ON THE GELFAND-KIRILLOV CONJECTURE FOR QUANTUM ALGEBRAS

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Abstract. Let $q$ be a complex not a root of unity and $g$ be a semi-simple Lie $C$-algebra. Let $U_q(g)$ be the quantized enveloping algebra of Drinfeld and Jimbo, $U_q(n^-) \otimes U_q(0) \otimes U_q(n)$ be its triangular decomposition, and $C_q[G]$ the associated quantum group. We describe explicitly Fract $U_q(n)$ and Fract $C_q[G]$ as a quantum Weyl field. We use for this a quantum analogue of the Taylor lemma.

0. Introduction

Let $q$ be a nonzero complex number which is not a root of unity. In this article, a $C$-algebra defined by generators $X_i, 1 \leq i \leq m$, and relations $X_i X_j = q^{a_{i,j}} X_j X_i, 1 \leq i < j \leq m, a_{i,j} \in \mathbb{Z},$ will be called “the algebra of regular functions on an affine quantum space”. Its skew field of fractions will be called the quantum Weyl field. The $X_i, 1 \leq i \leq m,$ will be called a system of $q$-commuting generators (SQCG).

Let $g$ be a semi-simple Lie $C$-algebra of rank $n$. Let $R$ be the root system associated to the choice of a Cartan subalgebra $\mathfrak{h}$. We denote by $\Delta = \{\alpha_i\}$ the set of simple roots of $R$, $P$ the lattice of associated weights generated by the fundamental weights $\varpi_i, 1 \leq i \leq n$, and $P^+: = \sum_i \mathbb{N}\varpi_i$ the lattice of dominant weights. Let $G$ be the simply connected group associated to $g$ and $U_q(g)$ the Drinfeld and Jimbo’s quantized enveloping algebra. We define as in the classical case its “nilpotent” subalgebra $U_q(n)$ and the quantum algebra of regular functions on the group $C_q[G]$.

A theorem of J. Alev and F. Dumas (cf. [1]) asserts that Fract $U_q(n)$ is a quantum Weyl field when $g$ is of type $A_n$. In [15], A. Joseph proves that this property is verified for all semi-simple Lie algebras $g$ when $q$ is generic. We prove in this article that Fract $U_q(n)$ and Fract $C_q[G]$ are quantum Weyl fields when $g$ is semi-simple and when $q$ is not a root of one; see [9] for the case where $q$ is a root of one. The method we used provides a system of $q$-commuting generators.

Inspired by [12, Theorem 3.2], we essentially used the quantum analogue of the Taylor lemma. This lemma asserts that if $1) \delta$ is a locally nilpotent $\sigma$-derivation (cf. 1.1) on a $C$-algebra $A$ and $2) \exists a \in A$ such that $\delta(a) = 1$, then $a$ is (right) transcendant on the invariant algebra $A^\delta$ and $A \simeq A^\delta[a]$.

Our results are proved as follows:

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As a first step, we give (cf. Proposition 2.1) a multi-parametered version of the Taylor lemma for the locally nilpotent action (as a bialgebra) of the Borel subalgebra $U_q(b)$ on an algebra $A$. The difficulty encountered in the quantum case is the following: the generators $E_\beta$ ($\beta$ being a positive root) of the Poincaré-Birkhoff-Witt base of $U_q(n)$ do not act as $\sigma$-derivations on $A$. To get round this problem, we can, from a reduced decomposition of the longest element $w_0$ in the Weyl group, define a total order on the set of these generators and obtain a decreasing sequence of subalgebras $U_q(n_\beta)$ of $U_q(n)$; cf. [10, Lemma 1.7]. With the help of a result of S.Z. Levendorskii and Y.S. Soibelman (cf. [17, 2.4.1]) we obtain that $E_\beta$ acts as a $\sigma$-derivation on the subalgebra of $U_q(n_\beta^\circ)$-invariants of $A$, $\beta^\circ$ being the root preceding $\beta$. So, we can inductively apply the Taylor lemma and prove Proposition 2.1.

As a second step, we apply Proposition 2.1, see also Assertion 2.2, to the (right) regular action of $U_q(b)$ on $C_q[G]$. Recall (cf. 1.4) that $C_q[G]$ is generated as a space by the coefficients $c_{\mu,\nu}^\lambda$ of the simple finite dimensional $U_q(g)$-modules $L_q(\lambda)$, $\lambda \in P^+$. Let $w_0 = s_{i_1} \ldots s_{i_N}$ be a reduced decomposition of $w_0$ into a product of elementary reflections. Let $\beta = h_1 := s_{i_1} \ldots s_{i_{n-1}}(\alpha_i)$ and $y_l = s_{i_1} \ldots s_{i_l}$. By using the Lusztig automorphisms and the Weyl character formula, we prove that $c_\beta := c_{\mu,-1,\nu,\pi,\pi}^\lambda$, is $U_q(n_\beta^\circ)$-invariant. Moreover, with the help of the $R$-matrix, we prove (cf. Proposition 2.3) that the $c_\beta$ $q$-commute, i.e. commute up to a power of $q$. By the quantized Taylor lemma and the Drinfeld duality, we obtain the claimed theorem for Fract $U_q(n)$. We may specify the description of Fract $U_q(n)$ as in [1 Théorème 2.15]; cf. Theorem 3.2. We give similar results for the quantum algebras $S_q^+$ of regular functions on a Schubert variety; cf. [14, 10.3.1 (3)]. On this subject, we remark that the elements $c_\beta$ belong to the Lakshmibai-Reshetikhin base of standard monomials [16]. After localization, they generate a polynomial base.

As a third step, we show that our method works for $C_q[G]$. If $\rho$ is the sum of fundamental weights, then the elements $d_\rho = c_{\mu,-1,\rho,-\mu,\rho}$, $d_\rho' = c_{\mu,\rho,-\mu,-1,\rho}$ and $c_{0,0,\pi,\pi}$ generate the quantum Weyl field Fract $C_q[G]$. This theorem is a consequence of the Taylor lemma for the regular action of $U_q(b) \otimes U_q(b)^{opp}$ on $C_q[G]$. Note that this result was proved by A.N. Panov for $G = SL_n$ and generic $q$ [21].

In the classical case, the Gelfand-Kirillov conjecture asks if the enveloping algebra of $g$ is a Weyl field. In [12], A. Joseph gives a generalization of the Gelfand-Kirillov conjecture, replacing the enveloping algebra of $g$ by an algebra on which $n$ acts by derivations. The title of our article must be understood in the sense of this generalization. At the present time, we do not know if Fract $U_q(g)$ is a quantum Weyl field. As for the classical case, this assertion may be shown when $g$ has type $A_n$ (see [19]).

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1. Preliminaries and notations

1.1. Let $g$ be a semi-simple Lie $C$-algebra of rank $n$. We fix a Cartan sub-algebra $h$ of $g$. Let $g = n^+ + h + n$ be the triangular decomposition and $\{\alpha_i\}$ be a base of the root system $\Delta$ resulting from this decomposition. We note $b = n + h$ and $b^- = n^+ + h$, the two opposed Borel sub-algebras. Let $P$ be the weight lattice generated by the fundamental weights $\varpi_i$, $1 \leq i \leq n$, and $P^+ := \sum_i N \varpi_i$ the
semigroup of integral dominant weights. We denote by $\rho$ the sum of fundamental weights. Let $W$ be the Weyl group, generated by the reflections corresponding to the simple roots $s_{\alpha_i}$. Let $w_0$ be the longest element of $W$. We denote by $(\cdot, \cdot)$ the $W$-invariant form on $P$. We have $(\alpha_j, w_i) = \delta_{ij} \frac{(\alpha_i, \alpha_i)}{2}$.

1.2. Let $q$ be a nonzero complex number not a root of unity and $U_q(\mathfrak{g})$ be the simply connected quantized enveloping algebra, defined as in [14, 3.2.9]. Let $U_q(n)$, resp. $U_q(n^-)$, be the subalgebra generated by the canonical generators $E_{\alpha_i}$, resp. $F_{\alpha_i}$, of positive, resp. negative, weights. For all $\lambda$ in $P$, let $\tau(\lambda)$ be the corresponding element in the algebra $U^0$ of the torus of $U_q(\mathfrak{g})$. We have the triangular decomposition $U_q(\mathfrak{g}) = U_q(n^-) \otimes U^0 \otimes U_q(n)$. We set

\begin{equation}
U_q(b) = U_q(n) \otimes U^0, \quad U_q(b^-) = U_q(n^-) \otimes U^0.
\end{equation}

$U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra with comultiplication $\Delta$, and antipode $S$.

We fix the following notations, where $t$ is a complex not root of one, $n$ a nonnegative integer and $\alpha$ a positive root : $[n]_t = \frac{t^n - 1}{t - 1}$, $[n]_t! = [n]_t [n-1]_t \cdots [1]_t$, $q_\alpha = q^{(\alpha, \alpha)}/2$.

1.3. For $w$ in $W$, let $T_w$ be the Lusztig automorphism [18] associated to $w$. We fix a decomposition of the longest element of the Weyl group $w_0 = s_{i_1} \cdots s_{i_N}$, where $N = \text{dim} n$. This decomposition settles an order, denoted $<$, into the set $\Delta^+$ of positive roots : $\beta_N = s_{i_N} \cdots s_{i_{N-1}}(\alpha_{i_N})$, $\beta_2 = s_{i_1}(\alpha_{i_2})$, $\beta_1 = \alpha_{i_1}$. Then, we introduce the following elements in $U_q(n) : E_{\beta_i} = T_{i_1} \cdots T_{i_{i-1}}(T_{i_1})$. We define in the same way $F_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}}(T_{i_1})$.

We know (cf. [18]) that these elements generate a Poincaré-Birkhoff-Witt base of $U_q(\mathfrak{n})$. We have, by [22], see also [10, Lemma 1.7]:

**Proposition.** Let $U_q(n_\beta)$ be the space generated by the ordered products $\prod E_{\alpha}$, $\alpha \in \Delta^+$, $\alpha \leq \beta$, $k_\alpha \in \mathbb{N}$. Then $U_q(n_\beta)$ is a subalgebra of $U_q(n)$. Moreover, if $\mu \leq \beta$, we have $E_\mu E_\beta - q^{-(\mu, \beta)} E_\beta E_\mu \in \sum_{\alpha < \beta} U_q(n_\beta) E_\alpha$.

1.4. The dual $U_q(\mathfrak{g})^*$ is endowed with a structure of a left, resp. right, $U_q(\mathfrak{g})$-module by $u.c(\alpha) = c(u\alpha)$, resp. $c(u) = c(u\alpha)$, $u, \alpha \in U_q(\mathfrak{g})$, $c \in U_q(\mathfrak{g})^*$. In the same way, if $M$ is a left $U_q(\mathfrak{g})$-module, we endow the dual $M^*$ with the structure of a right $U_q(\mathfrak{g})$-module by $\xi u(v) = \xi(uv)$, $u \in U_q(\mathfrak{g})$, $\xi \in M^*$, $v \in M$.

For all $\lambda$ in $P^+$, let $L_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight $\lambda$. We know that $L_q(\lambda)$ verifies the Weyl character formula, for all $w$ in $W$ we denote by $v_w w$ the extremal vector of weight $w\lambda$. For all integral dominant weight $\lambda$, we fix a weight base $(v_{\mu})$, $\mu \in \Omega(L_q(\lambda))$, of $L_q(\lambda)$. We denote by $(v_{\mu}^*)$ its dual base. From [13] 10.2, we have the assertion

**Assertion.** Let $\lambda$ be an integral dominant weight and $w$ an element of the Weyl group. Fix a space $M$ and an isomorphism $\phi : M \rightarrow L_q(\lambda)^*$. We can endow $M$ with the structure of a right $U_q(\mathfrak{g})$-module by : $v^*, u = \phi^{-1}(\phi(v^*) T_w(u))$, $v^* \in M$. Then the $U_q(\mathfrak{g})$-module $M$ is isomorphic to $L_q(\lambda)^*$ and $\phi^{-1}(v_{\mu}^*)$, resp. $\phi^{-1}(v_{\mu}^* w)$, is its highest weight, resp. lowest weight, vector.

For all $\xi$ in $L_q(\lambda)^*$ and $v$ in $L_q(\lambda)$, let $c_{\xi,v}^\lambda$ in $U_q(\mathfrak{g})^*$ given by $c_{\xi,v}^\lambda(u) = \xi(uv)$, $u \in U_q(\mathfrak{g})$. Then we have $u.c_{\xi,v}^\lambda = c_{\xi,uv}^\lambda$ and $c_{\xi,v}^\lambda u = c_{\xi,u}^\lambda$. If $\xi$, resp. $v$, has weight
Suppose that this action is locally finite. We set $\mathcal{V}$. We prove the result by induction on $p$. We recall the expression of the $R$-twist. From this property it follows easily that:

$$R = \mathbb{C}_q[G] = \bigoplus_{\lambda \in P^+} C(\lambda), \quad R^+ = \bigoplus_{\lambda \in P^+} C^+(\lambda).$$

For $w$ in $W$, we define the quantized algebra $S^+_w$ of regular functions on the Schubert variety (see [13, 15] for details): $S^+_w$ is the inductive limit of $(c_{\alpha,\lambda}^{-1}V_w^+(\lambda))^*$, for $\lambda$ in $P^+$, where $V_w^+(\lambda)^*$ is the dual of the Demazure module $V_w(\lambda)$, naturally identified as a quotient of $C^+(\lambda)$.

1.5. We know that $U_q(\mathfrak{g})$ is an almost cocommutative Hopf algebra; cf. [11]. Let $\mathcal{R} = \mathcal{R}(1) \otimes \mathcal{R}(2)$ be the $\mathcal{R}$-matrix of $U_q(\mathfrak{g})$. This satisfies $\mathcal{R}\Delta = \Delta \mathcal{R}$, where $t$ is the twist. From this property it follows easily that:

$$R = (\prod_{\alpha \in \Delta^+} \exp_{q^{-2}}((1 - q^{-2})E_{\alpha} \otimes F_{\alpha})) \tau(\gamma) \otimes \tau(\gamma),$$

where $\gamma \in P$, $\exp_{q^{-2}}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q!}$.

2. A Quantum Taylor Lemma

2.1. We have the following lemma, whose proof is an analogue to [20, 1.1], Proposition 1.1:

Lemma. Let $A$ be an $\mathbb{C}$-algebra, $\sigma$ a $\mathbb{C}$-automorphism of $A$, $\delta$ a $\sigma$-derivation of $A$, i.e. $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$, $a, b \in A$. Let $A^\delta$ be the algebra of $\delta$-invariants in $A$. Suppose that 1) $\delta$ is locally nilpotent, 2) $\sigma \delta \sigma^{-1} = Q \delta$, $Q \in \mathbb{C}^*$, $Q$ not root of one, 3) there exists $a$ in $A$ such that $\delta(a) \in \mathbb{C}^*$. Then $A = A^\delta[a]$ and $a$ is (right) transcendental on $A^\delta$, i.e. $A = \bigoplus_{p \geq 0} A^\delta a^p$.

Proof. By 2), $A^\delta$ is $\sigma$-stable. Moreover, we have : $\delta^p(a^p) = [p]_{q^{-1}} \delta^p(a)$. This implies the direct sum in the claimed equality. Let $u$ be in $A$, with degree $p$, i.e. $p$ is the greatest integer such that $u_0 := \delta^p(u) \neq 0$. Clearly, the element $u_0$ is in $A^\delta$. We prove the result by induction on $p$ by considering $u - \frac{[p]_{q^{-1}}}{[p]_{q^{-1}}M(a)} \sigma^{-p}(u_0)a^p$, of degree $\leq p - 1$.

Let $A$ be an $\mathbb{C}$-algebra such that $U_q(\mathfrak{b})$ acts (as a bialgebra) on $A$, i.e. $A$ is a $U_q(\mathfrak{b})$-module and $a(\mathfrak{b}) = a(1)u(2)v$, $u, v \in A$, $a \in U_q(\mathfrak{b})$, $\Delta(a) = a(1) \otimes a(2)$. Suppose that this action is locally finite. We set $A^\delta = A$ and we note $A^\delta l \leq \leq L$, the algebra of $U_q(\mathfrak{n}_\beta)$-invariants in $A$. This proposition follows from the lemma.

Proposition. Let $A$ be an algebra defined as above. Suppose that, for all $\beta$ in $\Delta^+$, there exists $a_\beta$ in $A$ such that $E_\alpha a_\beta = \delta_{\alpha\beta}$, $\alpha \leq \beta$, where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Then, for all $l$, $1 \leq l \leq N$, we have

$$A = \bigoplus_{(k_1, \ldots, k_l) \in \mathbb{N}^l} A^\delta a_{\beta_1}^{k_1} \cdots a_{\beta_l}^{k_l}.$$
Proof. We note $\phi : U_q(b) \to End(A)$, the natural morphism for this action. By Proposition 1.3 and [17, 2.4.1], $\delta := \phi(E_{\beta})$ is a $\phi(\tau(\beta))$-derivation on $A^{l^{-1}}$. The conditions of the previous lemma are satisfied because 1) $\delta$ is locally nilpotent on $A^{l^{-1}}$, 2) $\tau(\beta)$ and $E_{\beta}$ q-commute, 3) $a := a_{\beta}$ is in $A^{l^{-1}}$ and satisfies $\delta(a) = 1$ by the definition. The proposition is obtained by induction on $l$ using the previous lemma.

2.2. We shall see that if $A$ is one of the algebras considered in the introduction, then the elements $a_{\beta}$ of Proposition 2.1 exist in some localization of $A$, and not in the algebra $A$. For the classical case, cf. [12, Theorem 2.6], it is enough to localize by a set $S$ generated by $n$-invariant elements in $A$. We can then apply the Taylor lemma to $A_S$. In the quantum case, the Taylor lemma needs some refinements. We slightly modify Lemma 2.1 to get

Assertion. Let $C$ be a noetherian domain on $C$, $\sigma$ a $C$-automorphism of $C$ and $\delta$ be a $\sigma$-derivation on $C$ which satisfies 1) and 2) of Lemma 2.1. Suppose $s$ (nonzero) and $s'$ are q-commuting in $C$ and such that $\delta(s') = s = C^0$. Let $a = s's^{-1}$ in Fract $C$ and $M = \bigcup_{p \geq 0} Cs^{-p}$. Then $\delta$ acts on $M$ and (as spaces) $C \subset M = \bigoplus_{p \geq 0} M^p a^p$. 

Proof. The element $s$ in $\delta$-invariant, so $\delta$ extends as a locally nilpotent derivation on $M$. Clearly, $C$ is a subset of $M$ and $\delta(a) = 1$. The direct sum is proved as in Lemma 2.1. As $s$ and $s'$ q-commute, we have $\bigoplus_{p \geq 0} M^p a^p \subset M$. The reverse inclusion is an easy induction as in the proof of Lemma 2.1.

We now give a condition on $A$ which implies that Fract $A$ is isomorphic to a quantum Weyl field.

Definition. Let $A$ be a noetherian domain. We say that $A$ verifies the property $(P)$ if the following hypotheses are verified:

(i) $U_q(b)$ acts (as a Hopf algebra) on the $C$-algebra $A$ and this action is locally finite. Let $B = A^{N^l}$ be the subalgebra of $U_q(n)$-invariant elements in $A$.

(ii) $B$ is generated by elements $c_i$, $1 \leq i \leq m$, and $B = C[e^m][e^2][e^1]$ is an algebra of functions on a quantum affine space with SQCG $\{e^1, \ldots, e^m\}$.

(iii) There exist nonzero elements $c_{\beta}, \beta \in \Delta^+$, in $A$ which q-commute, q-commute with $c^i$, $1 \leq i \leq m$, and satisfy : $E_{\alpha} c_{\beta} = \delta_{\alpha\beta} c_{\beta}$, $\alpha \leq \beta$, where $c_{\beta}$ is either $c_\gamma$, $\gamma > \beta$, $E_{\beta}$-invariant, or $c^j$, $1 \leq j \leq m$.

Proposition. Let $A$ be a noetherian domain. If $A$ satisfies the property $(P)$, then Fract $A$ is isomorphic to a quantum Weyl field. To be precise, if $B$ is the algebra of $U_q(n)$-invariant elements in $A$, then Fract $A$ is isomorphic to the skew field of fractions of $B[c_{\beta_1} \ldots c_{\beta_i}]$.

Proof. Suppose that $A$ satisfies the property $(P)$. For all $\beta = \beta_i$ in $\Delta^+$, note $S_i$ the multiplicative set generated by $s_i := c_{\beta_i}$; cf. (iii). We can define the following elements : $a_{\beta} = (c_{\beta}^2)^{-1} c_{\beta} \in \text{Fract } A$.

In the context of the previous assertion, we set (improperly) $C_S = \bigcup_{p \geq 0} Cs^{-p}$, where $S$ is the multiplicative set generated by $s$. Recall that $A$ is a domain. The conditions of the assertion are easily verified from the property $(P)$: We have:

$$A = A^0 \subset A_{S_1}^1[a_{\beta_1}] \subset A_{S_2}^2[a_{\beta_2}][s_1^{-1}, a_{\beta_1}] \subset \ldots \subset A_{S_N}^N[a_{\beta_N}][s_{N-1}^{-1}, a_{\beta_{N-1}}, \ldots, s_1^{-1}, a_{\beta_1}].$$

(*)
Moreover, from (iii), \( c_{\beta_i}^2 \in B \), so \( c_{\beta_N} \in B[a_{\beta_N}] \). Inductively, we can prove that \( B[a_{\beta_N}] \cdots [a_{\beta_1}] \) contains all the \( s_l \). This and (*) imply that \( \text{Fract} \ A \) is isomorphic to the skew field of fractions of \( B[a_{\beta_N}] \cdots [a_{\beta_1}] \). The property \((P)\) (iii) asserts that the \( a_\alpha \) are in the skew field of fractions of \( B[c_{\beta_N}] \cdots [c_{\beta_1}] \). By the Taylor lemma, these extensions are (right) transcendental. Our proposition follows. \( \square \)

2.3. Fix \( 1 \leq l \leq N \) and \( \beta = \beta_l \). The reduced decomposition of \( w_0 \) being fixed as in 1.3, we define the elements \( y_l \) of the Weyl group: \( y_0 = \text{Id}, y_l = s_i s_{i_2} \cdots s_i, l > 0 \).

Then, we introduce in \( R^+ \) : \( c_{\beta} = c_{l_{y_{l-1}^{-1}w_i \cdots w_i}}, c_{\beta}^2 = c_{s_i} E_{\beta} \).

**Lemma.** Let \( I_l = \{ p, l < p \leq N | i_p = i_l \} \). If \( I_l \) is empty, \( c_{\beta_l} = c_{w_{y_{l-1}^{-1}w_i \cdots w_i}} \). If not, let \( l' \) be the minimal element in \( I_l \); then \( c_{\beta_l} = c_{\beta_{l'}} \) (up to a multiplicative scalar).

**Proof.** Fix \( l \). Set \( j = i_l \). We show the second assertion of the lemma; the first is similar. We have \( y_{l-1}(\varpi_j) = y_{l-1}s_{i_l} \cdots s_{i_l}(\varpi_j) = y_{l-1}\varpi_j = \varpi_j - \alpha_j = y_{l-1}(\varpi_j) - \beta \). From Assertion 1.4, with \( \lambda = \varpi_j \) and \( w = y_{l-1} \), it is enough to prove that \( v_{\varpi_j} \cdot E_{\varpi_j} = v_{s_{i_l} \varpi_j} \). This is clear by the Weyl character formula and we can conclude the lemma. \( \square \)

**Proposition.** Let \( S \) be the multiplicative set generated by the \( c_{\alpha}, \alpha \in \Delta^+ \). \( S \) is a Ore set in \( R^+ \). Let \( \alpha = (c_{\beta}^{-1})^{-1}c_{\beta} \in S^{-1}R^+ \). We have \( a_\alpha a_\alpha = q^{(\alpha, \alpha)} a_\alpha a_{\alpha'}, \alpha, \alpha' \in \Delta^+, \alpha' < \alpha \).

**Proof.** As in the previous proof, we fix \( \beta = \beta_l \) in \( \Delta^+ \), \( 1 \leq l \leq N \), and \( i_l = j \). From Assertion 1.4, the extremal vector \( v_{y_{l-1}^{-1}w_i}^* \) in \( L_q(\varpi) \) is annihilated by the right action of \( E_\alpha, \alpha > \beta \), and \( F_\alpha, \alpha \leq \beta \). So, this holds for \( c_{\beta} \). From (1.5.1) and (1.5.2), we can deduce that the \( c_{\beta}, \beta \in \Delta^+, q\)-commute. By Lemma 2.3 and by [14] Corollary 9.1.4, this is also true for the \( c_{\beta}^2, \beta \in \Delta^+ \). The first assertion of the proposition follows from loc. cit., [Lemma A.2.9], and loc. cit., [Lemma 9.1.10]. We used loc. cit., [Proposition 9.1.5] to calculate the exponent of \( q \) in the formula. \( \square \)

**Remark.** The fact that \( S \) is a Ore set is not essential for the next section. Indeed, the elements \( a_\beta \) defined above exist at least in \( \text{Fract} \ R^+ \), because \( R^+ \) is a noetherian domain, cf. [14] 9.1.11.

3. APPLICATIONS

3.1. In this section, we give a list of quantum algebras which satisfy the desired property.

**Theorem.** The skew field of fractions of the algebra \( R^+, \) resp. \( S^+_w, \) resp. \( U_q(u), \) resp. \( U_q(b), \) is isomorphic to a quantum Weyl field of dimension \( N + n, \) resp. \( l(w) + n, \) resp. \( N + n. \)

**Proof.** By [14] Chapter 7, Chapter 9], all the algebras in the claim are noetherian domains. Let’s verify the assertions of the property \( (P). \)

For \( R^+ \), the action of (i) is the right regular action, which is locally finite. The algebra of \( U_q(u) \) invariant elements in \( R^+ \) is generated by the \( c_{w_{\varpi_i}} \), \( 1 \leq i \leq n \), which \( q \)-commute by [14] 9.1.4. Then, (iii) is given by Lemma 2.3 and Proposition 2.3.

The assertion for \( S^+_w \) is similar. We may consider, without loss of generality, the case where \( w = s_{i_{l-1}^{-1}(w)+1} \cdots s_{i_N} \). Set \( \beta := \beta_{N-\ell(w)+1} \). We prove the theorem with the help of (i) the right action of \( U_q(u), \) (ii) the \( U_q(u) \)-invariant elements \( c_{w_{\varpi_i}} \),
If reduced decomposition of the longest element of the Weyl group previous label for the embedding sequence above. We can inductively define the morphic to a quantum Weyl field of dimension algebra of type almost maximal, and be noted . We know (cf. [14, 9.1.10, 9.2.11]) that restricts to an embedding from its surjective up to localization. It is also well known (cf. [3]) that there exists an algebra antihomomorphism which is true for almost maximal, it is true for . Let’s consider now the previous isomorphism extent to the skew field of fractions. We remark (cf. 4, Lemme 3.4, 5 I Prop. 4.2) that the image of is in Fract . Thus, we can conclude by (1.2.1).

As in [15 Corollaire 6], we have the following corollary.

**Corollary.** Let be a minimal primitive ideal of . Then Fract is isomorphic to a quantum Weyl field of dimension .

### 3.2
In this section, we make more explicit the system of -commuting generators (SQCG) of Fract for classical simple Lie algebras . Recall that these generators are, from the proof of Theorem 3.1, the images of the elements , , by the Drinfeld (anti)homomorphism, followed by the natural projection on .

We know [3] that there is a natural embedding from to which maps the lowest weight vector of to 1. It maps the highest weight vector to an element of the -center of ; cf. [7]. An element of will be called almost maximal, and be noted , if it is the image of a vector in . Remark that those elements can be explicitly computed with the help of 5 Lemme 3.3).

With the standard notations of [2] Planches I à IV], we recall the canonical embeddings of Dynkin diagrams:

\[ A_1 \subset \cdots \subset A_n, \quad A_1 \subset B_2 \subset \cdots \subset B_n, \quad A_1 \subset C_2 \subset \cdots \subset C_n, \quad A_3 \subset D_4 \subset \cdots \subset D_n. \]

If is a Dynkin label for a classical simple Lie algebra, we denote by the previous label for the embedding sequence above. We can inductively define the reduced decomposition of the longest element of the Weyl group for the Lie algebra of type by:

\[
\begin{align*}
\omega_0(X) &= \omega_0(X^\prime). & \text{if } X = A_n, \\
&= s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1 & \text{if } X = B_n \text{ or } C_n, \\
&= s_1 \cdots s_{n-2} s_n s_{n-1} \cdots s_1 & \text{if } X = D_n.
\end{align*}
\]

This decomposition of permits us to obtain inductively our SQCG.

**Theorem.** The system of -commuting generators corresponding to the simple classical Lie algebra of type is inductively given by

\[ SQCG(X) = SQCG(X^\prime) \cup \bigcup_{i=1}^n e_{\omega_i} \cup \bigcup_{i=1}^{n-1} e_{\omega_i} \cup \bigcup_{i=1}^{n-2} e_{\omega_i}. \]

**Remark.** This theorem is a generalization of 11 Théorème 2.15 for the classical Lie algebras. Note that, except for if has type , , and if has type , all those elements can be obtained as quantum determinants of a “basic
Moreover, up to a nonzero multiplicative scalar, \( E \) and of \( i; j \) \( B \) that these actions commute). Set \( B = R^{N,N} \).

3.3. Now, we give the proof of a similar theorem for the algebra \( R = C_q[G] \). For all \( i, j \), \( 1 \leq i, j \leq N \), we denote by \( R^{i,j} \) the subalgebra of elements in \( R \) which are invariant for the right action of \( U_q(n_{\beta_i}) \) and for the left action of \( U_q(n_{\beta_j}) \) (recall that these actions commute). Set \( B = R^{N,N} \).

Fix \( \beta = \beta_l \). We define the following elements in \( C(\rho) \):

\[
d_{\beta} = c_{y_{i-1}p,-y_{i}p}, \quad d'_{\beta} = c_{y_{i-1}p,-y_{i-2}p}.
\]

By Assertion 1.4, we prove that \( d_{\beta} \) is invariant for the left action of \( E_{\alpha} \), \( \alpha \leq \beta \), and of \( F_{\alpha} \), \( \alpha \leq \beta \). Moreover, \( d_{\beta} \) is invariant for the right action of \( E_{\alpha} \), \( \alpha \leq \beta \), and of \( F_{\alpha} \), \( \alpha \leq \beta \). In the same way, \( d'_{\beta} \) is invariant for the left and right action of \( E_{\alpha} \), \( \alpha < \beta \), and of \( F_{\alpha} \), \( \alpha < \beta \). Hence, by (1.5.1) and (1.5.2), the elements \( d_{\alpha} \) and \( d'_{\alpha} \), \( \alpha \in \Delta^+, \) \( q \)-commute. We have clearly \( d_{\beta} \in R^{i,i-1} \) and \( d_{\beta} \in R^{i,i-1} \).

Moreover, up to a nonzero multiplicative scalar, \( E_{\beta}d_{\beta} = d_{\beta}E_{\beta} = d'_{\beta}, \) \( l \neq N, \) \( d_{\beta}E_{\beta} = c_{w_{0}p,p} \in B, \) \( l = N. \) \( C_q[G] \) being noetherian [14, 9.2.2], we can easily modify the proof of Proposition 2.2 to obtain:

**Theorem.** Fract \( C_q[G] \) is isomorphic to a quantum Weyl field of dimension \( 2N+n \).

To be more precise, Fract \( C_q[G] \) is isomorphic to the skew Weyl field of fractions of \( B[d_{\beta}][d'_{\beta}] \ldots[d_{\beta}][d'_{\beta}][d_{\beta}][d'_{\beta}][d_{\beta}][d'_{\beta}]. \)

The generators of \( B \), the \( d_{\beta} \) and \( d'_{\beta} \), can be easily expressed as a product of elements of \( C(\varpi_i), \) \( 1 \leq i \leq n \). Consider the algebra \( C_q[SL_2] \) generated by the elements of the quantum matrix : \( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \). Then the system of \( q \)-commuting generators provided by the theorem is \( \{ b, a, c \} \).

**References**


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