PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 4, Pages 1241–1245
S 0002-9939(99)05106-0
Article electronically published on August 3, 1999

C^2-PERTURBATIONS OF HOPF'S BIFURCATION POINTS AND HOMOCLINIC TANGENCIES

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(Communicated by Mary Rees)

Abstract. In this note we show that a diffeomorphism which has a Hopf's bifurcation point, can be C^2 perturbed around the bifurcation point in order to get a diffeomorphism which exhibits homoclinic tangencies. In the C^3 case this is not possible because of the typical unfolding of a Hopf's bifurcation point.

1. Introduction

Homoclinic tangencies are associated to plenty of chaotic phenomena. The corresponding definitions will be given below. By making perturbations of diffeomorphisms which exhibit homoclinic tangencies, we can obtain hyperbolic sets [S], strange attractors [Mo], [MV], persistence of homoclinic tangencies and infinitely many sinks [N], [Ro]. Besides, for families which unfold a homoclinic tangency we have cascades of period doubling bifurcations [YA], [Ma1].

J. Palis has conjectured that diffeomorphisms which exhibit homoclinic tangencies in a compact surface are dense in the complement of the closure of structural stable diffeomorphisms. There are several results which support that conjecture and show that diffeomorphisms (or one parameter families) that exhibit some of the above mentioned phenomena can be approximated by diffeomorphisms which exhibit homoclinic tangencies. See for example the results of R. Ures for strange attractors [U], E. Catsigeras for cascades of period doubling bifurcations [Ct] and J.C. Martín for diffeomorphisms obtained via a Hopf's bifurcation from DA diffeomorphisms [Ma2]. Recently, E. Pujals and M. Sambarino have proved the above conjecture in the space of C^1 diffeomorphisms of a surface under the hypothesis that the set of sinks and the set of sources are separated. For a comprehensive exposition of the subject homoclinic tangencies see the book [PT].

This note addresses a result which is in the context of the conjecture. We prove here that a diffeomorphism f from a manifold M which has a Hopf's bifurcation point can be approximated in the C^2 topology by another one which exhibits homoclinic tangencies.

Let $f : M \to M$ be a C^r diffeomorphism from a smooth surface M and let $p \in M$. We define the stable manifold (resp. unstable manifold) of p by

$$W^s(p) = \{x \in M : \lim_{n\to\infty} \mathrm{dist}(f^n(x), f^n(p)) = 0\}$$

Received by the editors November 10, 1997 and, in revised form, June 4, 1998.
1991 *Mathematics Subject Classification.* Primary 58F12, 58F13; Secondary 58F14, 58F15.

(resp.

$$W^u(p) = \{x \in M : \lim_{n\to-\infty} \mathrm{dist}(f^n(x), f^n(p)) = 0\}).$$

If p is a periodic point of f, we say that q is a *homoclinic point* of p if $q \in W^s(p) \cap W^u(p) \setminus \{p\}$. If p is a hyperbolic periodic point (i.e. $D_p f^k$ has no eigenvalues on $\mathbb{S}^1$, where k is the period of p), then $W^s(p)$ and $W^u(p)$ are C^r immerse submanifolds of M. In such a case we say that a homoclinic point q associated to p is *transversal* if $T_q W^s(p) + T_q W^u(p) = T_q M$; in the other case we say that q is a *homoclinic tangency* associated to p.

Let $f : M \to M$ be a C^3 diffeomorphism from a smooth surface M and let $p \in M$ be a fixed point of f. Assume that $D_p f$ has complex eigenvalues λ and $\overline{\lambda}$ such that $\lambda^k \neq 1$ for $k = 1, \ldots, 4$. Under these hypotheses there exist C^∞ coordinates (see [R, proof of Theorem 13.1]), such that in these coordinates $p = 0$ and for z near p

$$f(z) = (1 - a\,|z|^2)\, e^{i\,(\theta + b\,|z|^2)} z + o(|z|^3) \tag{1}$$

where we identify as usual $\mathbb{R}^2$ and $\mathbb{C}$ and we let $\theta = \arg(\lambda)$. We say that p is a *Hopf's bifurcation point* of f if the above hypotheses hold and $a \neq 0$. If p is a periodic point of f with period m, we repeat the above definition but considering f^m instead of f. We observe that if $a > 0$ (resp. $a < 0$), then p is a nonhyperbolic attractor (resp. repeller).

In this note we show

Theorem. *Let $f \in \mathit{Diff}^3(M)$ and let $p \in M$ be a Hopf's bifurcation point of f. Then for all C^r neighborhoods $\mathcal{N}$ of f and for all neighborhoods U of p there exists $g \in \mathcal{N}$ exhibiting a homoclinic tangency associated to a periodic point $q \in U$ iff $r \leq 2$.*

It is clear that the sufficiency part of this theorem holds in higher dimensions (via using central manifolds). However, the necessity part is not true in dimension higher than 2, because in [Ma2] it is proved that a C^r diffeomorphism which has a Hopf's bifurcation point p and a homoclinic orbit associated to it can be approximated in the C^r topology by another one which exhibits a homoclinic tangency associated to a periodic point near to p.

2. Proof of the theorem

Proof. First, we are going to prove that given $\mathcal{N}$ and U, a C^2 neighborhood of f and a neighborhood of p respectively, there exist $g \in \mathcal{N}$ and $q \in U$, q a hyperbolic periodic point of g such that q has associated a homoclinic tangency.

So using polar coordinates we get from (1):

$$f(r,\varphi) = (r - a\,r^3 + g_1(r,\varphi), \varphi + \theta + b\,r^2 + g_2(r,\varphi))$$

where $g_i(r,\varphi) = o(r^3)$.

Let $\psi : \mathbb{R} \to [0,1]$ be a function C^∞ such that $\psi(x) = 0 \;\; \forall\, x \in (-\infty, 1/2]$ and $\psi(x) = 1 \;\; \forall\, x \in [1,\infty)$. Given $\epsilon > 0$, we define $\psi_\epsilon(x) = \psi(x/\epsilon)$. It is clear that if $\|\psi\|_k < C$, then for some constant $C > 0$ we have that $\|\psi_\epsilon - 1\|_k < C/\epsilon^k$. We let $\tilde{f}$ be

$$\tilde{f}(r,\varphi) = (r - \psi_\epsilon(r)\,(a\,r^3 - g_1(r,\varphi)), \varphi + \theta + b\,r^2 + \psi_\epsilon(r)\,g_2(r,\varphi)).$$

So $\tilde{f}$ is C^3, and

$$\tilde{f}(r,\varphi) = f(r,\varphi) \text{ if } r \geq \epsilon. \tag{2}$$

Let us see that if ϵ small enough, then $\tilde{f} \in \mathcal{N}$. If we put $f = (f_1, f_2)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$, then we have

$$|f_1(r,\varphi) - \tilde{f}_1(r,\varphi)| = |\psi_\epsilon(r) - 1|\,|a\,r^3 - g_1(r,\varphi)| \leq 2\,a\,C\,r^3, \tag{3}$$

$$\begin{aligned} |\frac{\partial f_1}{\partial r}(r,\varphi) - \frac{\partial \tilde{f}_1}{\partial r}(r,\varphi)| &= |\psi_\epsilon(r) - 1|\,|3\,a\,r^2 - \frac{\partial g_1}{\partial r}(r,\varphi)| \\ &\quad + |\psi'_\epsilon(r)|\,|a\,r^3 - g_1(r,\varphi)| \\ &\leq 4\,a\,C\,r^2 + 2\,a\,C\,r^3/\epsilon, \end{aligned} \tag{4}$$

and

$$\begin{aligned} |\frac{\partial^2 f_1}{\partial r^2}(r,\varphi) - \frac{\partial^2 \tilde{f}_1}{\partial r^2}(r,\varphi)| &= |\psi_\epsilon(r) - 1|\,|6\,a\,r - \frac{\partial^2 g_1}{\partial r^2}(r,\varphi)| \\ &\quad + 2\,|\psi'_\epsilon(r)|\,|3\,a\,r^2 - \frac{\partial g_1}{\partial r}(r,\varphi)| \\ &\quad + |\psi''_\epsilon(r)|\,|a\,r^3 - g_1(r,\varphi)| \\ &\leq 7\,a\,C\,r + 8\,a\,C\,r^2/\epsilon + 2\,a\,C\,r^3/\epsilon^2. \end{aligned} \tag{5}$$

We observe that from (3)–(5), and similar calculations for the other partial derivatives, we get that if $r < \epsilon$, then $\|f - \tilde{f}\|_2 \leq \tilde{C} r$, where $\tilde{C}$ does not depend on ϵ; from this and (2), $\|f - \tilde{f}\|_2 < \epsilon$ is obtained.

For $r < \epsilon/2$, $\tilde{f}|_{\{r<\epsilon/2\}}$ is a twist map which preserves area since

$$D_{(r,\varphi)}\tilde{f} = \begin{bmatrix} 1 & 0 \\ 2\,b\,r & 1 \end{bmatrix}.$$

On the other hand, the region $\{r < \epsilon/2\}$ is foliated by invariant circles under $\tilde{f}$ with rotation number $\theta + b\,r^2$. Let $\gamma = \{r = r_0\}$ with $r_0 < \epsilon/2$ and $\theta + b\,r_0^2 \in \mathbb{R}\backslash 2\,\pi\mathbb{Q}$; we have that $\tilde{f}|_{\{r<\epsilon/2\}}$ and γ satisfy the hypotheses in the following

Theorem ([Mo]). *Let $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ be a twist map C^k, $k \geq 3$, which preserves area. Let γ be an invariant C^k curve of f such that $f|\gamma$ has an irrational rotation number and let U be a neighborhood of γ. Then there exist*

(a) *a sequence, $\{f_n : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}\}$ of C^{k-1} maps such that $f_n|_{\mathbb{S}^1\times\mathbb{R}\backslash U} = f|_{\mathbb{S}^1\times\mathbb{R}\backslash U}$, f_n converge to f in the C^{k-1} topology, with $detDf_n < 1$ and f_n exhibits a saddle-node cycle with criticalities;*

(b) *a sequence $\{f'_n : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}\}$ of C^{k-1} maps such that $f'_n|_{\mathbb{S}^1\times\mathbb{R}\backslash U} = f|_{\mathbb{S}^1\times\mathbb{R}\backslash U}$, f'_n converge to f in the C^{k-1} with $detDf'_n < 1$ and f'_n exhibits a Birkhoff-Hénon attractor.*

So $\tilde{f}$ can be perturbed in the C^2 topology, in order to get a diffeomorphism $\tilde{g} \in \mathcal{N}$ exhibiting a saddle-node cycle with criticalities and included in a neighborhood of γ. Therefore, using [NPT, Remark 4.2], we get a perturbation g which is C^2 close to $\tilde{g}$ and exhibits a homoclinic tangency in a neighborhood of γ. So we are done.

Now we are going to prove that there exist $\mathcal{N}$, a C^3 neighborhood of f, and a neighborhood U of p such that, $\forall\, g \in \mathcal{N}$, g does not exhibit homoclinic orbits associated to periodic points in U.

Since the nature of the problem is local, we can assume that $M = \mathbb{R}^2$ and $p = (0,0)$. The Implicit Function Theorem implies there exist a C^3 neighborhood $\mathcal{N}'$ of f, a neighborhood U' of $(0,0)$ and a C^3 mapping $\phi : \mathcal{N}' \to U'$ such that $\phi(g)$ is the unique fixed point of g in U'. Let

$$\mathcal{B}' = \{g \in \mathcal{N}' : \ D_{\phi(g)}g \text{ has its eigenvalues with modulus equal to } 1\}.$$

We take $\mathcal{N}'' \subset \mathcal{N}'$ a small enough C^3 neighborhood of f so that $\mathcal{B} = \mathcal{N}'' \cap \mathcal{B}'$ is a C^2 codimension one submanifold of $\mathcal{N}''$ such that $f \in \mathcal{B}$. This submanifold is characterized by: $g \in \mathcal{N}'' \cap \mathcal{B}$ iff $\phi(g)$ is a Hopf's bifurcation point of g.

On the other hand there exists a C^3 map $F : (-1,1) \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_\mu = F(\mu, \cdot)$ is a local diffeomorphism on $(0,0)$ and such that the following conditions are satisfied:

(i) $F_0 = f$,
(ii) $F_\mu(0,0) = (0,0)$,
(iii) $\frac{\mathrm{d}|\lambda(\mu)|}{\mathrm{d}\mu}|_{\mu=0} > 0$ where $\lambda : (-1,1) \to \mathbb{C}$ is a continuous function such that $\lambda(\mu)$ is an eigenvalue of $D_{(0,0)}F_\mu$ and $\lambda(0) = \lambda$.

We assume that $a < 0$, the proof in the other case being similar. By Neĭmark–Sacker's Theorem as formulated in [R, Theorem 13.1, p. 74], there exist $\mu_0 > 0$ and a neighborhood U of $(0,0)$ such that $\Omega(F_\mu) \cap U = \{(0,0)\}$ (resp. $\{(0,0)\} \cup \mathcal{C}_\mu$) if $\mu \leq 0$ (resp. $\mu > 0$), where $\mathcal{C}_\mu$ is the invariant circle which appears from Hopf's bifurcation. Also $(0,0)$ is an attractor (resp. repeller) for $\mu \leq 0$ (resp. $\mu > 0$) and $\mathcal{C}_\mu$ is normally hyperbolic $\forall\, \mu > 0$. Furthermore, there exist $\epsilon > 0$, $\mu_0 > 0$ and U such that the above features hold for every family G such that $\|G - F\|_3 < \epsilon$ where $\|\cdot\|_3$ is the norm C^3 in the Banach space of mappings C^3 of $\mathbb{R}^2 \times (-1,1)$ in $\mathbb{R}^2$ (see the proof of Theorem 13.1 in [R]).

We take $\mathcal{N}''$ small enough in order that the following statements hold. Let $\Phi : \mathcal{B} \times (-1,1) \to \mathrm{Diff}^3(\mathbb{R}^2)$ given by $\Phi(g,\mu) = g + F_\mu - f$. For each $g \in \mathcal{B}$, $\|\{\Phi(g,\mu)\}_{\mu \in (-1,1)} - F\|_3 < \epsilon$. From (2) we know that $\Phi|_{\mathcal{B} \times (-\delta,\delta)}$ is a diffeomorphism onto its image if $\delta > 0$ is small enough. Let $\delta' = \min\{\delta, \mu_0\}$. Then $\mathcal{N} = \Phi(\mathcal{B} \times (-\delta', \delta'))$ is a neighborhood of f, and each $h \in \mathcal{N}$ can be written as G_μ con $\mu < \mu_0$ where G is a family which satisfies $\|G - F\|_3 < \epsilon$. In consequence h cannot exhibit a homoclinic orbit associated to a periodic point in U, which is what we want to prove. □

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