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# A NEW CHARACTERISATION OF THE ANALYTIC RADON-NIKODYM PROPERTY

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ABSTRACT. We show that a separable complex Banach space $X$ has the analytic Radon-Nikodym property if and only if there exists $1 \leq p < \infty$, such that the space consisting of all $L^p$-bounded $X$-valued analytic martingales is separable.

Let $(X, \|.\|)$ be a complex Banach space and let $1 \leq p \leq \infty$. $H^p(X)$ will denote the space consisting of all analytic functions $f: \mathbf{D} \to X$ verifying

$$\|f\|_p = Sup_{0<r<1}(\int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi})^{1/p} < \infty$$

for $1 \leq p < \infty$, and for $p = \infty$

$$\|f\|_\infty = Sup_{z \in \mathbf{D}} \|f(z)\| < \infty,$$

where $\mathbf{D}$ is the open unit disk of the complex plane. $H^p(X)$ equipped with the norm $\|.\|_p$ becomes a Banach space. $X$ is said to have the analytic Radon-Nikodym property (analytic RNP, in short), if there exists $1 \leq p \leq \infty$ (or equivalently for some $1 \leq p \leq \infty$), such that each $f \in H^p(X)$ has radial limits a.e. on $[0, 2\pi]$ in $X$; this means that for almost all $\theta \in [0, 2\pi]$, $\lim_{r \uparrow 1} f(re^{i\theta})$ exists in $X$ (see [1]).

An $X$-valued analytic martingale will be a sequence of integrable functions $f_n \in L^1([0, 2\pi]^n, X)$ such that $f_0 \equiv x_0 \in X$, and for every $n \in \mathbf{N}$, there exists $d_n \in L^1([0, 2\pi]^{n-1}, X)$ so that

$$f_n(\alpha_1, \alpha_2, \cdots, \alpha_n) - f_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$$

$$= d_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) e^{i\alpha_n}.$$

It is easy to see that each $X$-valued analytic martingale is an $X$-valued martingale in the usual sense. For $1 \leq p < \infty$, we shall denote by $\mathcal{A}_p(X)$ the space of all $L^p$-bounded $X$-valued analytic martingales. If $F = (f_n)_{n \geq 0} \in \mathcal{A}_p(X)$, define

$$\|F\|_p = Sup_{n \geq 0} \|f_n\|_p,$$

where $\|.\|_p$ is a norm on $\mathcal{A}_p(X)$ and it is not hard to verify that $\mathcal{A}_p(X)$, equipped with the norm $\|.\|_p$, becomes a complex Banach space.

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The analytic RNP has been extensively studied in the last ten years, for instance, it is shown by G.A. Edgar that a complex Banach space has the analytic RNP if and only if there exists $1 \leq p < \infty$, so that each $F = (f_n)_{n\geq 0} \in \mathcal{A}_p(X)$ converges in the $L^p$-norm (see [2]), and there exists an important relation between $H^p(X)$ and $\mathcal{A}_p(X)$ (see [3]). For more information about the analytic RNP, we refer to [3], [4], and [5].

M. Daher has established the following elegant characterisation of the analytic RNP for separable complex Banach spaces (see [6]).

**Theorem 1.** *Let $X$ be a separable complex Banach space. $X$ has the analytic RNP if and only if there exists $1 \leq p < \infty$ so that $H^p(X)$ is separable.*

The purpose of this paper is to establish the analogue of this result in the "analytic martingale" setting. Precisely we shall show the following

**Theorem 2.** *Let $X$ be a separable complex Banach space. $X$ has the analytic RNP if and only if there exists $1 \leq p < \infty$ such that $\mathcal{A}_p(X)$ is separable.*

Let $X$ be a complex Banach space, $F = (f_n)_{n\geq 0}$ an element of $\mathcal{A}_p(X)$, $f_0 \equiv x_0 \in X$ and

$$f_n(\alpha_1, \alpha_2, \cdots, \alpha_n) - f_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = d_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})e^{i\alpha_n}.$$

For fixed $(\theta_1, \theta_2, \cdots) \in [0, 2\pi]^{\mathbf{N}}$, $G = (g_n)_{n\geq 0}$ defined by $g_0 \equiv x_0$ and

$$g_n(\alpha_1, \alpha_2, \cdots, \alpha_n) - g_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$$

$$= d_n(\theta_1\alpha_1, \theta_2 + \alpha_2,\ \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\alpha_n}e^{i\theta_n}$$

is an $X$-valued analytic martingale and $G = (g_n)_{n\geq 0} \in \mathcal{A}_p(X)$. So for each $F = (f_n)_{n\geq 0} \in \mathcal{A}_p(X)$, one can define a function from $[0, 2\pi]^{\mathbf{N}}$ into $\mathcal{A}_p(X)$ by $S(\theta_1, \theta_2, \cdots) = (S_n(\theta_1, \theta_2, \cdots))_{n\geq 0} \in \mathcal{A}_p(X)$, by $S_0 \equiv x_0$ and

$$S_n(\theta_1, \theta_2, \cdots)(\alpha_1, \alpha_2, \cdots, \alpha_n) - S_{n-1}(\theta_1, \theta_2, \cdots)(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$$

$$= d_n(\theta_1\alpha_1, \theta_2 + \alpha_2,\ \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\alpha_n}e^{i\theta_n}.$$

The proof of Theorem 2 will use the following lemma.

**Lemma.** *Let $X$ be a complex Banach space, $1 \leq p < \infty$, and let $F = (f_n)_{n\geq 0}$ be an element in $\mathcal{A}_p(X)$; then $F = (f_n)_{n\geq 0}$ converges in the $L^p$-norm if and only if the function $S$ defined above is measurable.*

*Proof of the Lemma.* Let $F = (f_n)_{n\geq 0} \in \mathcal{A}_p(X)$; assume that $F = (f_n)_{n\geq 0}$ converges in the $L^p$-norm. There exists then $f \in L^p([0, 2\pi]^{\mathbf{N}}, X)$ such that for every $n \in \mathbf{N}$, if $\mathcal{F}_n$ is the $\sigma$-algebra on $[0, 2\pi]^{\mathbf{N}}$ generated by the first $n$ coordinates, then $f_n = \mathbf{E}(f|\mathcal{F}_n)$, where $\mathbf{E}(f|\mathcal{F}_n)$ denotes the expectation of $f$ with respect to the $\sigma$-algebra $\mathcal{F}_n$. The function

$$[0, 2\pi]^{\mathbf{N}} \times [0, 2\pi]^{\mathbf{N}} \to X,$$

$$((\alpha_i)_{i\geq 1}, (\theta_i)_{i\geq 1}) \to f(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots)$$

is clearly measurable and belongs to $L^p([0, 2\pi]^{\mathbf{N}} \times [0, 2\pi]^{\mathbf{N}}, X)$. Hence the function

$$[0, 2\pi]^{\mathbf{N}} \to \mathcal{A}_p(X),$$

$$(\theta_i)_{i\geq 1} \to (\mathbf{E}(f(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots)|\mathcal{F}_n))_{n\geq 0}$$

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is measurable, where the expectation in the expression above is taken for the variables $(\alpha_1, \alpha_2, \cdots) \in [0, 2\pi]^{\mathbf{N}}$. But $(\mathbf{E}(f(\alpha_1+\theta_1, \alpha_2+\theta_2, \cdots)|\mathcal{F}_n))_{n\geq 0} = S(\theta_1, \theta_2, \cdots)$; the function $S$ is therefore measurable.

Inversely, assume that the function $S$ is measurable; then for each $(\theta_i)_{i\geq 1} \in [0, 2\pi]^{\mathbf{N}}$, $\|S((\theta_i)_{i\geq 1})\|_p = \|F\|_p$, $S$ is a bounded measurable function. We have $S_n = \mathbf{E}(S|\mathcal{F}_n)$, where $S_n$ is defined by $S_0 \equiv x_0 \in \mathcal{A}_p(X)$ and for $n \in \mathbf{N}$, $(\theta_1, \theta_2, \cdots, \theta_n) \in [0, 2\pi]^n$ $S_n(\theta_1, \theta_2, \cdots, \theta_n)$ is an $X$-valued analytic martingale which only depends on the first $n$ coordinates and

$$(S_n(\theta_1, \theta_2, \cdots, \theta_n) - S_{n-1}(\theta_1, \theta_2, \cdots, \theta_{n-1}))(\alpha_1, \alpha_2, \cdots, \alpha_n)$$

$$= d_n(\theta_1\alpha_1, \theta_2 + \alpha_2, \ \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\theta_n}e^{i\alpha_n}.$$

Indeed, if $S(\theta_1, \theta_2, \cdots) = (T_n(\theta_1, \theta_2, \cdots))_{n\geq 0}$, then $T_0 \equiv x_0 \in \mathcal{A}_p(X)$ and

$$(T_n(\theta_1, \theta_2, \cdots) - T_{n-1}(\theta_1, \theta_2, \cdots))(\alpha_1, \alpha_2, \cdots, \alpha_n)$$

$$= d_n(\theta_1\alpha_1, \theta_2 + \alpha_2, \ \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\theta_n}e^{i\alpha_n}.$$

We have to show that for each $A \in \mathcal{F}_n$

$$(*) \qquad \int_A S(\Theta)d\mu(\Theta) = \int_A S_n(\Theta)d\mu(\Theta),$$

where $\Theta = (\theta_1, \theta_2, \cdots) \in [0, 2\pi]^{\mathbf{N}}$ and $\mu$ denotes normalized Lebesgue measure on the product space $[0, 2\pi]^{\mathbf{N}}$. Define

$$Q: \ \mathcal{A}_p(X) \to (X \times L^p([0, 2\pi], X) \times L^p([0, 2\pi]^2, X) \times \cdots)_\infty,$$

$$G = (g_n)_{n\geq 0} \to (g_0, \ g_1 - g_0, \ g_2 - g_1, \cdots),$$

where $Q$ is an injective continuous linear application. To show that the equality $(*)$ holds true, it will suffice to show that

$$Q\left(\int_A S(\Theta)d\mu(\Theta)\right) = Q\left(\int_a S_n(\Theta)d\mu(\Theta)\right)$$

or equivalently

$$(**) \qquad \int_A Q(S(\Theta))d\mu(\Theta) = \int_A Q(S_n(\Theta))d\mu(\Theta).$$

The above equality is an equality between elements in

$$(X \times L^p([0, 2\pi], X) \times L^p([0, 2\pi]^2, X) \times \cdots)_\infty.$$

To show that $(**)$ holds true, it is sufficient to show that the corresponding coordinates of $\int_A Q(S(\Theta))d\mu(\Theta)$ and $\int_A Q(S_n(\Theta))d\mu(\Theta)$ coincide. Let $Q(S(\Theta))_{(m)}$ be the $m^{th}$ coordinate of $Q(S(\Theta))$ and let $Q(S_n(\theta))_{(m)}$ be the $m^{th}$ coordinate of $Q(S_n(\Theta))$.

If $1 \leq m \leq n$,

$$Q(S(\Theta))_{(m)}(\alpha_1, \alpha_2, \cdots, \alpha_m) = d_m(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots, \alpha_{m-1} + \theta_{m-1})e^{i\theta_m}e^{i\alpha_m},$$

$$Q(S_n(\Theta))_{(m)}(\alpha_1, \alpha_2, \cdots, \alpha_m) = d_m(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots, \alpha_{m-1} + \theta_{m-1})e^{i\theta_m}e^{i\alpha_m};$$

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hence

$$\int_A Q(S(\Theta))_{(m)} d\mu(\Theta) = \int_A Q(S_n(\Theta))_{(m)} d\mu(\Theta).$$

If $m > n$, then

$$Q(S(\Theta))_{(m)}(\alpha_1, \alpha_2, \cdots, \alpha_m) = d_m(\alpha_1+\theta_1, \alpha_2+\theta_2, \cdots, \alpha_{m-1}+\theta_{m-1}) e^{i\theta_m} e^{i\alpha_m},$$

$$Q(S_n(\Theta))_{(m)}(\alpha_1, \alpha_2, \cdots, \alpha_m) = 0.$$

We get

$$\int_A Q(S(\Theta))_{(m)} d\mu(\Theta) = \int_A Q(S_n(\Theta))_{(m)} d\mu(\Theta) = 0,$$

which shows that for every $1 \le m < \infty$

$$\int_A Q(S(\Theta))_{(m)} d\mu(\Theta) = \int_A Q(S_n(\Theta))_{(m)} d\mu(\Theta),$$

and hence the equality $(**)$ holds true. As $S_n = \mathbf{E}(S|\mathcal{F}_n)$, $S_n$ converges to $S$ in $L^p([0,2\pi]^n, \mathcal{A}_p(X))$. $S_n$ is then a Cauchy sequence in $L^p([0,2\pi]^{\mathbf{N}}, \mathcal{A}_p(X))$. For $n, m \in \mathbf{N}$, we have

$$\|S_n - S_{n+m}\|_p$$

$$= \left( \int\int \| \sum_{k=n+1}^{n+m} d_k(\alpha_1+\theta_1, \cdots, \alpha_{k-1}+\theta_{k-1}) e^{i\theta_k} e^{i\alpha_k} \|^p d\mu(\Theta) d\mu(\alpha_1, \alpha_2, \cdots) \right)^{1/p}$$

$$= \left( \int \| \sum_{k=n+1}^{n+m} d_k(\theta_1, \theta_2, \cdots, \theta_{n-1}) e^{i\theta_k} \|^p d\mu(\Theta) \right)^{1/p}$$

$$= \|f_n - f_{n+m}\|_p;$$

hence $F = (f_n)_{n\ge 0}$ is a Cauchy sequence in $L^p([0,2\pi]^{\mathbf{N}}, X)$ and therefore converges in the $L^p$-norm in $X$. This finishes the proof of the Lemma.

*Proof of Theorem* 2. Suppose that $X$ is separable. If $X$ has the analytic RNP, every $F = (f_n)_{n\ge 0} \in \mathcal{A}_p(X)$ converges in the $L^p$-norm to an element $f$ of $L^p([0,2\pi]^{\mathbf{N}}, X)$ and $\|F\|_p = \|f\|_p$. $\mathcal{A}_p(X)$ is then identified with a closed subspace of $L^p([0,2\pi]^{\mathbf{N}}, X)$. As $L^p([0,2\pi]^{\mathbf{N}}, X)$ is separable, $\mathcal{A}_p(X)$ is separable.

Inversely, suppose that there exists $1 \le p < \infty$ such that $\mathcal{A}_p(X)$ is separable, and let $F = (f_n)_{n\ge 0} \in \mathcal{A}_p(X)$. We have to show that $F$ converges in the $L^p$-norm in $X$. By the lemma it is sufficient to show that the function $S$ defined above is measurable.

Let $G = (g_n)_{n\ge 0} \in \mathcal{A}_p(X)$ be fixed and $\epsilon > 0$; consider the ball

$$B(G,\ \epsilon) = \{H = (h_n)_{n\ge 0} \in \mathcal{A}_p(X) : \quad \|H - G\|_p \le \epsilon\}$$

$$= \bigcap_{n\ge 1} \{H = (h_n)_{n\ge 0} \in \mathcal{A}_p(X) : \quad \|g_n - f_n\|_p \le \epsilon\}.$$

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We get

$$S^{-1}(B(G,\ \epsilon)) = \bigcap_{n\geq 1} S^{-1}(\{H = (h_n)_{n\geq 0} \in \mathcal{A}_p(X) : \quad \|g_n - f_n\|_p \leq \epsilon\}).$$

But for fixed $n \in \mathbf{N}$

$$\begin{aligned} &S^{-1}(\{H = (h_n)_{n\geq 0} \in \mathcal{A}_p(X) : \|g_n - h_n\|_p \leq \epsilon\}) \\ &\quad = \{(\theta_1, \theta_2, \cdots) \in [0, 2\pi]^{\mathbf{N}} : \Big( \int \|g_n(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &\qquad\qquad - h_n(\theta_1 + \alpha_1, \theta_2 + \alpha_2, \cdots, \theta_n + \alpha_n)\|^p d\mu(\alpha_1, \alpha_2, \cdots) \Big)^{1/p} \leq \epsilon\} \end{aligned}$$

which is clearly a measurable subset of $[0, 2\pi]^{\mathbf{N}}$; hence $S^{-1}(B(G,\ \epsilon))$ is a measurable subset of $[0, 2\pi]^{\mathbf{N}}$. As the Borel sets of $\mathcal{A}_p(X)$ are generated by balls ($\mathcal{A}_p(X)$ is separable), the function $S$ is measurable. This finishes the proof.

Let $X$ be a complex Banach space. We shall denote by $H^p_0([0, 2\pi], X)$ the subspace of $L^1([0, 2\pi], X)$ consisting of all $f$, so that the Fourier coefficient $\hat{f}(n) = 0$ for a negative integer $n \in \mathbf{Z}$. An $X$-valued integrable sequence $F = (f_n)_{n\geq 0}$ is called an $X$-valued Hardy martingale (see [7]) if $f_0 \equiv x_0 \in X$, for each $n \in \mathbf{N}$, $f_n \in L^1([0, 2\pi]^n, X)$, and the function $\alpha_n \to f_n(\alpha_1, \alpha_2, \cdots, \alpha_n) - f_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ belongs to $H^1_0([0, 2\pi], X)$ for almost all $(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \in [0, 2\pi]^{n-1}$. It is easy to see that each analytic martingale is a Hardy martingale and every $X$-valued Hardy martingale is an $X$-valued martingale in the usual sense. $X$ has the analytic RNP if and only if there exists $1 \leq p < \infty$ such that every $L^p$-bounded $X$-valued Hardy martingale converges in the $L^p$-norm (see [5] and [7]). We denote by $\mathcal{H}_p(X)$ the space of all $L^p$-bounded $X$-valued Hardy martingales. For $F = (f_n)_{n\geq 0} \in \mathcal{H}_p(X)$, define $\|F\|_p = Sup_{n\geq 1}\|f_n\|_p$; $\|.\|_p$ thus defined is a norm on $\mathcal{H}_p(X)$. It is not hard to verify that $\mathcal{H}_p(X)$ equipped with this norm becomes a Banach space.

**Theorem 3.** *Let $X$ be a separable complex Banach space. $X$ has the analytic RNP if and only if there exists $1 \leq p < \infty$, such that $\mathcal{H}_p(X)$ is separable.*

*Proof of Theorem* 3. Let $X$ be a separable complex Banach space. If there exists $1 \leq p < \infty$ such that $\mathcal{H}_p(X)$ is separable, as $\mathcal{A}_p(X)$ is a closed subspace of $\mathcal{H}_p(X)$, $\mathcal{A}_p(X)$ is a separable Banach space, by Theorem 2, $X$ has the analytic RNP. Inversely, if $X$ has the analytic RNP, then, for every $1 \leq p < \infty$, each $L^p$-bounded $X$-valued Hardy martingale converges in the $L^p$-norm. For each $F = (f_n)_{n\geq 0} \in \mathcal{H}_p(X)$, there exists $f \in L^p([0, 2\pi]^{\mathbf{N}}, X)$ such that $\lim_{n\to\infty}\|f_n - f\|_p = 0$, $f_n = \mathbf{E}(f|\mathcal{F}_n)$, and $\|f\|_p = \|F\|_p$. $\mathcal{H}_p(X)$ is then identified with a subspace of $L^p([0, 2\pi]^{\mathbf{N}}, X)$. As $X$ is separable, $L^p([0, 2\pi]^{\mathbf{N}}, X)$ is separable; hence $\mathcal{H}_p(X)$ is a separable complex Banach space.

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