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PSEUDO-DIFFERENTIAL OPERATORS AND MAXIMAL REGULARITY RESULTS FOR NON-AUTONOMOUS PARABOLIC EQUATIONS

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Abstract. In this paper, we show that a pseudo-differential operator associated to a symbol $a \in L^\infty(\mathbb{R} \times \mathbb{R}, \mathcal{L}(H))$ (H being a Hilbert space) which admits a holomorphic extension to a suitable sector of $\mathbb{C}$ acts as a bounded operator on $L^2(\mathbb{R}, H)$. By showing that maximal L^p-regularity for the non-autonomous parabolic equation $u'(t)+A(t)u(t) = f(t), u(0) = 0$ is independent of $p \in (1,\infty)$, we obtain as a consequence a maximal $L^p([0,T], H)$-regularity result for solutions of the above equation.

1. Introduction

A classical result in the theory of pseudo-differential operators states that an operator associated to a symbol belonging to the class S^0 acts as a bounded operator on $L^2(\mathbb{R}^N)$ (see e.g. [12], Ch.VI). It was observed in recent years that pseudo-differential operators with operator-valued symbols (i.e. symbols which take values in the space of bounded linear operators on a Banach space X) are very useful in proving so called maximal regularity results for autonomous parabolic evolution equations. For details and more information in this direction we refer to [2], [8], [4], [10], [3] and [7]. In this paper we examine maximal L^p-regularity results for *non-autonomous* equations of the form

$$\begin{aligned} u'(t) + A(t)u(t) &= f(t), \qquad t \in [0, T], \\ u(0) &= 0 \end{aligned}$$

via the technique of pseudo-differential operators with operator-valued symbols. Since operators $A(t)$ associated to specific boundary value problems arising in applications very often show non-smooth dependence on t, we are in particular interested in symbols $a(x, \xi)$ having non-smooth dependence on x.

It is one aim of this paper to show, roughly speaking, that a pseudo-differential operator associated to a symbol $a \in L^\infty(\mathbb{R} \times \mathbb{R}, \mathcal{L}(H))$, where H is a Hilbert space, which admits a bounded, holomorphic extension to a suitable sector of the complex plane, acts as a bounded operator on $L^2(\mathbb{R}, H)$. Considering in particular the symbol a given by $a(t, \tau) := A(t)(i\tau+A(t))^{-1}$ we obtain as a consequence a maximal $L^2([0, T], H)$-regularity result for (1.1) provided the family $A(t)_{t\in[0,T]}$ satisfies the so called Acquistapace-Terreni condition. Note that our result generalizes in particular

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the result of de Simon [5] on $L^2(0,T;H)$-regularity for the autonomous case, i.e. $A(t) = A$ for all $t \in [0,T]$, to equations of the form (1.1).

Observe that we allow that the domains $D(A(t))$ of $A(t)$ may vary with $t \in [0,T]$. Hence maximal regularity results for (1.1) cannot be obtained from those for the autonomous equation by simple perturbation techniques.

We remark that the maximal $L^2([0,T];H)$-regularity result for (1.1) is the first cornerstone in establishing mixed $L^p([0,T];L^q(\Omega))$-estimates $(1 < p,q < \infty)$ for equations of the form (1.1). The Calderón-Zygmund theory for operator-valued kernels as developed for instance in [11] allows us to prove that, for arbitrary Banach spaces X and $p \in (1,\infty)$, there is maximal $L^p(0,T;X)$-regularity for (1.1) if and only if there is maximal $L^2(0,T;X)$-regularity for (1.1). Hence we obtain maximal $L^p(0,T;H)$-regularity for (1.1).

In [6] we prove mixed $L^p - L^q$ estimates for the solution of (1.1) (under suitable assumptions on the heat kernels on the semigroups generated by $A(t)$), by interpolating between the $L^1 - L^1_w$ result proved in [6] and the $L^2 - L^2$ result stated as Theorem 2.1 below and by applying the fact that the property of maximal L^p-regularity is independent of p . We finally remark that our maximal regularity results may be used to prove existence and uniqueness results for semilinear problems of the form $u'(t) + A(t)u(t) = f(t,u(t)), u(0) = 0$. For details we refer to [6].

Throughout this paper we denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y, whenever X and Y are Banach spaces and by H a Hilbert space. If A is a linear operator in X, we denote its domain by $D(A)$, its resolvent set by $\rho(A)$ and its spectrum by $\sigma(A)$. Furthermore, we denote by $\mathcal{S}(\mathbb{R};X)$ the space of all rapidly decreasing smooth functions on $\mathbb{R}$. The Fourier transform $\widehat{f}$ of a function $f \in \mathcal{S}(\mathbb{R};X)$ is defined by

$$(\mathcal{F}f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x)dx, \qquad \xi \in \mathbb{R}.$$

Finally, we denote by C various constants which may differ from occurrence to occurrence but are always independent of the free variable of a given formula.

2. Pseudo-differential operators with non-smooth operator-valued symbols

For $\theta \in (0,\pi)$ set $\Sigma_\theta := \{z \in \mathbb{C}\backslash\{0\}; |\arg z| < \theta\}$. Let $a \in L^\infty(\mathbb{R}\times\mathbb{R};\mathcal{L}(H))$ and define the pseudo-differential operator

$$Op(a) : \mathcal{S}(\mathbb{R},H) \to BC(\mathbb{R},H)$$

with operator-valued symbol a by

$$(Op(a)u)(x) := \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{ix\xi}a(x,\xi)\hat{u}(\xi)d\xi, \quad x \in \mathbb{R},$$

where $\mathcal{S}(\mathbb{R},H)$ denotes the Schwartz space of rapidly decreasing smooth H-valued functions on $\mathbb{R}$.

2.1. Theorem. *Let $a \in L^\infty(\mathbb{R}\times\mathbb{R},\mathcal{L}(H))$ and assume that $\xi \mapsto a(x,\xi)$ admits a holomorphic $\mathcal{L}(H)$-valued extension $z \mapsto a(x,z)$ to Σ_θ and $-\Sigma_\theta$ for some $\theta \in (0,\frac{\pi}{2})$ such that $\sup_{z\in\Sigma_\theta, z\in-\Sigma_\theta}\sup_{x\in\mathbb{R}}\|a(x,z)\|_{\mathcal{L}(H)} < \infty$. Then the operator $Op(a)$, initially defined on $S(\mathbb{R},H)$, extends to a bounded operator on $L^2(\mathbb{R},H)$.*

Proof. Let $\alpha := \frac{\sin\theta}{2}$ and set $R := 1 + \frac{\alpha}{2}$. Choose $\varphi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}\varphi \subset (R^{-1}, R)$ such that

$$\int_{\mathbb{R}} \frac{\varphi^2(\tau)}{|\tau|} d\tau = 1.$$

For $u \in S(\mathbb{R}, H)$ we then have

$$(Op(a)u)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} a(x,\xi) \varphi^2(\frac{\xi}{\tau}) \hat{u}(\xi) d\xi \frac{1}{|\tau|} d\tau. \tag{2.1}$$

Furthermore, let $\Gamma := \{z \in \mathbb{C}; |z-1| = \alpha\}$ be positively oriented. By Cauchy's theorem we have

$$a(x,\xi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{a(x,\tau z)}{z - \xi/\tau} dz \qquad (x \in \mathbb{R})$$

for those $(\xi, \tau) \in \mathbb{R} \times \mathbb{R}$ satisfying $\varphi(\frac{\xi}{\tau}) \neq 0$. Inserting this in (2.1) we obtain by Fubini's theorem

$$(Op(a)u)(x) = \frac{1}{2\pi i} \int_{\Gamma} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi} e^{ix\xi} \frac{\varphi^2(\frac{\xi}{\tau})}{z - \frac{\xi}{\tau}} a(x,\tau z) \hat{u}(\xi) d\xi \frac{d\tau}{|\tau|} dz.$$

Setting $g_{z,\tau}(\xi) := \frac{\varphi(\frac{\xi}{\tau})}{z - \frac{\xi}{\tau}}$, $h_\tau(\xi) := \varphi(\frac{\xi}{\tau})$ and denoting the inner integral above by $I_{\tau,z}u$ we obtain

$$(I_{\tau,z}u)(x) = (\mathcal{F}^{-1}(g_{z,\tau}) * \mathcal{F}^{-1}(h_\tau) * a(x,\tau z)u)(x), \quad x \in \mathbb{R}.$$

Since $\mathrm{supp}\varphi \subset (R^{-1}, R)$ it follows from Plancherel's theorem that

$$(I_{\tau,z}u, I_{\rho,z}u)_{L^2(\mathbb{R},H)} = 0 \tag{2.2}$$

provided $\frac{\rho}{\tau} \in \mathbb{R} \setminus \{(R^{-2}, R^2)\}$. In order to estimate $Op(a)u$ in $L^2(\mathbb{R}, H)$ notice that

$$\|Op(a)u\|_{L^2(\mathbb{R},H)} \leq \frac{1}{2\pi} \int_{\Gamma} \|H_z u\|_{L^2(\mathbb{R},H)} dz, \tag{2.3}$$

where $H_z u := \int_{\mathbb{R}} I_{\tau,z} u \frac{d\tau}{|\tau|}$ is understood as an inproper integral in $L^2(\mathbb{R}, H)$. It follows from (2.2) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
\|H_z u\|^2_{L^2(\mathbb{R},H)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} (I_{\tau,z}u, I_{\rho,z}u)_{L^2(\mathbb{R},H)} \frac{d\rho}{|\rho|} \frac{d\tau}{|\tau|} \\
&= \int_{R^{-2}}^{R^2} \int_{\mathbb{R}} (I_{\tau,z}u, I_{\tau\rho,z}u)_{L^2(\mathbb{R},H)} \frac{d\tau}{|\tau|} \frac{d\rho}{|\rho|} \\
&\leq \int_{R^{-2}}^{R^2} \int_{\mathbb{R}} \|I_{\tau,z}u\|^2_{L^2(\mathbb{R},H)} \frac{d\tau}{|\tau|} \frac{d\rho}{|\rho|} \\
&= 4 \log R \int_{\mathbb{R}} \|I_{\tau,z}u\|^2_{L^2(\mathbb{R},H)} \frac{d\tau}{|\tau|}.
\end{aligned}$$

Observe that by Plancherel's theorem we have

$$\|I_{\tau,z}u\|_{L^2(\mathbb{R},H)} \leq \sup_{x\in\mathbb{R}} \|a(x,\tau z)\|_{\mathcal{L}(H)} \sup_{\eta\in\mathbb{R}} |\frac{\varphi(\eta)}{z-\eta}| \|h_\tau \hat{u}\|_{L^2(\mathbb{R},H)}.$$

Therefore there exists a constant $C > 0$ such that

$$\begin{aligned}\|H_z u\|^2_{L^2(\mathbb{R},H)} &\leq C\int\limits_{\mathbb{R}} \|h_\tau \hat{u}\|^2_{L^2(\mathbb{R},H)} \frac{d\tau}{|\tau|} \\ &= C\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} |\varphi(\frac{\xi}{\tau})|^2 \frac{d\tau}{|\tau|} \|\hat{u}(\xi)\|^2_H d\xi = C\|u\|^2_{L^2(\mathbb{R},H)}.\end{aligned}$$

Combining this estimate with (2.3) it follows that

$$\|Op(a)u\|_{L^2(\mathbb{R},H)} \leq C\|u\|_{L^2(\mathbb{R},H)}$$

for $u \in S(\mathbb{R}, H)$ and by density for all $u \in L^2(\mathbb{R}, H)$. □

3. Maximal regularity for non-autonomous parabolic equations

Let $T > 0$ and let $(A(t))_{t\in[0,T]}$ be a family of densely defined linear operators in X satisfying the following two assumptions:

A1) There exists $\theta \in (0, \pi/2)$ such that $\sigma(A(t)) \subset \Sigma_\theta$ for all $t \in [0, T]$ and for $\varphi \in (\theta, \pi)$ there exists $M > 0$ such that

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|}, \quad t \in [0,T], \lambda \in \mathbb{C}\backslash\Sigma_\varphi.$$

A2) There exist constants $\alpha, \beta \in [0, 1], \alpha < \beta$, $\omega \in (\theta, \pi/2)$, $c > 0$ such that

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\|_{\mathcal{L}(X)} \leq c\frac{|t-s|^\beta}{1 + |\lambda|^{1-\alpha}}$$

for $s, t \in [0, T], \lambda \in \mathbb{C}\backslash\Sigma_\omega$.

We remark that the above conditions A1), A2) on $A(t)$ were introduced and investigated by Acquistapace, Terreni [1] and Yagi [13] in order to construct the evolution operator associated with $A(t), t \in [0, T]$.

Let $1 < p < \infty$ and $f : [0, T] \to X$ be a function. We consider the following non-autonomous initial value problem:

$$\begin{aligned} u'(t) + A(t)u(t) &= f(t), \qquad t \in [0,T], \\ u(0) &= 0. \end{aligned} \tag{3.1}$$

The family $\{A(t), t \in [0, T]\}$ is said to belong to the *class* $MR(p, X)$ and we say that there is *maximal* L^p *regularity* for (3.1) if for each $f \in L^p(0, T; X)$ there exists a unique

$$u \in W^{1,p}(0,T;X) \quad \text{with} \quad t \mapsto A(t)u(t) \in L^p(0,T;X)$$

satisfying (3.1) in the $L^p(0, T; X)$-sense.

The following two theorems are the main results of this section.

3.1. Theorem. *Let X be a Banach space, $T > 0$, and assume that $\{A(t), t \in [0, T]\}$ satisfies A1) and A2). Suppose that there exists $p \in (1, \infty)$ such that the family $\{A(t), t \in [0, T]\}$ belongs to the class $MR(p, X)$. Then $\{A(t), t \in [0, T]\}$ belongs to $MR(q, X)$ for all $q \in (1, \infty)$.*

3.2. Theorem. *Let H be a Hilbert space, $1 < p < \infty$, $T > 0$ and assume that $\{A(t), t \in [0,T]\}$ satisfies A1) and A2). Then $\{A(t), t \in [0,T]\}$ belongs to $MR(p,H)$.*

We start the proof of the two theorems above with the following observation. It follows from the results in [1], [9] that if u is a solution of (3.1), then u fulfills

$$\begin{aligned} (3.2)\quad A(t)u(t) \quad &= \quad \int_0^t A(t)^2 e^{-(t-s)A(t)}(A(t)^{-1} - A(s)^{-1})A(s)u(s)ds \\ &+ \quad \int_0^t A(t)e^{-(t-s)A(t)}f(s)ds \end{aligned}$$

for $t \in [0,T]$. For the time being let $q \in (1,\infty)$ and define the operator $Q \in \mathcal{L}(L^q(0,T;X))$ by

$$(Qg)(t) := \int_0^t A(t)^2 e^{-(t-s)A(t)}(A(t)^{-1} - A(s)^{-1})g(s)ds, \qquad t \in [0,T].$$

The results in [1] and [9] imply that $\|Q\|_{\mathcal{L}(L^q(0,T;X))} \le 1/2$ provided the constant c in A2) is sufficiently small. Observe, however, that the family $\{A(t), t \in [0,T]\}$ belongs to the class $MR(q;X)$ if and only if this holds true for $\{A(t)+K, t \in [0,T]\}$, where K denotes an arbitrary constant. Hence, there is no loss of generality in choosing c as small as we want. It follows that the operator $Id - Q$ is invertible in $L^q(0,T;X)$. Moreover, by (3.2) we know that

$$(Id - Q)A(\cdot)u = Sf, \quad \text{where} \quad (Sf)(t) := \int_0^t A(t)e^{-(t-s)A(t)}f(s)ds$$

provided u is a solution of (3.1). Summarizing, we proved the following fact.

3.3. Proposition. *The family $\{A(t), t \in [0,T]\}$ belongs to the class $MR(q;X)$ if and only if S acts a bounded operator on $L^q(0,T;X)$.*

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. By assumption and Proposition 3.1 we know that S acts boundedly on $L^p(0,T;X)$. In order to show that S is bounded on $L^q(0,T;X)$ for $q \in (1,\infty)$, if suffices to verify (see [11], Theorems III.1.2, III.1.3) that

$$(3.3) \qquad \sup_{s,s' \in (0,T)} \int_{|s-s'| \le \frac{|t-s|}{2}} \|k(t,s) - k(t,s')\| dt < \infty,$$

$$(3.4) \qquad \sup_{s,s' \in (0,T)} \int_{|s-s'| \le \frac{|t-s|}{2}} \|k(s,t) - k(s',t)\| dt < \infty$$

where $k(t,s) := A(t)e^{-(t-s)A(t)}1_{(0,t)(s)}$.

To this end, note that for $s, s' \in (0,T)$ we have

$$\begin{aligned}
&\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} \|k(t,s)-k(t,s')\|dt \\
= &\int\limits_{|s-s'|\leq\frac{1}{2}(t-s)} \|A(t)e^{-(t-s)A(t)}1_{(0,t)}(s) - A(t)e^{-(t-s')A(t)}1_{(0,t)}(s')\|dt \\
= &\int\limits_{|s-s'|\leq\frac{1}{2}(t-s)} \|\int\limits_s^{s'} A(t)^2 e^{-(t-\sigma)A(t)}d\sigma\|dt \\
\leq &\int\limits_{|s-s'|\leq\frac{1}{2}(t-s)} |\int\limits_s^{s'} \frac{M}{(t-\sigma)^2}d\sigma|dt = M\int\limits_{|s-s'|\leq\frac{1}{2}(t-s)} |\frac{1}{t-s}-\frac{1}{t-s'}|dt \\
< \ &\infty.
\end{aligned}$$

Moreover, for $s, s' \in (0,T)$, we have

$$\begin{aligned}
&\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} \|k(s,t)-k(s',t)\|dt \\
= &\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} \|A(s)e^{-(s-t)A(s)}1_{s\geq t} - A(s')e^{-(s'-t)A(s')}1_{s'\geq t}\|dt \\
\leq &\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} \|A(s)e^{-(s-t)A(s)} - A(s)e^{-(s'-t)A(s)}\|dt \\
+ &\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} \|A(s)e^{-(s'-t)A(s)} - A(s')e^{-(s'-t)A(s')}\|dt \\
\leq &\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} \|\int\limits_s^{s'} A(s)^2 e^{-(\sigma-t)A(s)}d\sigma\|dt \\
+ &\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} \|\frac{1}{2\pi i}\int\limits_{\Gamma_\theta} \lambda e^{-(s'-t)\lambda}((\lambda-A(s))^{-1}-(\lambda-A(s'))^{-1})d\lambda\|dt \\
\leq &\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} |\int\limits_s^{s'} \frac{M}{|t-\sigma|^2}d\sigma|dt \\
+ &\int\limits_{|s-s'|\leq\frac{1}{2}|t-s|} (\frac{1}{\pi}\int\limits_0^{\infty} re^{-(s'-t)r\cos\theta}\frac{c(M+1)|s-s'|^\beta}{(1+r)^{1-\alpha}}dr)dt \\
< \ &\infty.
\end{aligned}$$

The proof is complete. ∎

Proof of Theorem 3.2. Observe that the symbol a defined by

$$a(t,\tau) := \begin{cases} A(0)(i\tau + A(0))^{-1}, & t < 0, \\ A(t)(i\tau + A(t))^{-1}, & t \in [0,T], \\ A(T)(i\tau + A(T))^{-1}, & t > T, \end{cases}$$

satisfies, thanks to A1), the assumptions of Theorem 2.1. Hence it follows from this theorem and Proposition 3.3 that the family $\{A(t), t \in [0,T]\}$ belongs to the class $MR(2;H)$. Theorem 3.1 implies now the assertion. □

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