A NEW CHARACTERISATION OF THE ANALYTIC RADON-NIKODYM PROPERTY

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ABSTRACT. We show that a separable complex Banach space $X$ has the analytic Radon-Nikodym property if and only if there exists $1 < p < \infty$, such that the space consisting of all $L^p$-bounded $X$-valued analytic martingales is separable.

Let $(X, \|\cdot\|)$ be a complex Banach space and let $1 \leq p \leq \infty$. $H^p(X)$ will denote the space consisting of all analytic functions $f: D \to X$ verifying

$$\|f\|_p = \sup_{0 < r < 1} \left( \int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty$$

for $1 \leq p < \infty$, and for $p = \infty$

$$\|f\|_\infty = \sup_{z \in D} \|f(z)\| < \infty,$$

where $D$ is the open unit disk of the complex plane. $H^p(X)$ equipped with the norm $\|\cdot\|_p$ becomes a Banach space. $X$ is said to have the analytic Radon-Nikodym property (analytic RNP, in short), if there exists $1 \leq p \leq \infty$ (or equivalently for some $1 < p \leq \infty$), such that each $f \in H^p(X)$ has radial limits a.e. on $[0, 2\pi]$ in $X$; this means that for almost all $\theta \in [0, 2\pi]$, $\lim_{r \downarrow 0} f(re^{i\theta})$ exists in $X$ (see [1]).

An $X$-valued analytic martingale will be a sequence of integrable functions $f_n \in L^1([0, 2\pi]^n, X)$ such that $f_0 \equiv x_0 \in X$, and for every $n \in \mathbb{N}$, there exists $d_n \in L^1([0, 2\pi]^{n-1}, X)$ so that

$$f_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = f_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) + d_n(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})e^{i\alpha_n}.$$

It is easy to see that each $X$-valued analytic martingale is an $X$-valued martingale in the usual sense. For $1 \leq p < \infty$, we shall denote by $\mathcal{A}_p(X)$ the space of all $L^p$-bounded $X$-valued analytic martingales. If $F = (f_n)_{n \geq 0} \in \mathcal{A}_p(X)$, define

$$\|F\|_p = \sup_{n \geq 0} \|f_n\|_p,$$

where $\|\cdot\|_p$ is a norm on $\mathcal{A}_p(X)$ and it is not hard to verify that $\mathcal{A}_p(X)$, equipped with the norm $\|\cdot\|_p$, becomes a complex Banach space.

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The analytic RNP has been extensively studied in the last ten years, for instance, it is shown by G.A. Edgar that a complex Banach space has the analytic RNP if and only if there exists $1 \leq p < \infty$, so that each $F = (f_n)_{n \geq 0} \in A_p(X)$ converges in the $L^p$-norm (see [2]), and there exists an important relation between $H^p(X)$ and $A_p(X)$ (see [3]). For more information about the analytic RNP, we refer to [3, 4], and [5].

M. Daher has established the following elegant characterisation of the analytic RNP for separable complex Banach spaces (see [2]).

**Theorem 1.** Let $X$ be a separable complex Banach space. $X$ has the analytic RNP if and only if there exists $1 \leq p < \infty$ so that $H^p(X)$ is separable.

The purpose of this paper is to establish the analogue of this result in the “analytic martingale” setting. Precisely we shall show the following

**Theorem 2.** Let $X$ be a separable complex Banach space. $X$ has the analytic RNP if and only if there exists $1 \leq p < \infty$ such that $A_p(X)$ is separable.

Let $X$ be a complex Banach space, $F = (f_n)_{n \geq 0}$ an element of $A_p(X)$, $f_0 \equiv x_0 \in X$ and

$$f_n(\alpha_1, \alpha_2, \cdots, \alpha_n) - f_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = d_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})e^{i\alpha_n}.$$  

For fixed $(\theta_1, \theta_2, \cdots) \in [0, 2\pi]^N$, $G = (g_n)_{n \geq 0}$ defined by $g_0 \equiv x_0$ and

$$g_n(\alpha_1, \alpha_2, \cdots, \alpha_n) - g_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = d_n(\theta_1 \alpha_1, \theta_2 + \alpha_2, \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\alpha_n}e^{i\theta_n}$$

is an $X$-valued analytic martingale and $G = (g_n)_{n \geq 0} \in A_p(X)$. So for each $F = (f_n)_{n \geq 0} \in A_p(X)$, one can define a function from $[0, 2\pi]^N$ into $A_p(X)$ by $S(\theta_1, \theta_2, \cdots) = (S_n(\theta_1, \theta_2, \cdots))_{n \geq 0} \in A_p(X)$, by $S_0 \equiv x_0$ and

$$S_n(\theta_1, \theta_2, \cdots)(\alpha_1, \alpha_2, \cdots, \alpha_n) - S_{n-1}(\theta_1, \theta_2, \cdots)(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = d_n(\theta_1 \alpha_1, \theta_2 + \alpha_2, \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\alpha_n}e^{i\theta_n}.$$  

The proof of Theorem 2 will use the following lemma.

**Lemma.** Let $X$ be a complex Banach space, $1 \leq p < \infty$, and let $F = (f_n)_{n \geq 0}$ be an element in $A_p(X)$; then $F = (f_n)_{n \geq 0}$ converges in the $L^p$-norm if and only if the function $S$ defined above is measurable.

**Proof of the Lemma.** Let $F = (f_n)_{n \geq 0} \in A_p(X)$; assume that $F = (f_n)_{n \geq 0}$ converges in the $L^p$-norm. There exists then $f \in L^p([0, 2\pi]^N, X)$ such that for every $n \in \mathbb{N}$, if $\mathcal{F}_n$ is the $\sigma$-algebra on $[0, 2\pi]^N$ generated by the first $n$ coordinates, then $f_n = \mathbb{E}(f|\mathcal{F}_n)$, where $\mathbb{E}(f|\mathcal{F}_n)$ denotes the expectation of $f$ with respect to the $\sigma$-algebra $\mathcal{F}_n$. The function

$$[0, 2\pi]^N \times [0, 2\pi]^N \to X,$$

$$((\alpha_i)_{i \geq 1}, (\theta_i)_{i \geq 1}) \to f(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots)$$

is clearly measurable and belongs to $L^p([0, 2\pi]^N \times [0, 2\pi]^N, X)$. Hence the function

$$[0, 2\pi]^N \to A_p(X),$$

$$(\theta_i)_{i \geq 1} \to (\mathbb{E}(f(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots)|\mathcal{F}_n))_{n \geq 0}$$
is measurable, where the expectation in the expression above is taken for the variables \((\alpha_1, \alpha_2, \cdots) \in [0, 2\pi]^N\). But \(\mathbf{E}(f(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots) | F_n))_{n \geq 0} = S(\theta_1, \theta_2, \cdots)\); the function \(S\) is therefore measurable.

Conversely, assume that the function \(S\) is measurable; then for each \((\theta_i)_{i \geq 1} \in [0, 2\pi]^N\), \(\|S((\theta_i)_{i \geq 1})\|_p = \|F\|_p\). \(S\) is a bounded measurable function. We have \(S_n = \mathbf{E}(S | F_n)\), where \(S_n\) is defined by \(S_0 \equiv x_0 \in \mathcal{A}_p(X)\) and for \(n \in \mathbb{N}\), \((\theta_1, \theta_2, \cdots, \theta_n) \in [0, 2\pi]^n\) \(S_n(\theta_1, \theta_2, \cdots, \theta_n)\) is an \(X\)-valued analytic martingale which only depends on the first \(n\) coordinates and

\[
(S_n(\theta_1, \theta_2, \cdots, \theta_n) - S_{n-1}(\theta_1, \theta_2, \cdots, \theta_{n-1}))(\alpha_1, \alpha_2, \cdots, \alpha_n) = d_n(\theta_1 \alpha_1, \theta_2 + \alpha_2, \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\theta_n}e^{i\alpha_n}.
\]

Indeed, if \(S(\theta_1, \theta_2, \cdots) = (T_n(\theta_1, \theta_2, \cdots))_{n \geq 0}\), then \(T_0 \equiv x_0 \in \mathcal{A}_p(X)\) and

\[
(T_n(\theta_1, \theta_2, \cdots) - T_{n-1}(\theta_1, \theta_2, \cdots))(\alpha_1, \alpha_2, \cdots, \alpha_n) = d_n(\theta_1 \alpha_1, \theta_2 + \alpha_2, \cdots, \theta_{n-1} + \alpha_{n-1})e^{i\theta_n}e^{i\alpha_n}.
\]

We have to show that for each \(A \in \mathcal{F}_n\)

\[
(*) \quad \int_A S(\Theta) d\mu(\Theta) = \int_A S_n(\Theta) d\mu(\Theta),
\]

where \(\Theta = (\theta_1, \theta_2, \cdots) \in [0, 2\pi]^N\) and \(\mu\) denotes normalized Lebesgue measure on the product space \([0, 2\pi]^N\). Define

\[
Q : \mathcal{A}_p(X) \to (X \times L^p([0, 2\pi], X) \times L^p([0, 2\pi]^2, X) \times \cdots)_\infty,
\]

\[
G = (g_n)_{n \geq 0} \mapsto (g_0, g_1 - g_0, g_2 - g_1, \cdots),
\]

where \(Q\) is an injective continuous linear application. To show that the equality \((*)\) holds true, it will suffice to show that

\[
Q \left( \int_A S(\Theta) d\mu(\Theta) \right) = Q \left( \int_A S_n(\Theta) d\mu(\Theta) \right)
\]

or equivalently

\[
(**) \quad \int_A Q(S(\Theta)) d\mu(\Theta) = \int_A Q(S_n(\Theta)) d\mu(\Theta).
\]

The above equality is an equality between elements in

\[
(X \times L^p([0, 2\pi], X) \times L^p([0, 2\pi]^2, X) \times \cdots)_\infty.
\]

To show that \((**)\) holds true, it is sufficient to show that the corresponding coordinates of \(\int_A Q(S(\Theta)) d\mu(\Theta)\) and \(\int_A Q(S_n(\Theta)) d\mu(\Theta)\) coincide. Let \(Q(S(\Theta))_{(m)}\) be the \(m\)th coordinate of \(Q(S(\Theta))\) and let \(Q(S_n(\Theta))_{(m)}\) be the \(m\)th coordinate of \(Q(S_n(\Theta))\).

If \(1 \leq m \leq n\),

\[
Q(S(\Theta))_{(m)}(\alpha_1, \alpha_2, \cdots, \alpha_m) = d_m(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots, \alpha_{m-1} + \theta_{m-1})e^{i\theta_m}e^{i\alpha_m},
\]

\[
Q(S_n(\Theta))_{(m)}(\alpha_1, \alpha_2, \cdots, \alpha_m) = d_m(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \cdots, \alpha_{m-1} + \theta_{m-1})e^{i\theta_m}e^{i\alpha_m};
\]
hence

\[ \int_A Q(S(\Theta))(m) d\mu(\Theta) = \int_A Q(S_n(\Theta))(m) d\mu(\Theta). \]

If \( m > n \), then

\[ Q(S(\Theta))(m) = d_m(\alpha_1 + \theta_1, \alpha_2 + \theta_2, \ldots, \alpha_{m-1} + \theta_{m-1}) e^{i\theta_m} e^{i\alpha_m}, \]

\[ Q(S_n(\Theta))(m) = 0. \]

We get

\[ \int_A Q(S(\Theta))(m) d\mu(\Theta) = \int_A Q(S_n(\Theta))(m) d\mu(\Theta) = 0, \]

which shows that for every \( 1 \leq m < \infty \)

\[ \int_A Q(S(\Theta))(m) d\mu(\Theta) = \int_A Q(S_n(\Theta))(m) d\mu(\Theta), \]

and hence the equality (***) holds true. As \( S_n = E(S|F_n) \), \( S_n \) converges to \( S \) in \( L^p([0,2\pi]^N, \mathcal{A}_p(X)) \). \( S_n \) is then a Cauchy sequence in \( L^p([0,2\pi]^N, \mathcal{A}_p(X)) \). For \( n, m \in \mathbb{N} \), we have

\[ \|S_n - S_{n+m}\|_p \]

\[ = \left( \int \int \| \sum_{k=n+1}^{n+m} d_k(\alpha_1 + \theta_1, \ldots, \alpha_{k-1} + \theta_{k-1}) e^{i\theta_k} e^{i\alpha_k} \|_p d\mu(\Theta) d\mu(\alpha_1, \alpha_2, \ldots) \right)^{1/p} \]

\[ = \left( \int \int \| \sum_{k=n+1}^{n+m} d_k(\theta_1, \theta_2, \ldots, \theta_{n-1}) e^{i\theta_k} \|_p d\mu(\Theta) \right)^{1/p} \]

\[ = \|f_n - f_{n+m}\|_p; \]

hence \( F = (f_n)_{n \geq 0} \) is a Cauchy sequence in \( L^p([0,2\pi]^N, X) \) and therefore converges in the \( L^p \)-norm in \( X \). This finishes the proof of the Lemma.

**Proof of Theorem 2.** Suppose that \( X \) is separable. If \( X \) has the analytic RNP, every \( F = (f_n)_{n \geq 0} \in \mathcal{A}_p(X) \) converges in the \( L^p \)-norm to an element \( f \) of \( L^p([0,2\pi]^N, X) \) and \( \|F\|_p = \|f\|_p \). \( \mathcal{A}_p(X) \) is then identified with a closed subspace of \( L^p([0,2\pi]^N, X) \). As \( L^p([0,2\pi]^N, X) \) is separable, \( \mathcal{A}_p(X) \) is separable.

Inversely, suppose that there exists \( 1 \leq p < \infty \) such that \( \mathcal{A}_p(X) \) is separable, and let \( F = (f_n)_{n \geq 0} \in \mathcal{A}_p(X) \). We have to show that \( F \) converges in the \( L^p \)-norm in \( X \). By the lemma it is sufficient to show that the function \( S \) defined above is measurable.

Let \( G = (g_n)_{n \geq 0} \in \mathcal{A}_p(X) \) be fixed and \( \epsilon > 0 \); consider the ball

\[ B(G, \epsilon) = \{ H = (h_n)_{n \geq 0} \in \mathcal{A}_p(X) : \|H - G\|_p \leq \epsilon \} \]

\[ = \bigcap_{n \geq 1} \{ H = (h_n)_{n \geq 0} \in \mathcal{A}_p(X) : \|g_n - f_n\|_p \leq \epsilon \}. \]
We get
\[ S^{-1}(B(G, \epsilon)) = \bigcap_{n \geq 1} S^{-1}(\{ H = (h_n)_{n \geq 0} \in A_p(X) : \| g_n - f_n \|_p \leq \epsilon \}). \]

But for fixed \( n \in \mathbb{N} \)
\[ S^{-1}(\{ H = (h_n)_{n \geq 0} \in A_p(X) : \| g_n - h_n \|_p \leq \epsilon \}) = \{ (\theta_1, \theta_2, \cdots) \in [0, 2\pi]^N : \left( \int \| g_n(\alpha_1, \alpha_2, \cdots, \alpha_n) \right. \]
\[ \left. - h_n(\theta_1 + \alpha_1, \theta_2 + \alpha_2, \cdots, \theta_n + \alpha_n) \right\|^p d\mu(\alpha_1, \alpha_2, \cdots) \right)^{1/p} \leq \epsilon \]

which is clearly a measurable subset of \([0, 2\pi]^N\); hence \( S^{-1}(B(G, \epsilon)) \) is a measurable subset of \([0, 2\pi]^N\). As the Borel sets of \( A_p(X) \) are generated by balls \( (A_p(X) \) is separable), the function \( S \) is measurable. This finishes the proof.

Let \( X \) be a complex Banach space. We shall denote by \( H^p_0([0, 2\pi], X) \) the subspace of \( L^1([0, 2\pi], X) \) consisting of all \( f \), so that the Fourier coefficient \( \hat{f}(n) = 0 \) for a negative integer \( n \in \mathbb{Z} \). An \( X \)-valued integrable sequence \( F = (f_n)_{n \geq 0} \) is called an \( X \)-valued Hardy martingale (see [7]) if \( f_0 \equiv x_0 \in X \), for each \( n \in \mathbb{N} \), \( f_n \in L^1([0, 2\pi]^N, X) \), and the function \( \alpha_n \mapsto f_n(\alpha_1, \alpha_2, \cdots, \alpha_n) = f_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \) belongs to \( H^p_0([0, 2\pi], X) \) for almost all \((\alpha_1, \alpha_2, \cdots, \alpha_n) \in [0, 2\pi]^{n-1} \). It is easy to see that each analytic martingale is a Hardy martingale and every \( X \)-valued Hardy martingale is an \( X \)-valued martingale in the usual sense. \( X \) has the analytic RNP if and only if there exists \( 1 \leq p < \infty \) such that every \( L^p \)-bounded \( X \)-valued Hardy martingale converges in the \( L^p \)-norm (see [5] and [7]). We denote by \( H_p(X) \) the space of all \( L^p \)-bounded \( X \)-valued Hardy martingales. For \( F = (f_n)_{n \geq 0} \in H_p(X) \), define \( \| F \|_p = \sup_{n \geq 1} \| f_n \|_p \). \( \| \cdot \|_p \) thus defined is a norm on \( H_p(X) \). It is not hard to verify that \( H_p(X) \) equipped with this norm becomes a Banach space.

**Theorem 3.** Let \( X \) be a separable complex Banach space. \( X \) has the analytic RNP if and only if there exists \( 1 \leq p < \infty \) such that \( H_p(X) \) is separable.

**Proof of Theorem 3.** Let \( X \) be a separable complex Banach space. If there exists \( 1 \leq p < \infty \) such that \( H_p(X) \) is separable, as \( A_p(X) \) is a closed subspace of \( H_p(X) \), \( A_p(X) \) is a separable Banach space, by Theorem 2, \( X \) has the analytic RNP. Inversely, if \( X \) has the analytic RNP, then, for every \( 1 \leq p < \infty \), each \( L^p \)-bounded \( X \)-valued Hardy martingale converges in the \( L^p \)-norm. For each \( F = (f_n)_{n \geq 0} \in H_p(X) \), there exists \( f \in L^p([0, 2\pi]^N, X) \) such that \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \), \( f_n = E(f|F_n) \), and \( \| f \|_p = \| F \|_p \). \( H_p(X) \) is then identified with a subspace of \( L^p([0, 2\pi]^N, X) \). As \( X \) is separable, \( L^p([0, 2\pi]^N, X) \) is separable; hence \( H_p(X) \) is a separable complex Banach space.

**References**


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