = (\text{Im } \phi)^-$ . Since  $A$  is FC,  $\phi^* : \sigma(C) \rightarrow \sigma(A)$ ,  $\phi^*(f) = f \circ \phi$  is well defined. It is also continuous with respective Gelfand topologies. Since the algebra  $C$  is a locally convex  $Q$ -algebra,  $\sigma(C)$  is compact [Ma, p. 187]. Then  $\phi^*(\sigma(C))$  is a compact subset of  $\sigma(A)$ . We have  $\phi^*(\sigma(C)) = \sigma(A)$ . Indeed, if  $\phi^*(\sigma(C)) \neq \sigma(A)$ , there exists  $x \neq 0$  in  $A$  such that  $\chi(\phi(x)) = 0$  for all  $\chi$  in  $\sigma(C)$ .

Since  $C$  is commutative semisimple and lmc,  $\phi(x) = 0$ ; and then  $x = 0$  for  $\phi$  is one-to-one. Thus  $\phi^*(\sigma(C)) = \sigma(A)$ . Now let  $P = (p_\alpha)$  be a family of uniform seminorms defining the topology  $\tau$  of  $A$ . Since  $A$  is advertibly complete, the spectrum  $\text{Sp}_A(x) = \{\chi(x) : \chi \in \sigma(A)\}$  and the spectral radius  $\rho_A(x) = \sup_\alpha \{\lim_{n \rightarrow \infty} (p_\alpha(x^n))^{1/n}\}$  for all  $x$  in  $A$  [Ma, p. 104, p. 99]. Since  $\sigma(A)$  is compact,  $\text{Sp}_A(x)$  is bounded. Since  $p_\alpha(x^2) = p_\alpha(x)^2$  for all  $x$  and  $\alpha$ ,  $\rho_A(x) = \sup_\alpha p_\alpha(x)$  for all  $x \in A$ . Also,  $\sigma(A) = \phi^*(\sigma(C))$  gives  $\text{Sp}_A(x) = \{f(\phi(x)) : f \in \sigma(C)\} \subset \text{Sp}_C(\phi(x))$ . Further, as  $C$  is a  $Q$ -algebra,  $s(C) = \{x \in C : \rho_C(x) \leq 1\}$  is a neighbourhood of 0 by [Mi, Prop. 13.5, p. 58]; and there exists a convex balanced open set  $W$  such that  $0 \in W \subseteq s(C)$ . The Minkowski functional  $q$  of  $W$  in  $C$  is a continuous seminorm satisfying  $\rho_C(y) \leq q(y)$  for all  $y \in C$ . Hence for each  $\alpha$ , for each  $x \in A$ ,  $p_\alpha(x) \leq \rho_A(x) \leq \rho_C(\phi(x)) \leq q(\phi(x))$ . This proves that  $\phi^{-1}/\text{Im } \phi$  is continuous.

(2) Suppose  $\phi$  is continuous. Then  $\phi^* : \sigma(C) \rightarrow \sigma(A)$  is well defined even if  $A$  is not FC. Then  $\phi^{-1}/\text{Im } \phi$  is continuous as above making  $\phi$  a topological isomorphism. Thus  $\phi(A)$  is advertibly complete; and hence inverse closed in its completion. Whence it is inverse closed in the  $Q$ -algebra  $C$ , for the completion of  $\phi(A)$  is contained in  $C$ . Therefore  $\phi(A)$ , and so  $A$ , is a  $Q$ -algebra. Hence the topology on  $A$  given by the algebra norm  $\rho_A$  is finer than  $\tau$ . Now since  $A$  is a  $Q$ -algebra,  $s(A) = \{x \in A : \rho_A(x) \leq 1\}$  is a neighbourhood of 0 on  $(A, \tau)$ . Thus  $\rho$  determines  $\tau$ . □

*Remark.* Once  $\phi^*$  is defined, the full strength of weak regularity has not been used. In fact, one has only to find a nonzero element vanishing on a given compact set.

We now give a result in the absence of FC. We consider the space  $\sigma^*(A)$  consisting of all nonzero multiplicative functionals on  $A$  endowed with the weak topology  $\sigma(A^*, A)$ . We then introduce the following notion of weak  $\sigma^*$ -compact-regular weakened in the sense of the previous remark.

**Definition.** A commutative topological algebra  $A$  is called *weakly  $\sigma^*$ -compact-regular* if for a compact subset  $K$  of  $\sigma^*(A)$ ,  $K \neq \sigma^*(A)$ , there exists a nonzero  $x \in A$  such that  $\chi(x) = 0$  for all  $\chi \in K$ .

**Theorem 2.** *Let  $A$  be a weakly  $\sigma^*$ -compact-regular advertibly complete uniform algebra,  $B$  a locally convex algebra and  $\phi : A \rightarrow B$  a one-to-one homomorphism such that  $C = (\text{Im } \phi)^-$  is a strongly semisimple  $Q$ -algebra. Then  $\phi^{-1}/\text{Im } \phi$  is continuous. If  $\phi$  is continuous, then the topology of  $A$  is normable.*

For the proof, consider the map  $\phi^{**} : \sigma(C) \rightarrow \sigma^*(A)$ ,  $\phi^{**}(\chi) = \chi \circ \phi$ . If  $\chi \circ \phi$  is identically zero, then by the continuity of  $\chi$ , one obtains that  $\chi$  is also identically zero. This contradicts  $\chi \in \sigma(C)$ . Thus  $\phi^{**}$  is defined; and then it is continuous. Now one obtains  $\phi^{**}(\sigma(C)) = \sigma^*(A)$ ; and the proof can be completed as in Theorem 1.

We conjecture that the semisimplicity of  $(\text{Im } \phi)^-$  in Theorem 1 (and strong semisimplicity in Theorem 2) can be omitted. The following supports this.

**Theorem 3.** *Let  $A$  be a uniform lmca,  $B$  a locally convex algebra, and  $\phi : A \rightarrow B$  a one-to-one homomorphism such that  $(\text{Im } \phi)^-$  is a  $Q$ -algebra. Assume that at least one of the following holds.*

- (a)  *$A$  is advertibly complete and  $\text{Im } \phi$  is FC with continuous product.*
- (b)  *$A$  is FC, Ptak (as a l.c. space), regular, having locally equicontinuous spectrum  $\sigma(A)$  (in particular,  $A$  is FC, Frechet, regular, having locally compact spectrum*

$\sigma(A)$ ), and  $B$  is *lmca*. Then  $\phi^{-1}/\text{Im } \phi$  is continuous. If  $\phi$  is continuous, then the topology of  $A$  is normable.

*Proof.* (1) Assume (a). Then  $\sigma^*((\text{Im } \phi)^-) = \sigma((\text{Im } \phi)^-)$  (since a  $Q$ -algebra is FC)  $= \sigma(\text{Im } \phi)$  (by the joint continuity of multiplication in  $\text{Im } \phi$ )  $= \sigma^*(\text{Im } \phi)$  and  $\phi^*(\sigma^*(\text{Im } \phi)) = \sigma^*(A)$  as  $\phi$  is one-to-one. Then, for all  $x \in A$ ,  $\text{Sp}_A(x) = \{\chi(x) : \chi \in \sigma(A)\} = \{\chi(x) : \chi \in \sigma^*(A)\} = \{f(\phi(x)) : f \in \sigma^*(\text{Im } \phi)\} = \{f(\phi(x)) : f \in \sigma^*(C)\}$ . Hence for some continuous seminorm  $q$ ,  $\rho_A(x) = \rho_C(x) \leq q(\phi(x))$  ( $x \in A$ ).

(2) Assume (b). By [Ma, Coro. 1.3, p. 184], local equicontinuity of  $\sigma(A)$  implies continuity of the Gelfand map  $x \rightarrow \hat{x}$  and local compactness of  $\sigma(A)$ . We show that  $\sigma(A) = \phi^*(\sigma(C))$ . Note that  $\phi^*(\sigma(C)) \subset \sigma(A)$ . Suppose  $\chi \in \sigma(A) \setminus \phi^*(\sigma(C))$ . By the local compactness, there exists a compact set  $K \subseteq \sigma(A)$  and disjoint open sets  $U, V$  in  $\sigma(A)$  such that  $\chi \in K \subset U$ ,  $\phi^*(\sigma(C)) \subset V$ . As  $A$  is Ptak, regular and having continuous Gelfand map, [Ma, Coro. 4.4, p. 344] implies that there exist  $x, y \in A$  such that  $g(x) = 1$  ( $g \in \phi^*(\sigma(C))$ ),  $g(x) = 0$  ( $g \in \sigma(A) \setminus V$ );  $g(y) = 1$  ( $g \in K$ ),  $g(y) = 0$  ( $g \in \sigma(A) \setminus U$ ). Then  $g(x)g(y) = 0$  for all  $g \in \sigma(A)$ . By the semisimplicity of  $A$ ,  $xy = 0 = \phi(x)\phi(y)$ . On the other hand, for all  $f \in \sigma(C)$ ,  $f(\phi(x)) = 1$ . Thus  $0 \notin \{f(\phi(x)) : f \in \sigma(C)\} = \text{Sp}_C(\phi(x))$ ,  $C$  being *lmc* and a  $Q$ -algebra. Thus  $\phi(x)$  is invertible in  $C$ . Hence  $\phi(y) = \phi(x)^{-1}\phi(x)\phi(y) = 0$ , so that  $y = 0$ , a contradiction. It follows that  $\phi^*(\sigma(C)) = \sigma(A)$ . Now the proof can be completed as in Theorem 1. Note that if  $A$  is Frechet, then every compact subset of  $\sigma(A)$  is equicontinuous [Mi, Prop. 4.2, p. 17]. Hence by [Ma, Th. 1.1, p. 182], the Gelfand map is continuous. Further if  $\sigma(A)$  is locally compact, then it is locally equicontinuous [Ma, Cor. 1.3, p. 184].  $\square$

*Remarks.* (1) If  $B$  is *lmca* and  $C = (\text{Im } \phi)^-$  is a semisimple  $Q$ -algebra, then  $C$  is strongly semisimple.

(2) Actually in the above theorems,  $\phi^{-1} : \phi(A) \rightarrow (A, \|\cdot\|)$  is continuous, where  $\|\cdot\|$  is the uniform norm given by  $\|x\| = \sup\{p(x) : p \text{ is a continuous uniform seminorm}\}$ . The existence of this norm implies that  $A$  is spectrally bounded.

(3) Theorem 2 also applies to Example 1. Indeed, the algebra  $(C[0, \Omega], \tau)$  is a complete uniform algebra. It is weakly  $\sigma^*$ -compact-regular, for  $\sigma^*(C[0, \Omega])$  is homeomorphic to the Stone-Cech compactification  $\beta[0, \Omega]$  of  $[0, \Omega]$  and  $C[0, \Omega]$  is isomorphic to the algebra  $C(\beta[0, \Omega])$  [GJ].

(4) The hypothesis  $(\text{Im } \phi)^-$  is a  $Q$ -algebra cannot be omitted. Let  $A = C_b(\mathbb{R})$  be the algebra of all continuous bounded functions on the real line. Endowed with the sup norm, it is a uniform Banach algebra. By the same arguments as in (3), one shows that  $A$  is weakly  $\sigma^*$ -compact-regular. Consider  $B = C(\mathbb{R})$  to be the algebra of all continuous functions with the compact open topology. Consider  $\phi : A \rightarrow B$ ,  $\phi(f) = f$ . Then  $(\text{Im } \phi)^- = C(\mathbb{R})$ . It is well known that it is not a  $Q$ -algebra. Clearly  $\phi^{-1}$  is discontinuous.

(5) The referee has asked: (In above theorems) does the automatic continuity of  $\phi^{-1}$  on  $\text{Im } \phi$  necessitate  $(\text{Im } \phi)^-$  a  $Q$ -algebra? The following answers this.

**Proposition 4.** *Let  $A$  be a complete  $Q$ -lmca,  $B$  an *lmca*, and  $\phi : A \rightarrow B$  a one-to-one homomorphism such that  $\phi^{-1}/\text{Im } \phi$  is continuous. Then  $(\text{Im } \phi)^-$  is a  $Q$ -algebra.*

*Proof.* We may assume that  $B$  is complete, hence  $C = (\text{Im } \phi)^-$  is complete. Then  $\phi^{-1}/\text{Im } \phi$  extends as a continuous homomorphism  $\psi : C \rightarrow A$ . By assumption, there exists a continuous seminorm  $p$  on  $A$  such that for all  $x$  in  $A$ ,  $r_C(\phi(x)) \leq r_{\text{Im } \phi}(\phi(x)) \leq r_A(x) \leq p(x) \leq p(\psi(\phi(x)))$ . Since  $\psi$  is continuous, there exists a

continuous seminorm  $q$  on  $C$  such that  $p(\psi(y)) \leq q(y)$  ( $y \in C$ ). Hence in the above,  $r_C(\phi(x)) \leq p(\psi(\phi(x))) \leq q(\phi(x))$  for all  $x$  in  $A$ . Now let  $y \in C$ ,  $y = \lim \phi(x_\alpha)$  for some net  $(x_\alpha)$  in  $A$ . For any continuous multiplicative functional  $f$  on  $C$ ,  $|f(y)| \leq |f(y - \phi(x_\alpha))| + |f(\phi(x_\alpha))| \leq |f(y - \phi(x_\alpha))| + q(\phi(x_\alpha)) \rightarrow q(y)$ . Hence  $r_C(y) = \sup |f(y)| \leq q(y)$  ( $y \in C$ ) showing that  $C$  is a  $Q$ -algebra.

If  $A$  is not a  $Q$ -algebra, then this does not hold. For the open unit disc  $U$  in the complex plane, let  $A = H(U)$  be the uniform Frechet algebra consisting of holomorphic functions on  $U$  with the compact-open topology,  $B = C(U)$  with the compact-open topology, and  $\phi : A \rightarrow B$  be  $\phi(f) = f$ . Clearly  $\phi$  is a homeomorphism and  $(\text{Im } \phi)^- = B$  fails to be a  $Q$ -algebra. □

APPLICATIONS

**(1) Proposition.** *Let  $A$  be an advertibly complete lmca. Let  $\| \cdot \|$  be any continuous norm on  $A$ . Then  $A$  cannot be simultaneously weakly regular and uniform unless the topology of  $A$  is normable.*

*Proof.* Let  $\tau$  denote the lmc topology on  $A$ . Let  $P = (p_\alpha)$  be a family of submultiplicative seminorms on  $A$  defining  $\tau$ . Then  $P_0 = P \cup \{ \| \cdot \| \}$  also determines  $\tau$ . Suppose  $(A, \tau)$  is uniform. Then  $\tau$  is defined by a family  $S = (q_i)$  of uniform seminorms. By closing  $S$  with maxima of finite subfamilies and applying the continuity of  $\| \cdot \|$ , there exists a  $q$  in  $S$  which is a norm. Let  $A_q$  be the uniform Banach algebra obtained by completing  $(A, q)$ . Let  $\phi : A \rightarrow A_q$  be  $\phi(x) = x$ . Now if  $A$  is weakly regular, then Theorem 1 applied to  $\phi$  implies that  $\tau$  is normable.

It follows that an advertibly complete non-normed weakly regular uniform algebra cannot support a continuous norm. Let  $X$  be a compact Hausdorff space. By a well known result of Kaplansky, if  $| \cdot |$  is any norm on  $C(X)$  making it a normed algebra, then the supnorm  $\| \cdot \| \leq | \cdot |$ . The following has a bearing with this. A norm on an algebra  $A$  is *semisimple* if the completion of  $(A, | \cdot |)$  is semisimple [BD]. □

**(2) Corollary.** *Let  $\| \cdot \|$  be a uniform norm on an algebra  $A$  such that  $(A, \| \cdot \|)$  is a weakly regular  $Q$ -normed algebra. Let  $| \cdot |$  be any submultiplicative norm on  $A$ .*

- (i) *If  $| \cdot |$  is semisimple, then  $\| \cdot \| \leq | \cdot |$ . Further if  $| \cdot |$  is continuous, then  $| \cdot |$  is equivalent to  $\| \cdot \|$ .*
- (ii) *Let  $(A, \| \cdot \|)$  be complete and regular. Then  $\| \cdot \| \leq | \cdot |$  for any submultiplicative norm  $| \cdot |$ .*

Indeed let  $\phi : (A, \| \cdot \|) \rightarrow (\tilde{A}, | \cdot |)$  ( $\tilde{A}$  = completion of  $(A, | \cdot |)$ ),  $\phi(x) = x$ . Theorem 1 implies that there exists  $k > 0$  such that  $\| \cdot \| \leq k | \cdot |$ . Since  $(A, \| \cdot \|)$  is  $Q$ ,  $\rho_A(x) = \inf \|x^n\|^{1/n} = \lim \|x^n\|^{1/n} = \|x\| \leq \lim |x^n|^{1/n} \leq |x|$  for all  $x \in A$ . (ii) follows by Theorem 3(b).

(3) Let  $A = \mathbb{C} \times C_c^\infty(\mathbb{R})$  (resp.  $B = \mathbb{C} \times C_c(\mathbb{R})$ ) be the algebra of all complex  $C^\infty$ -functions (resp. continuous functions) on  $\mathbb{R}$  which are constant outside some compact set (depending on the function). We endow  $A$  (resp.  $B$ ) with the inductive limit topology  $\tau_D$  (resp.  $\tau_K$ ). The algebra  $(A, \tau_D)$  is a complete regular lmca,  $(B, \tau_K)$  is a lmc  $Q$ -algebra [Ma, p. 128] and  $A$  is dense in  $B$  [K, p. 148]. Consider  $\phi : A \rightarrow B$ ,  $\phi(f) = f$ . Since the topology  $\tau_D$  is finer than  $\tau_K$  on  $A$ ,  $\phi$  is continuous. It is classical that  $A$  is not normable. Hence by Theorem 1,  $(A, \tau_D)$  cannot be uniform.

Let  $A$  be the algebra  $\mathbb{C} \times C_c^\infty(\mathbb{R})$  endowed with the compact open topology  $\tau$ . It is a weakly  $\sigma^*$ -compact-regular uniform lmca. Since it is inverse closed in  $C(\mathbb{R})$ , it

is advertibly complete. Take  $(B, \tau_K)$  as above. Let  $\phi : (A, \tau) \rightarrow (B, \tau_K)$ ,  $\phi(f) = f$ . By Theorem 1,  $\phi^{-1}$  is continuous.

(4) Let  $U \subset \mathbb{C}^d$  be open. Let  $H(U)$  be the uniform Frechet algebra of all holomorphic functions on  $U$  with the compact open topology. Let  $H^\infty(U) = \{f \in H(U) : f \text{ is bounded}\}$ , a uniform Banach algebra. Let  $X \subset \mathbb{C}^d$  be compact. Let  $H(X)$  be the algebra of holomorphic germs on  $X$ . Choose a decreasing sequence  $(U_n)$  of open neighbourhoods of  $X$  such that  $\bar{U}_{n+1} \subset U_n$  and  $\bar{U}_{n+1}$  is compact. In view of the continuous embeddings

$$\cdots \rightarrow H^\infty(U_n) \rightarrow H(U_n) \rightarrow H^\infty(U_{n+1}) \rightarrow H(U_{n+1}) \rightarrow \cdots,$$

$H(X)$  can be realized as inductive limits  $H(X) = \varinjlim H(U_n) = \varinjlim H^\infty(U_n)$ , its topology  $\tau$  being the finest locally convex topology making all  $\phi_n : H(U_n) \rightarrow H(X)$ ,  $\phi_n(f) = f/X$  and similarly making all  $\psi_n : H^\infty(U_n) \rightarrow H(X)$ ,  $\psi_n(f) = f/X$  continuous.

None of  $H^\infty(U)$  and  $H(U)$  is weakly regular. Note that  $(H(X), \tau)$  is a complete semisimple  $Q$ -algebra [Ma, p. 134]. If  $H(U_n)$  is weakly regular, then by Theorem 1, it becomes a Banach algebra and  $\phi_n$  becomes a homeomorphism. If  $H^\infty(U_n)$  is weakly regular, then  $\psi_n$  becomes a homeomorphism. Either of these forces  $H(X)$  to be a uniform Banach algebra. Being a complete, non-normed  $Q$ -algebra,  $H(X)$  is not a uniform algebra [BD].

(5) Let  $D = \{z \in \mathbb{C} : |z| < 1\}$ ,  $Y = D \times [0, 1]$ ,  $Z = \bar{D} \times [0, 1]$ . Let  $A = \{f \in C(Y) : f \text{ is holomorphic on } D \times \{0\}\}$ ,  $B = \{f \in C(Z) : f \text{ is holomorphic on } D \times \{0\}\}$ . Let  $0 < r < 1$ . Let  $\|f\|_r = \sup\{|f(x)| : x \in \bar{D}_r \times [0, 1]\}$  ( $f \in A$ );  $|f|_r = \sup\{|f(x)| : x \in \bar{D} \times [0, r]\}$  ( $f \in B$ ). Each of  $A$  and  $B$  with the topology defined respectively by  $\{\|\cdot\|_r : 0 < r < 1\}$  and  $\{|\cdot|_r : 0 < r < 1\}$  is a uniform Frechet algebra. Any  $f \in C(Y)$  (resp.  $f \in C(Z)$ ) vanishing on  $D \times \{0\}$  is in  $A$  (resp. in  $B$ ). This implies that both  $A$  and  $B$  are weakly regular, not regular. Thus each of  $A$  and  $B$  fails to support a continuous norm.

#### ACKNOWLEDGEMENT

A query by the referee resulted in Proposition 4. The referee has also made several suggestions regarding the exposition. We sincerely thank the referee for these.

#### REFERENCES

- [BK] S. J. Bhatt and D. J. Karia, *Uniqueness of the uniform norm with an application to topological algebras*, Proc. Amer. Math. Soc. **116** (2) (1992), 499–503. MR **92m**:46068
- [B] S. J. Bhatt, *Automatic continuity of homomorphisms in topological algebras*, Proc. Amer. Math. Soc. **119** (1) (1993), 135–139. MR **93k**:46042
- [BD] S. J. Bhatt and H. V. Dedania, *Banach algebras with unique uniform norm*, Proc. Amer. Math. Soc. **124** (2) (1996), 579–584. MR **96d**:46067
- [GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Springer-Verlag, New York, Heidelberg, Berlin, 1976. MR **53**:11352
- [K] Vo-Khac Khoan, *Distributions Analyse de Fourier Operateurs aux dérivées partielles*, vol. T. 1, Vuibert, Paris, 1972.
- [L] R. J. Loy, *Math. Reviews*, 93k:46042, p. 6116, Amer. Math. Soc., 1993.
- [Ma] A. Mallios, *Topological Algebras*, Selected Topics, North-Holland, 1986. MR **87m**:46099
- [Me] M. J. Meyer, *Spectral extension property and extension of multiplicative linear functionals*, Proc. Amer. Math. Soc. **112** (1991), 855–861. MR **91j**:46059
- [Mi] E. Michael, *Locally multiplicatively-convex algebras*, Mem. Amer. Math. Soc., vol. 11, Providence, RI, 1952. MR **14**:482a

- [S] J. Schmets, *Espaces de fonctions continues*, Lecture Notes in Mathematics 519, Springer-Verlag, 1976. MR **54**:11041
- [Z] W. Zelazko, *Banach algebras*, Elsevier Publishing Co. Amsterdam, London, New York, 1973. MR **56**:6389

(A. Beddaa and M. Oudadess) ECOLE NORMALE SUPERIEURE, B.P. 5118 TAKADDIUM, RABAT, MOROC

(S. J. Bhatt) DEPARTMENT OF MATHEMATICS, SARDAR PATEL UNIVERSITY, VALLABH VIDYANAGAR 388120, GUJARAT, INDIA