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ALL NON-P-POINTS ARE THE LIMITS OF NONTRIVIAL SEQUENCES IN SUPERCOMPACT SPACES

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ABSTRACT. A Hausdorff topological space is called *supercompact* if there exists a subbase such that every cover consisting of this subbase has a subcover consisting of two elements. In this paper, we prove that every non-P-point in any continuous image of a supercompact space is the limit of a nontrivial sequence. We also prove that every non-P-point in a closed G_δ-subspace of a supercompact space is a cluster point of a subset with cardinal number $\leq c$. But we do not know whether this statement holds when replacing c by the countable cardinal number. As an application, we prove in ZFC that there exists a countable stratifiable space which has no supercompact compactification.

1. INTRODUCTION

In this paper, all spaces are assumed to be Hausdorff topological spaces. The notation of supercompactness was introduced by de Groot [6]. A space X is called *supercompact* if there exists a subbase $\mathcal{S}$ for X such that every cover of X consisting of elements of $\mathcal{S}$ has a subcover consisting of two elements. By the Alexander subbase lemma (we recently gave a simple proof for this lemma [12]), every supercompact space is compact. All continuous images of linearly ordered compacta are supercompact [2]. But the Čech-Stone compactification $\beta\omega$ of the infinite countable discrete space ω is not supercompact (see [1] or Section 3 in the present paper). In a space X a point p is called a *P-point* if $x \notin (\bigcup \mathcal{C})^- \setminus \bigcup \mathcal{C}$ for any countable family $\mathcal{C}$ of closed subsets of X; a point p is called a *weak P-point* if $x \notin C^- \setminus C$ for any countable subset C of X. It is trivial that every P-point is a weak P-point. However, there exists a non-P-point weak P-point in $\beta\omega\setminus\omega$ [8]. In 1994, the first author of this paper in [11] proved that in a continuous image of a closed G_δ-subspace of a supercompact space every non-weak-P-point is the limit of a nontrivial sequence and answered some problems in [4] and [9]. In the present paper, we prove the following theorems:

Theorem 1. *Let Y be a continuous image of a supercompact space and y a non-P-point in Y. Then y is the limit of a nontrivial sequence in Y.*

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Theorem 2. *Let Y be a closed G_δ-subspace of a supercompact space and y a non-P-point in Y. Then there exists a subset A of Y such that $p \in A^- \backslash A$ and $| A |\leq c$, where c is the cardinal number of the set of all real numbers.*

Thus we propose the following problem:

Problem 1. Under the assumptions of Theorem 2, we ask if there must be a countable subset A of Y such that p is a cluster point of A. That is, are P-point and weak-P-point equivalent in any closed G_δ-subspace of a supercompact space?

Remark 1. The statement in Theorem 2 does not hold for *any* compact Hausdorff space. In fact, Theorem 3.2 and Proposition 4.8 in Dow [5] imply that for any cardinal number κ there exists a compact Hausdorff space X such that X contains a non-P-point which is not a cluster point of any set in X with size at most κ.[1]

2. Proofs of the main theorems

Now we give proofs of the above theorems. At first, let us list some notation. Let $\mathcal{S}$ be a family of subsets in a topological space X. If the family $\{X\backslash S : S \in \mathcal{S}\}$ is a subbase for X, then $\mathcal{S}$ is called a *closed subbase for* X. If every pair of elements of $\mathcal{S}$ has a nonempty intersection, then $\mathcal{S}$ is called *linked*. If every linked subfamily of $\mathcal{S}$ has a nonempty intersection, then $\mathcal{S}$ is called *binary*. Obviously, a space is supercompact if and only if it has a binary closed subbase. Furthermore, we can assume that this closed subbase is closed with respect to arbitrary intersection. The following lemma proved in [11] is necessary to prove our theorems.

Lemma 1. *Let $\mathcal{S}$ be a closed subbase for a compact space X which is closed with arbitrary intersection, F a closed set and U an open set in X with $F \subset U$. Then there exists a finite subfamily $\mathcal{F}$ of $\mathcal{S}$ such that $F \subset int(\bigcup \mathcal{F}) \subset \bigcup \mathcal{F} \subset U$. Furthermore, if $F = \{p\}$ is a single point set, then $\mathcal{F}$ satisfies also that $p \in \bigcap \mathcal{F}$.*

Proof of Theorem 1. Let X be a supercompact space with a binary closed subbase $\mathcal{S}$ which is closed with respect to arbitrary intersection and $X \in \mathcal{S}$. Let $f : X \longrightarrow Y$ be a continuous mapping from X onto Y. Suppose $\mathcal{B}$ is a countable family of closed sets of Y such that

$$y \in (\bigcup \mathcal{B})^- \backslash \bigcup \mathcal{B}.$$

Let $\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{B}\}$. Then there exists $p \in f^{-1}(y)$ such that $p \in (\bigcup \mathcal{A})^- \backslash \bigcup \mathcal{A}$ because f is a closed mapping. By Lemma 1, for every $A \in \mathcal{A}$, there exists a finite subfamily $\mathcal{S}(A)$ of $\mathcal{S}$ such that $A \subset \bigcup \mathcal{S}(A) \subset X\backslash f^{-1}(y)$. Let $\mathcal{F} = \bigcup\{\mathcal{S}(A) : A \in \mathcal{A}\}$. Then $\mathcal{F}$ is a countable subfamily of $\mathcal{S}$ and $p \in (\bigcup \mathcal{F})^- \backslash \bigcup \mathcal{F}$. Now for every $F \in \mathcal{F}$, the family

$$\{F\} \cup \{S \in \mathcal{S} : S \cap F \neq \emptyset \text{ and } p \in S\}$$

is a linked subfamily of $\mathcal{S}$ and hence it has a nonempty intersection. Choose a point x_F in this intersection and let $C = \{x_F : F \in \mathcal{F}\}$. Then C is a countable set of X and $f^{-1}(y) \cap C = \emptyset$. In order to prove y is a cluster point of the countable set $f(C)$, it remains to verify that $p \in C^-$. In fact, if $p \notin C^-$, then, by Lemma 1, there exists a finite subfamily $\mathcal{S}_0$ of $\mathcal{S}$ such that

$$p \in int(\bigcup \mathcal{S}_0) \cap \bigcap \mathcal{S}_0 \subset \bigcup \mathcal{S}_0 \subset X\backslash C^-. \tag{1}$$

[1]This remark is due to Professor M. G. Bell in University of Manitoba (Canada).

Because $\bigcup \mathcal{S}_0$ is a neighborhood of p and $p \in (\bigcup \mathcal{F})^- \backslash \bigcup \mathcal{F}$, there exists $F \in \mathcal{F}$ such that $\bigcup \mathcal{S}_0 \cap F \neq \emptyset$. Hence there exists $S \in \mathcal{S}_0$ such that $F \cap S \neq \emptyset$. It follows from the definition of x_F that $x_F \in S$. This contradicts with (1). Thus we have proved that y is not a weak-P-point in Y. It follows from the theorem in [11] that y is the limit of a nontrivial sequence in Y. □

Proof of Theorem 2. Let X be a supercompact space with a binary closed subbase $\mathcal{S}$ which is closed with respect to arbitrary intersection and $X \in \mathcal{S}$. Let $Y \subset X$ be a closed G_δ-subspace of X. Then there exists a sequence $\{U_1, U_2, \cdots\}$ of open sets of X such that $U_1 \supset U_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} U_n = Y$. Since y is not a P-point in Y, there exists a countable family $\mathcal{C}$ of closed sets in Y (hence in X) such that $y \in (\bigcup \mathcal{C})^- \backslash \bigcup \mathcal{C}$. Now for every n and $C \in \mathcal{C}$, by Lemma 1, there exists a finite subfamily $\mathcal{S}(C, n)$ of $\mathcal{S}$ such that

$$C \subset \bigcup \mathcal{S}(C, n) \subset U_n \backslash \{y\}.$$

Hence,

$$C \subset \bigcap_{n=1}^{\infty} \bigcup \mathcal{S}(C, n) = \bigcup \{\bigcap_{n=1}^{\infty} f(n) : f \in \prod_{n=1}^{\infty} \mathcal{S}(C, n)\}.$$

For every $f \in \prod_{n=1}^{\infty} \mathcal{S}(C, n)$, let

$$S(C, f) = \bigcap_{n=1}^{\infty} f(n).$$

Then $S(C, f) \subset \bigcap_{n=1}^{\infty}(U_n \backslash \{y\}) = Y \backslash \{y\}$ and $S(C, f) \in \mathcal{S}$ since $\mathcal{S}$ is closed with respect to arbitrary intersection. Furthermore,

$$C \subset \bigcup \{S(C, f) : f \in \prod_{n=1}^{\infty} \mathcal{S}(C, n)\}.$$

Thus,

$$y \in (\bigcup \{S(C, f) : C \in \mathcal{C} \text{ and } f \in \prod_{n=1}^{\infty} \mathcal{S}(C, n)\})^-.$$

Hence, similar to Theorem 1, we may choose $x(C, f) \in S(C, f)$ satisfying that y is a cluster point of the set A of all $x(C, f)$'s. It is trivial that $| A | \leq c$. Thus we complete the proof of Theorem 2. □

Remark 2. It is not difficult to extend our theorems from the countable cardinal number to any cardinal number.

3. An application

It is an important topic to give some classes of Tychonoff spaces having supercompact compactifications. All separable metrizable spaces have supercompact compactifications since all compact metrizable spaces are supercompact [3]. But it seem to be yet open whether *all* metrizable spaces have supercompact compactifications [7]. Van Mill [7] proved that if $p \in \beta\omega \backslash \omega$ is a P-point in $\beta\omega \backslash \omega$, then the space $\omega \cup \{p\}$ has no supercompact compactification. However, S. Shelah proved that the existence of a P-point in $\beta\omega \backslash \omega$ is only a consistent result but not a theorem in ZFC (see [10]). Thus van Mill's theorem cannot imply in ZFC that there exists a stratifiable space having no supercompact compactification. Applying Theorem 1 in the present paper we, however, can obtain many countable stratifiable spaces which have no supercompact compactification. In particular, the space $\omega \cup \{p\}$ has no supercompact compactification for every $p \in \beta\omega \backslash \omega$.

The following simple lemma seems to be known:

Lemma 2. *Let X be a Tychonoff space and $p \in \beta X \backslash X$. Then for every compactification $\gamma(X \cup \{p\})$ of the space $X \cup \{p\} \subset \beta X$, p is the limit of a nontrivial sequence in $\gamma(X \cup \{p\})$ if and only if so is p in βX.*

Proof. It suffices to verify the following fact:

For any compactification $\gamma(X \cup \{p\})$ of the space $X \cup \{p\}$ and the unique extension $f : \beta X = \beta(X \cup \{p\}) \longrightarrow \gamma(X \cup \{p\})$ of the embedding $i : X \cup \{p\} \longrightarrow \gamma(X \cup \{p\})$ we have $f^{-1}(p) = \{p\}$.

In fact, if $f(q) = p$ for some $q \in \beta X$ but $q \neq p$, then there exist open sets $U,\ V \subset \beta X$ such that $p \in U,\ q \in V$ and $U^-_{\beta X} \cap V^-_{\beta X} = \emptyset$. It follows that

$$p \in f(V^-_{\beta X}) = f((V \cap X)^-_{\beta X}) = (f(V \cap X))^-_{\gamma(X \cup \{p\})} = (V \cap X)^-_{\gamma(X \cup \{p\})}.$$

Thus

$$p \in (V \cap X)^-_{\gamma(X \cup \{p\})} \cap (X \cup \{p\}) = (V \cap X)^-_{X \cup \{p\}} \subset (V \cap X)^-_{\beta X} = V^-_{\beta X}.$$

A contradiction occurs. □

Theorem 3. *Let X be a Tychonoff space with a dense subset which may be represented as a union of countably many compact sets. If $p \in \beta X \backslash X$ is not the limit of any nontrivial sequence in βX, then there exists no supercompact compactification of the space $X \cup \{p\}$.*

Proof. It follows from Theorem 1 and Lemma 2 since p is not a P-point in any compactification of the space $X \cup \{p\}$. □

Corollary 1. *There exists a countable space with only one nonisolated point having no supercompact compactification.*

Proof. $\omega \cup \{p\}$ is such a space for every $p \in \beta\omega \backslash \omega$. □

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