

GROUPS ACTING TRANSITIVELY ON COMPACT CR MANIFOLDS OF HYPERSURFACE TYPE

ANDREA SPIRO

(Communicated by Leslie Saper)

ABSTRACT. Let $M = G/L$ be a compact homogeneous manifold with G acting effectively and with a G -invariant CR structure of hypersurface type; then any maximal compact subgroup $K \subset G$ acts transitively on M .

1. INTRODUCTION

A CR manifold of codimension p is a manifold M endowed with a pair (\mathcal{D}, J) , where $\mathcal{D} \subset TM$ is a regular distribution of codimension p and J is a smoothly defined complex structure on each subspace $\mathcal{D}_p \subset T_pM$ such that

$$(1.1) \quad J([JX, Y] + [X, JY]) \in \mathcal{D},$$

$$(1.2) \quad [JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0.$$

In case (\mathcal{D}, J) is of codimension one (usually said of *hypersurface type*), we say that it is *Levi nondegenerate* if \mathcal{D} is a contact distribution, i.e. for any 1-form θ on M such that $\ker \theta = \mathcal{D}$, we have $\ker d\theta_p \cap \mathcal{D}_p = \{0\}$ at all points. If M is a real hypersurface in \mathbb{C}^n and (\mathcal{D}, J) is the CR structure inherited by the complex structure of \mathbb{C}^n , then \mathcal{D} is a contact distribution if and only if the classical Levi form of M is nondegenerate.

The purpose of this paper is the proof of the following fact on compact homogeneous CR manifolds.

Theorem 1.1. *Let $M = G/L$ be a compact homogeneous manifold and let (\mathcal{D}, J) be a G -invariant CR structure of hypersurface type and Levi nondegenerate. If G acts effectively, then any maximal compact subgroup $K \subset G$ acts transitively on G/L .*

From the main theorem, we get that any compact homogeneous CR manifold of codimension one and Levi nondegenerate is in the class of homogeneous manifolds of compact Lie groups with invariant contact structure. Hence, Theorem 1.1 combines nicely with the following proposition, whose proof can be easily obtained from the results in [Al] and it furnishes some interesting facts on the topology of homogeneous CR manifolds (see [AS]).

Received by the editors December 30, 1997 and, in revised form, June 5, 1998.

1991 *Mathematics Subject Classification.* Primary 32C16; Secondary 53C30, 57S26.

Key words and phrases. Homogeneous CR manifolds, real hypersurfaces, actions of compact Lie groups.

Proposition 1.2. *Let $M = K/L$ be a compact homogeneous manifold with K compact and let \mathcal{D} be a K -invariant contact structure. If K acts effectively, then the center $Z \subset K$ is at most 1-dimensional and*

- (1) $\pi_1(M)$ is finite if and only if Z is 0-dimensional;
- (2) $\pi_1(M) = \mathbb{Z}$ if and only if Z is 1-dimensional; in this case M is a trivial S^1 -bundle over a flag manifold.

Another important consequence of Theorem 1.1 is that the classification of non-degenerate CR structures on compact homogeneous manifolds is reduced to the analysis of those on homogeneous spaces of compact Lie groups admitting an invariant contact structure. This classification can be found in [AS].

Finally, we remark that Theorem 1.1 has been proved in [AHR] for certain orbits of algebraic linear groups. However, the reader can also check that the arguments used in that paper imply that our result holds for *any* linear group orbit in $\mathbb{C}P^N$. Since it can be shown that *any homogeneous CR manifold with nondegenerate Levi form can be realized as an orbit in some $\mathbb{C}P^N$* (see [AS]), this could be an alternative path for the proof of Theorem 1.1. Nevertheless, one of the clear advantages of the approach we present here is that it is much more direct and well suited for generalizations to CR structures of higher codimension.

2. PROOF OF THEOREM 1.1

Let $M = G/L$ be a homogeneous manifold with nondegenerate invariant CR structure (\mathcal{D}, J) of codimension r . The integrable complex structures are considered as CR structures of codimension zero.

For any point $p \in M$, we will denote by \mathcal{D}'_p and \mathcal{D}''_p the holomorphic and anti-holomorphic subspaces of $\mathcal{D}_p^{\mathbb{C}}$, respectively. Moreover, for any $X \in \mathfrak{g} = \text{Lie}(G)$, we will denote by \hat{X} the corresponding infinitesimal transformation on M .

For any homogeneous CR manifold there exist two natural subspaces of \mathfrak{g} , characterizing the CR structure: if we set $p_o = eL$, then

$$(2.1) \quad \mathfrak{g}^+ \stackrel{\text{def}}{=} \{X \in \mathfrak{g}_{\mathbb{C}} : \hat{X}_{p_o} \in \mathcal{D}'_{p_o}\}, \quad \mathfrak{g}^- \stackrel{\text{def}}{=} \{X \in \mathfrak{g}_{\mathbb{C}} : \hat{X}_{p_o} \in \mathcal{D}''_{p_o}\}.$$

It is clear that $\mathfrak{g}^- = \overline{\mathfrak{g}^+}$. Moreover from (1.1) and (1.2) it follows that \mathfrak{g}^+ and \mathfrak{g}^- are both subalgebras of $\mathfrak{g}_{\mathbb{C}}$.

Vice versa, it is well known that any pair of subalgebras \mathfrak{g}^+ and $\mathfrak{g}^- = \overline{\mathfrak{g}^+}$ in $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}^+ \cap \mathfrak{g}^- = \mathfrak{l}_{\mathbb{C}} = \text{Lie}(L)_{\mathbb{C}}$ determines a G -invariant, integrable CR structure on G/L .

We will say that a diffeomorphism $\varphi: M \rightarrow M'$ between two CR manifolds (M, \mathcal{D}, J) and (M', \mathcal{D}', J') is *holomorphic* if: a) $\varphi_*(\mathcal{D}) \subset \mathcal{D}'$; b) $\varphi_*(Jv) = J'\varphi_*(v)$ for all $v \in \mathcal{D}$.

From the properties of the subalgebras \mathfrak{g}^+ and \mathfrak{g}^- , the following can be easily obtained:

Proposition 2.1. *Let $M = G/L$ be a homogeneous CR manifold with CR structure (\mathcal{D}, J) of codimension $r \geq 0$, and let N be a connected normal subgroup of G such that $N \cdot L$ is closed. Then $M' = G/(L \cdot N)$ admits a unique G -invariant CR structure (\mathcal{D}', J') of codimension $r' \leq r$ such that the fibering $\pi: M \rightarrow M'$ is holomorphic. The typical fiber $F = (L \cdot N)/L$ admits a natural invariant CR structure of codimension $r - r'$, so that the immersion $\iota: F \rightarrow M$ is holomorphic.*

Proof. Let \mathfrak{g}^+ and \mathfrak{g}^- be as in (2.1) and let \mathfrak{n} be the ideal of \mathfrak{g} corresponding to N . The projection π is holomorphic if and only if on M' we consider a CR structure associated to the subalgebras

$$\tilde{\mathfrak{g}}^+ = \mathfrak{g}^+ + \mathfrak{n}_{\mathbb{C}}, \quad \tilde{\mathfrak{g}}^- = \mathfrak{g}^- + \mathfrak{n}_{\mathbb{C}}.$$

Since $\tilde{\mathfrak{g}}^- = \overline{\tilde{\mathfrak{g}}^+}$ and $\tilde{\mathfrak{g}}^- \cap \tilde{\mathfrak{g}}^+ = \mathfrak{l}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$, there actually exists an invariant CR structure associated to $\tilde{\mathfrak{g}}^{\pm}$ and this proves its existence and uniqueness.

The regular distribution \mathcal{D}'' on F , given by the $\mathcal{D}''_p = \text{kernel}(\pi_*) \cap \mathcal{D}_p$, is invariant by the action of J and hence it corresponds to an almost CR structure on F . The corresponding subspaces are $\hat{\mathfrak{g}}^{\pm} = \mathfrak{g}^{\pm} \cap (\mathfrak{l}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}})$. Since they are subalgebras, $(\mathcal{D}'', J|_{\mathcal{D}''})$ is an integrable CR structure. It is clear that the immersion ι is holomorphic. □

For any invariant CR structure on $M = G/L$, there is a simple way to define a projective action of G on some $\mathbb{C}P^N$ and a G -equivariant holomorphic map between G/L and an orbit $G(v) \subset \mathbb{C}P^N$. This map is called *anticanonical map* and it has been introduced in [AHR]. We give here a different but equivalent definition in place of the original one.

Recall that any subspace $V \subset \mathfrak{g}_{\mathbb{C}}$ of dimension k can be considered as an element of $Gr(\mathfrak{g}_{\mathbb{C}}, k) \subset \mathbb{C}P^N$ (with $N = \binom{\dim \mathfrak{g}_{\mathbb{C}}}{k} - 1$) and the corresponding point in $\mathbb{C}P^N$ is given by the projective class $[w] \in \mathbb{C}P^N$ of a polyvector

$$w = e_1 \wedge \dots \wedge e_k$$

determined by any basis $\{e_1, \dots, e_k\}$ of V . We will use the symbol w_V to denote a polyvector associated to a subspace V . If \mathfrak{g}^{\pm} are the subalgebras corresponding to the CR structure on G/L , then the anticanonical map is defined as

$$\varphi: G/L \rightarrow G \cdot [w_{(\mathfrak{g}^-)}] \subset \mathbb{C}P^N,$$

$$\varphi(gL) = [Ad(g) \cdot w_{\mathfrak{g}^-}].$$

It is clear that the image of the anticanonical map is $\varphi(G/L) = G/H$, with $H = N_G(\mathfrak{g}^-)$. Furthermore:

Lemma 2.2. *The anticanonical map is holomorphic.*

Proof. It suffices to show that for $p_o = eL$ and all $v \in \mathcal{D}_{p_o}$ $\varphi_*(Jv)_{p_o} = i\varphi_*(v)_{p_o}$. Consider $Y \in \mathfrak{g}^+$ such that $X = Y + \bar{Y}$ satisfies $v = \hat{X}_{p_o}$. Since the adjoint action of \bar{Y} on $w_{\mathfrak{g}^-}$ is trivial,

$$\begin{aligned} \varphi_*(Jv)_{p_o} &= \left. \frac{d}{dt} \right|_0 [\exp(t(J(Y + \bar{Y}))) \cdot w_{\mathfrak{g}^-}] = \left. \frac{d}{dt} \right|_0 [\exp(tiY) \cdot w_{\mathfrak{g}^-}] \\ &= i \left. \frac{d}{dt} \right|_0 [\exp(t(Y + \bar{Y})) \cdot w_{\mathfrak{g}^-}] = i\varphi_*(v)_{p_o}. \end{aligned}$$

□

At last, we need the following lemma.

Lemma 2.3. *If G is abelian, then the distribution \mathcal{D} is trivial and $\dim G/L = 1$.*

Proof. Suppose that \mathcal{D} is not trivial and let θ be a defining form for \mathcal{D} . We want to show that θ is not a contact form and from this we reach a contradiction.

Let $v, w \in \mathcal{D}_p$ and pick two infinitesimal transformations \hat{X} and \hat{Y} of G so that $\hat{X}_p = v$ and $\hat{Y} = w$. Since \mathcal{D} is G -invariant, $\mathcal{L}_{\hat{X}}\theta = f_X\theta$ and $\mathcal{L}_{\hat{Y}}\theta = f_Y\theta$ for some nonvanishing functions f_X and f_Y . Then

$$d\theta_p(v, w) = (\mathcal{L}_{\hat{X}}\theta)_p(\hat{Y}) - (\mathcal{L}_{\hat{Y}}\theta)_p(\hat{X}) + \theta_p([\hat{X}, \hat{Y}]) = f_X\theta_p(w) - f_Y\theta_p(v) = 0.$$

In other words $d\theta$ vanishes identically on \mathcal{D} . Contradiction. □

We may now start the proof of Theorem 1.1. When $\dim M = 1$, the claim is easily seen to be satisfied. In fact, consider the connected component G^o of the identity and write it as $G^o = K \cdot E$, where K is a maximal compact subgroup and E is simply connected. If K does not act transitively on M , then it fixes all points: in this case, by effectivity, $K = \{e\}$ and there exists a 1-parameter subgroup ℓ of E which acts transitively on G/L . Since it has effective action, it has no normal subgroup acting trivially on M and hence it must be compact and diffeomorphic to S^1 : contradiction.

By the inductive hypothesis, we may now suppose that, for any compact homogeneous CR manifold G'/L' with $\dim G'/L' < \dim M$, at least one (and hence all) maximal compact subgroup $\tilde{K}' \subset G'$ acts transitively on G'/L' .

Consider the anticanonical map φ from G/L to $G/H_0 \subset \mathbb{C}P^N$, with $H_0 = N_G(\mathfrak{g}^-)$. Since it is a holomorphic map, the CR structures on G/H_0 and on H_0/L are of codimension at most one. From this fact and the nondegeneracy of (\mathcal{D}, J) we can prove also the following.

Lemma 2.4. $\varphi_*|_{\mathcal{D}}$ is injective at all points and $\dim H_0/L \leq 1$.

Proof. Suppose that $0 \neq v \in \mathcal{D}_{p_o} \cap \ker \varphi_{0*}$. Hence there is some $X \in \mathfrak{h}_0 = \text{Lie}(H_0)$ such that $X = X' + \overline{X'}$, with $X' \in \mathfrak{g}^+$ and $v = \hat{X}_{p_o}$. Since $X \in \mathfrak{h}_0$, we also have that $[X', \mathfrak{g}^-] \subset \mathfrak{g}^-$.

Let θ be a defining form for \mathcal{D} . Then for any $w \in \mathcal{D}_{p_o}$, we may write $w = \hat{Y}_{p_o}$, with $Y = Y' + \overline{Y'}$ for some $Y' \in \mathfrak{g}^+$ and we get

$$d\theta(v, w) = \frac{1}{2}d\theta(v - iJv, w + iJw) = -\theta([\hat{X}', \overline{Y'}]) \in \theta(\mathfrak{g}^-) = \{0\}$$

and the Levi form is degenerate: contradiction. This proves that $\mathcal{D}_{p_o} \cap \ker \varphi_{0*} = \{0\}$ and that the CR structure on the typical fiber H_0/L must be trivial. □

If $\dim H_0/L = 1$, the base G/H_0 is a complex submanifold of $\mathbb{C}P^N$, which is homogeneous with respect to the projective action of G ; therefore the base is a flag manifold and in particular it is simply connected. Hence, by Montgomery's theorem ([Mon]), for any maximal compact subgroup \tilde{K} in H_0 , there exists a maximal compact subgroup $K \supset \tilde{K}$ acting transitively on G/H_0 . If we consider a maximal compact subgroup $\tilde{K} \subset H_0$ acting transitively on H_0/L , we are done.

If $\dim H_0/L = 0$, M is a covering space of G/H_0 and we are reduced to proving that there exists a maximal compact subgroup K , which acts transitively on G/H_0 . Recall that G/H_0 is an orbit $G \cdot [w]$ in $\mathbb{C}P^N$ and hence there is no loss of generality if we assume that G is a linear algebraic group.

Let R be the radical of G and let

$$R^0 = R \quad , \quad R^1 = R' \quad , \quad \dots \quad , \quad R^{p+1} = (R^p)' = \{e\}$$

be the associated sequence of commutator groups. Also let \mathfrak{r}^p be the Lie algebra of the last nontrivial group R^p . Observe that G being an algebraic group, R^p and $H_0 \cdot R^p$ are closed algebraic subgroups in G . Then, by Proposition 2.1, there is a natural CR structure on $\tilde{M} = G/(H_0 \cdot R^p)$ such that $\pi: G/H_0 \rightarrow G/(H_0 \cdot R^p)$ and $\iota: (H_0 \cdot R^p)/H_0 \rightarrow G/H_0$ are holomorphic.

Now, note that $(H_0 \cdot R^p)/H_0$ is of positive dimension, since, by effectivity, $R^p \not\subset H_0$. Similarly \tilde{M} is not a point, because otherwise the abelian normal subgroup R^p would act transitively on G/H_0 and by Lemma 2.3 this would imply that $\dim M = 1$.

Therefore only the following cases may occur.

Case 1. The fiber $H_0 \cdot R^p/H_0$ is a compact, homogeneous, complex manifold of $\mathbb{C}P^N$ and $\tilde{M} = G/(H_0 \cdot R^p)$ is a compact, homogeneous, CR manifold of hypersurface type and its dimension is $1 \leq \dim G/(H_0 \cdot R^p) < \dim M$.

Case 2. The fiber $H_0 \cdot R^p/H_0$ is a compact, homogeneous, nondegenerate CR manifold of hypersurface type.

If Case 1 occurs, the fiber $H_0 \cdot R^p/H_0$ is an orbit in $\mathbb{C}P^N$ and hence a flag manifold. In particular, there exists a maximal compact subgroup K' acting transitively on it. At the same time, by the inductive hypothesis, there exists a maximal compact subgroup $K \supset K'$ acting transitively on $G/(H_0 \cdot R^p)$ and the theorem is proved.

Case 2 cannot occur because of the following contradiction. On the nondegenerate CR manifold $H_0 \cdot R^p/H_0$, the abelian group R^p acts transitively and hence $H_0 \cdot R^p/H_0$ is 1-dimensional by Lemma 2.3. By the fact that R^p is normal, all orbits of R^p are equivalent to $H_0 \cdot R^p/H_0$; furthermore, the action of R^p is effective and hence R^p is a compact normal subgroup of G diffeomorphic to S^1 . Consider on \mathfrak{g} an $Ad(R^p)$ -invariant metric $\langle \cdot, \cdot \rangle$ and let

$$\mathfrak{g} = \mathfrak{g}' + \mathfrak{r}^p$$

be the corresponding decomposition in orthogonal subspaces. Then

$$\langle [\mathfrak{r}^p, \mathfrak{g}'], \mathfrak{r}^p \rangle = -\langle \mathfrak{g}', [\mathfrak{r}^p, \mathfrak{r}^p] \rangle = 0$$

and this shows that $[\mathfrak{r}^p, \mathfrak{g}'] \subset \mathfrak{r}^p \cap \mathfrak{g}' = \{0\}$. But then \mathfrak{r}^p is in the center of \mathfrak{g} and $\mathfrak{r}^p \subset Lie(N_G(\mathfrak{g}^-)) = \mathfrak{h}_0 = \mathfrak{l}$. This contradicts the effectivity of the action of G on M .

REFERENCES

- [Al] D. V. Alekseevsky, *Contact homogeneous spaces*, Engl. transl. in. *Funct. Anal. Appl.* **24** (4) (1991), 324–325. MR **91j**:53027
- [AHR] H. Azad, A. Huckleberry and W. Richthofer, *Homogeneous CR manifolds*, *J. Reine und Angew. Math.* **358** (1985), 125–154. MR **87g**:32035
- [AS] D. V. Alekseevsky and A. F. Spiro, *Invariant CR structures on compact homogeneous manifolds*, preprint.
- [Mon] D. Montgomery, *Simply Connected Homogeneous spaces*, *Proc. A.M.S.* **1** (1950), 467–469. MR **12**:242c

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ANCONA, 60131 ANCONA, ITALY
E-mail address: spiro@anvax1.unian.it