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A CHARACTERIZATION OF MÖBIUS TRANSFORMATIONS

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ABSTRACT. Let $n \geq 2$ be an integer and let $\mathcal{D}$ be a domain of $\mathbb{R}^n$. Let $f : \mathcal{D} \to \mathbb{R}^n$ be an injective mapping which takes hyperspheres whose interior is contained in $\mathcal{D}$ to hyperspheres in $\mathbb{R}^n$. Then f is the restriction of a Möbius transformation.

1. Introduction

Let $n \geq 2$ be an integer. A theorem of A.D. Alexandrov [1] states that any bijective transformation of $\mathbb{R}^{n+1}$ which preserves the Lorentz distance 0 between pairs of points in both directions is the product of a Lorentz transformation and a dilatation. The following Theorem 1.3 is due to A.D. Alexandrov [2], J.A. Lester [7], and I. Popovici and D.C. Rădulescu [9] and generalizes Alexandrov's theorem.

Definition 1.1. Let $n \in \mathbb{N}$, $n \geq 2$. For $x, y \in \mathbb{R}^n$ let $x \cdot y$ denote the standard euclidean product between x and y. The Lorentz product, resp. Lorentz distance, between $x, y \in \mathbb{R}^{n+1}$ is defined by

$$\begin{aligned} x \diamond y &:= x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}, \\ d(x, y) &:= (y - x) \diamond (y - x). \end{aligned}$$

Definition 1.2 (cf. [6])**.** Let $n \in \mathbb{N}$, $n \geq 2$.

a) Let $\mathcal{D} \subset \mathbb{R}^n$. A mapping $f : \mathcal{D} \to \mathbb{R}^n$ is the restriction of a *Möbius transformation* if $\mathbb{R}\sigma_1\big(f(x)\big) = \mathbb{R}\big(\sigma_1(x) A_1\big)$ is satisfied for all $x \in \mathcal{D}$, where

$$\sigma_1(z) := \left(\frac{1 - z \cdot z}{2}, z, \frac{1 + z \cdot z}{2} \right)$$

for all $z \in \mathbb{R}^n$, and where A_1 is an $(n+2) \times (n+2)$-Lorentz matrix, $A_1 M_1 A_1^T = M_1 := \operatorname{diag}(1, \ldots, 1, -1)$.

b) Let $\mathcal{D} \subset \mathbb{R}^{n+1}$. A mapping $f : \mathcal{D} \to \mathbb{R}^{n+1}$ is the restriction of a *Lie transformation* if $\mathbb{R}\sigma_2\big(f(x)\big) = \mathbb{R}\big(\sigma_2(x) A_2\big)$ for all $x \in \mathcal{D}$, where

$$\sigma_2(z) := \left(\frac{1 - z \diamond z}{2}, z, \frac{1 + z \diamond z}{2} \right)$$

for all $z \in \mathbb{R}^{n+1}$, and where A_2 is an $(n+3) \times (n+3)$-matrix with $A_2 M_2 A_2^T = M_2 := \operatorname{diag}(1, \ldots, 1, -1, -1)$.

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Theorem 1.3. *Let $\mathcal{D}$ be a domain (i.e. an open, connected subset) of $\mathbb{R}^{n+1}$, $n \geq 2$. Let $f : \mathcal{D} \to \mathbb{R}^{n+1}$ be a mapping such that*

$$d(x, y) = 0 \quad \Leftrightarrow \quad d\big(f(x), f(y)\big) = 0$$

for all $x, y \in \mathcal{D}$. Then f is the restriction of a Lie transformation.

Alexandrov's theorem and Theorem 1.3 are important results in a modern field of geometrical research which is called *characterizations of geometrical mappings under mild hypotheses* [3], [4], [8]. In particular no regularity assumptions such as differentiability or even continuity are needed in these kinds of characterizations. In the same sense, C. Carathéodory proved [5] that any injective mapping of a domain $\mathcal{D}$ of $\mathbb{R}^2$ to $\mathbb{R}^2$ is the restriction of a Möbius transformation if the following condition is satisfied:

The image of any circle contained with its interior in $\mathcal{D}$, is itself a circle.

2. Results

There is a close connection between Carathéodory's theorem and Theorem 1.3 ($n = 2$). In fact we will generalize Carathéodory's theorem to arbitrary dimensions with the help of Theorem 1.3.

Theorem 2.1. *Let $n \geq 2$ be an integer and let $\mathcal{D}$ be a domain of $\mathbb{R}^n$. Let $f : \mathcal{D} \to \mathbb{R}^n$ be an injective mapping such that $f(H)$ is a hypersphere, whenever $H \subset \mathcal{D}$ is a hypersphere and the interior of H is contained in $\mathcal{D}$. Then f is the restriction of a Möbius transformation.*

Definition 2.2. A *similarity* of $\mathbb{R}^n$, $n \geq 2$, is a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = kxQ + t$ where $k > 0$, $t \in \mathbb{R}^n$, and Q is an orthogonal $n \times n$-matrix, $QQ^T = E$.

It is well known that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ which is induced by a Möbius transformation is a similarity. Hence, Theorem 2.1 implies the following corollary.

Corollary 2.3. *Let $n \geq 2$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an injective mapping such that images of euclidean hyperspheres are euclidean hyperspheres. Then f is a similarity.*

Now let $\mathcal{D}$ be the set $I^n := \big\{x \in \mathbb{R}^n \mid x \cdot x < 1\big\}$ of points in Poincaré's sphere model of n-dimensional hyperbolic geometry, $n \geq 2$. A hyperbolic hypersphere in I^n is a euclidean hypersphere which is contained in I^n. If $f : I^n \to I^n$ is induced by a Möbius transformation and if f is surjective, then f is a hyperbolic motion.

Corollary 2.4. *Let $n \geq 2$. Let $f : I^n \to I^n$ be a bijection of n-dimensional hyperbolic space which maps hyperbolic hyperspheres onto hyperbolic hyperspheres. Then f is a hyperbolic motion.*

3. Proof of Theorem 2.1

We show that, whenever H is a hypersphere contained in $\mathcal{D}$ such that the interior I of H is also contained in $\mathcal{D}$, then $f|_I$ is the restriction of a Möbius transformation. This implies Theorem 2.1 since

a) Any Möbius transformation is uniquely determined by its restriction to any non-empty open subset of $\mathbb{R}^n$.

b) For any two points $x, y \in \mathcal{D}$, there is a finite sequence $I_1, \ldots, I_k \subset \mathcal{D}$ of interiors of hyperspheres with $x \in I_1$, $y \in I_k$, $I_j \cap I_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$.

Let H be a hypersphere contained in $\mathcal{D}$ such that the interior I of H is also contained in $\mathcal{D}$.

1. Let I' denote the interior of the hypersphere $H' := f(H)$. Then either $f(I) \subset I'$ or $f(I) \subset \mathbb{R}^n \setminus (H' \cup I')$.

Proof. Let $x, y \in I$. Then there is a hypersphere $H_1 \subset I$ which contains x and y. Since f is injective and $f(H_1)$ is a hypersphere, either $f(H_1) \subset I'$ or $f(H_1) \subset \mathbb{R}^n \setminus (H' \cup I')$. Thus $f(x), f(y)$ are on the same side of $f(H_1)$.

2. Let $\mu : \mathbb{R}^n \setminus I' \to \mathbb{R}^n$ denote the restriction of a Möbius transformation which satisfies $\mu(H') = H'$ and $\mu\big(\mathbb{R}^n \setminus (I' \cup H')\big) \subset I'$. Let $g : H \cup I \to \mathbb{R}^n$ be defined by $g := f|_{H \cup I}$ if $f(I) \subset I'$, and $g := \mu \circ f|_{H \cup I}$ if $f(I) \subset \mathbb{R}^n \setminus (H' \cup I')$. Then $g(I) \subset I'$, $g(H) = H'$, and g is an injective mapping which takes hyperspheres in $H \cup I$ to hyperspheres in $H' \cup I'$.

3. Let $H_1 \subset I$ be a hypersphere with interior I_1. Then $g(I_1)$ is contained in the interior I_1' of $H_1' := g(H_1)$, and $g\big(I \setminus (H_1 \cup I_1)\big)$ is contained in the exterior of H_1'.

Proof. Let $z \in I \setminus (H_1 \cup I_1)$. There is a hypersphere $H_2 \subset H \cup I$ with $z \in H_2$, $\#(H \cap H_2) = 1$ and $H_1 \cap H_2 = \emptyset$. Then $H_2' := g(H_2) \subset H' \cup I'$, $g(z) \in H_2'$, $\#(H' \cap H_2') = 1$ and $H_1' \cap H_2' = \emptyset$. Hence $g(z) \notin I_1'$, and $g\big(I \setminus (H_1 \cup I_1)\big) \subset I' \setminus (H_1' \cup I_1')$. From the proof of 1. we know that $g(I_1)$ is either contained in the interior or in the exterior of H_1'. We take a hypersphere $H_3 \subset I$, $\#(H_1 \cap H_3) > 1$. Then $H_3 \cap I_1 \neq \emptyset$ and $\#\big(H_1' \cap g(H_3)\big) > 1$. Hence $g(H_3 \cap I_1) \cap I_1' \neq \emptyset$ and $g(I_1) \subset I_1'$.

Definition 3.1. Two hyperspheres $H_1, H_2 \subset \mathbb{R}^n$ are in *interior (exterior) contact* if $\#(H_1 \cap H_2) = 1$ and H_i is contained in the interior (exterior) of H_j where $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$.

4. Two hyperspheres $H_1, H_2 \subset I$ are in interior (exterior) contact iff $g(H_1), g(H_2)$ are in interior (exterior) contact.

Proof. Since g is injective, $\#(H_1 \cap H_2) = 1$ iff $\#\big(g(H_1) \cap g(H_2)\big) = 1$. The assertion now follows from 3.

Definition 3.2. For any hypersphere H_1 let $\gamma(H_1) \in \mathbb{R}^n$, $\rho(H_1) > 0$ denote the euclidean center and radius of H_1. Let $\lambda(H_1) := \big(\gamma(H_1), \rho(H_1)\big) \in \mathbb{R}^n \times \mathbb{R}_{>0}$.

5. Two distinct hyperspheres H_1, H_2 of $\mathbb{R}^n$ are in interior contact iff the Lorentz distance between $\lambda(H_1)$ and $\lambda(H_2)$ is zero.

6. The set $\mathcal{C} := \big\{\lambda(H_1) \mid H_1 \subset I \text{ is a hypersphere}\big\}$ is a domain of $\mathbb{R}^{n+1}$.

Proof. $\mathcal{C} = \big\{x \in \mathbb{R}^n \times]0, \rho(H)[\mid d\big(x, \lambda(H)\big) < 0\big\}$ is open and connected.

7. The mapping $\varphi := \lambda \circ g \circ \lambda^{-1} : \mathcal{C} \to \mathcal{C}' := \big\{\lambda(g(H_1)) \mid H_1 \subset I \text{ is a hypersphere}\big\}$ satisfies $d(x, y) = 0$ iff $d\big(\varphi(x), \varphi(y)\big) = 0$ for all $x, y \in \mathcal{C}$.

Proof. From 5. and 4., for all distinct hyperspheres $H_1, H_2 \subset I$,

$$\begin{aligned} d\big(\lambda(H_1), \lambda(H_2)\big) = 0 \quad &\Leftrightarrow \quad H_1 \text{ and } H_2 \text{ are in interior contact} \\ &\Leftrightarrow \quad g(H_1) \text{ and } g(H_2) \text{ are in interior contact} \\ &\Leftrightarrow \quad d\big(\lambda(g(H_1)), \lambda(g(H_2))\big) = 0. \end{aligned}$$

8. From 7. and Theorem 1.3, φ is the restriction of a Lie transformation, i.e. there is an $(n+3) \times (n+3)$-matrix $A_2 =: (a_{ij})_{i,j=1,\dots,n+3}$ as in Definition 1.2 b), such that $\mathbb{R}\sigma_2(y) = \mathbb{R}\big(\sigma_2(x)A_2\big)$ for all $x \in \mathcal{C}$, $y = \varphi(x)$.

Definition 3.3. A *light line* of $\mathbb{R}^{n+1}$ is a line $u + \mathbb{R}v$, $u, v \in \mathbb{R}^{n+1}$, $v \neq 0$, where $d(v,v) = 0$.

9. $f|_I$ is the restriction of a Möbius transformation.

Proof. Let $x \in I$. Let l_1, l_2 be two distinct light lines which contain $(x, 0)$. Then $\{(x,0)\} = l_1 \cap l_2 \subset \partial\mathcal{C}$. The images $\varphi(l_i \cap \mathcal{C}) \neq \emptyset$ are contained in uniquely determined light lines l_i', $i = 1, 2$. Since φ is continuous, $\{(g(x), 0)\} = l_1' \cap l_2'$ is contained in $\partial\mathcal{C}'$. Hence for all $x \in I$, we have $\mathbb{R}\sigma_2\big((g(x),0)\big) = \mathbb{R}\big(\sigma_2\big((x,0)\big)A_2\big)$ which implies

$$\mathbb{R}\sigma_1\big(g(x)\big) = \mathbb{R}\big(\sigma_1(x)A_1\big), \tag{3.1}$$

$$\sigma_1(x) \cdot (a_{1,n+2}, \dots, a_{n+1,n+2}, a_{n+3,n+2}) = 0 \tag{3.2}$$

where A_1 is the $(n+2)\times(n+2)$-matrix obtained from A_2 by deleting the $(n+2)$th row and $(n+2)$th column. Equation (3.2) is a quadratic equation in $x = (x_1, \dots, x_n)$ which holds for any $x \in I$, and we obtain $a_{1,n+2} = \dots = a_{n+1,n+2} = a_{n+3,n+2} = 0$. Together with $A_2M_2A_2^T = M_2$ we have $A_1M_1A_1^T = M_1$, where M_1 is chosen as in Definition 1.2 a). Equation (3.1) implies that g is the restriction of a Möbius transformation. Hence also $f|_I$ is the restriction of a Möbius transformation. ☐

Remark 3.4. It is possible to prove Theorem 2.1 by Carathéodory's theorem. If $n = 3$ and $f : \mathcal{D} \to \mathbb{R}^3$ is injective and has the sphere preserving property, then we can apply Carathéodory's theorem to any hypersphere $H \subset \mathcal{D}$ whose interior is contained in $\mathcal{D}$, after removing a point $p \in H$ and $f(p) \in f(H)$, to show that f is a Möbius transformation between H and its image $f(H)$. This Möbius transformation is the restriction of the same Möbius transformation for all hyperspheres. Induction proves the result for all $n \geq 2$.

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