A CHARACTERIZATION OF MöBIUS TRANSFORMATIONS

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Abstract. Let \( n \geq 2 \) be an integer and let \( D \) be a domain of \( \mathbb{R}^n \). Let \( f : D \to \mathbb{R}^n \) be an injective mapping which takes hyperspheres whose interior is contained in \( D \) to hyperspheres in \( \mathbb{R}^n \). Then \( f \) is the restriction of a Möbius transformation.

1. INTRODUCTION

Let \( n \geq 2 \) be an integer. A theorem of A.D. Alexandrov [1] states that any bijective transformation of \( \mathbb{R}^{n+1} \) which preserves the Lorentz distance \( 0 \) between pairs of points in both directions is the product of a Lorentz transformation and a dilatation. The following Theorem 1.3 is due to A.D. Alexandrov [2], J.A. Lester [7], and I. Popovici and D.C. Rădulescu [9] and generalizes Alexandrov’s theorem.

Definition 1.1. Let \( n \in \mathbb{N}, n \geq 2 \). For \( x, y \in \mathbb{R}^n \) let \( x \cdot y \) denote the standard euclidean product between \( x \) and \( y \). The Lorentz product, resp. Lorentz distance, between \( x, y \in \mathbb{R}^{n+1} \) is defined by
\[
x \circ y := x_1y_1 + \ldots + x_ny_n - x_{n+1}y_{n+1},
\]
\[
d(x, y) := (y - x) \circ (y - x).
\]

Definition 1.2 (cf. [6]). Let \( n \in \mathbb{N}, n \geq 2 \).

a) Let \( D \subset \mathbb{R}^n \). A mapping \( f : D \to \mathbb{R}^n \) is the restriction of a Möbius transformation if \( \mathbb{R}\sigma_1(f(x)) = \mathbb{R}(\sigma_1(x)A_1) \) is satisfied for all \( x \in D \), where
\[
\sigma_1(z) := \left( \frac{1}{2} - \frac{z}{2}, \frac{z}{2}, \frac{1 + z \cdot z}{2} \right)
\]
for all \( z \in \mathbb{R}^n \), and where \( A_1 \) is an \((n + 2) \times (n + 2)\)-Lorentz matrix, \( A_1M_1A_1^T = M_1 := \text{diag}(1, \ldots, 1, -1) \).

b) Let \( D \subset \mathbb{R}^{n+1} \). A mapping \( f : D \to \mathbb{R}^{n+1} \) is the restriction of a Lie transformation if \( \mathbb{R}\sigma_2(f(x)) = \mathbb{R}(\sigma_2(x)A_2) \) for all \( x \in D \), where
\[
\sigma_2(z) := \left( \frac{1}{2} - \frac{z \circ z}{2}, z, \frac{1 + z \circ z}{2} \right)
\]
for all \( z \in \mathbb{R}^{n+1} \), and where \( A_2 \) is an \((n + 3) \times (n + 3)\)-matrix with \( A_2M_2A_2^T = M_2 := \text{diag}(1, \ldots, 1, -1, -1) \).

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Theorem 1.3. Let $D$ be a domain (i.e. an open, connected subset) of $\mathbb{R}^{n+1}$, $n \geq 2$. Let $f : D \to \mathbb{R}^{n+1}$ be a mapping such that
d(x,y) = 0 \iff d(f(x), f(y)) = 0
for all $x,y \in D$. Then $f$ is the restriction of a Lie transformation.

Alexandrov’s theorem and Theorem 1.3 are important results in a modern field of geometrical research which is called characterizations of geometrical mappings under mild hypotheses [3], [4], [8]. In particular no regularity assumptions such as differentiability or even continuity are needed in these kinds of characterizations. In the same sense, C. Carathéodory proved [5] that any injective mapping of a domain $D$ of $\mathbb{R}^2$ to $\mathbb{R}^2$ is the restriction of a Möbius transformation if the following condition is satisfied:
The image of any circle contained with its interior in $D$, is itself a circle.

2. RESULTS

There is a close connection between Carathéodory’s theorem and Theorem 1.3 ($n = 2$). In fact we will generalize Carathéodory’s theorem to arbitrary dimensions with the help of Theorem 1.3.

Theorem 2.1. Let $n \geq 2$ be an integer and let $D$ be a domain of $\mathbb{R}^n$. Let $f : D \to \mathbb{R}^n$ be an injective mapping such that $f(H)$ is a hypersphere, whenever $H \subset D$ is a hypersphere and the interior of $H$ is contained in $D$. Then $f$ is the restriction of a Möbius transformation.

Definition 2.2. A similarity of $\mathbb{R}^n$, $n \geq 2$, is a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = kxQ + t$ where $k > 0$, $t \in \mathbb{R}^n$, and $Q$ is an orthogonal $n \times n$-matrix, $QQ^T = E$.

It is well known that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ which is induced by a Möbius transformation is a similarity. Hence, Theorem 2.1 implies the following corollary.

Corollary 2.3. Let $n \geq 2$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an injective mapping such that images of euclidean hyperspheres are euclidean hyperspheres. Then $f$ is a similarity.

Now let $D$ be the set $I^n := \{ x \in \mathbb{R}^n \mid x \cdot x < 1 \}$ of points in Poincaré’s sphere model of $n$-dimensional hyperbolic geometry, $n \geq 2$. A hyperbolic hypersphere in $I^n$ is a euclidean hypersphere which is contained in $I^n$. If $f : I^n \to I^n$ is induced by a Möbius transformation and if $f$ is surjective, then $f$ is a hyperbolic motion.

Corollary 2.4. Let $n \geq 2$. Let $f : I^n \to I^n$ be a bijection of $n$-dimensional hyperbolic space which maps hyperbolic hyperspheres onto hyperbolic hyperspheres. Then $f$ is a hyperbolic motion.

3. PROOF OF THEOREM 2.1

We show that, whenever $H$ is a hypersphere contained in $D$ such that the interior $I$ of $H$ is also contained in $D$, then $f|_I$ is the restriction of a Möbius transformation. This implies Theorem 2.1 since
a) Any Möbius transformation is uniquely determined by its restriction to any non-empty open subset of $\mathbb{R}^n$.

b) For any two points $x,y \in D$, there is a finite sequence $I_1, \ldots, I_k \subset D$ of interiors of hyperspheres with $x \in I_1$, $y \in I_k$, $I_j \cap I_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$.

Let $H$ be a hypersphere contained in $D$ such that the interior $I$ of $H$ is also contained in $D$. 
1. Let \( I' \) denote the interior of the hypersphere \( H' := f(H) \). Then either \( f(I) \subset I' \) or \( f(I) \subset \mathbb{R}^n \setminus (H' \cup I') \).

**Proof.** Let \( x, y \in I \). Then there is a hypersphere \( H_1 \subset I \) which contains \( x \) and \( y \). Since \( f \) is injective and \( f(H_1) \) is a hypersphere, either \( f(H_1) \subset I' \) or \( f(H_1) \subset \mathbb{R}^n \setminus (H' \cup I') \). Thus \( f(x), f(y) \) are on the same side of \( f(H_1) \).

2. Let \( \mu : \mathbb{R}^n \setminus I' \to \mathbb{R}^n \) denote the restriction of a Möbius transformation which satisfies \( \mu(H') = H' \) and \( \mu(\mathbb{R}^n \setminus (I' \cup H')) \subset I' \). Let \( g : H \cup I \to \mathbb{R}^n \) be defined by \( g := f|_{H \cup I} \) if \( f(I) \subset I' \), and \( g := \mu \circ f|_{H \cup I} \) if \( f(I) \subset \mathbb{R}^n \setminus (H' \cup I') \). Then \( g(I) \subset I' \), \( g(H) = H' \), and \( g \) is an injective mapping which takes hyperspheres in \( H \cup I \) to hyperspheres in \( H' \cup I' \).

3. Let \( H_1 \subset I \) be a hypersphere with interior \( I_1 \). Then \( g(I_1) \) is contained in the interior \( I_1' \) of \( H_1' := g(H_1) \), and \( g(I \setminus (H_1 \cup I_1)) \) is contained in the exterior of \( H_1' \).

**Proof.** Let \( z \in I \setminus (H_1 \cup I_1) \). There is a hypersphere \( H_2 \subset H \cup I \) with \( z \in H_2 \), \#(\( H \cap H_2 \)) = 1 and \( H_1 \cap H_2 = \emptyset \). Then \( H_2' := g(H_2) \subset H' \cup I' \), \( g(z) \in H_2' \), \#(\( H' \cap H_2' \)) = 1 and \( H_1' \cap H_2' = \emptyset \). Hence \( g(z) \notin I_1' \), and \( g(I \setminus (H_1 \cup I_1)) \subset I' \setminus (H_1' \cup I_1') \). From the proof of 1, we know that \( g(I_1) \) is either contained in the interior or in the exterior of \( H_1' \). We take a hypersphere \( H_3 \subset I \), \#(\( H_1 \cap H_3 \)) > 1. Then \( H_3 \cap I_1 = \emptyset \) and \#(\( H_1' \cap g(H_3) \)) > 1. Hence \( g(H_3 \cap I_1) \cap I_1' \neq \emptyset \) and \( g(I_1) \subset I_1' \).

**Definition 3.1.** Two hyperspheres \( H_1, H_2 \subset \mathbb{R}^n \) are in interior (exterior) contact if \#(\( H_1 \cap H_2 \)) = 1 and \( H_1 \) is contained in the interior (exterior) of \( H_2 \) where \( (i, j) = (1, 2) \) or \( (i, j) = (2, 1) \).

4. Two hyperspheres \( H_1, H_2 \subset I \) are in interior (exterior) contact iff \( g(H_1), g(H_2) \) are in interior (exterior) contact.

**Proof.** Since \( g \) is injective, \#(\( H_1 \cap H_2 \)) = 1 iff \#(\( g(H_1) \cap g(H_2) \)) = 1. The assertion now follows from 3.

**Definition 3.2.** For any hypersphere \( H_1 \) let \( \gamma(H_1) \in \mathbb{R}^n \), \( \rho(H_1) > 0 \) denote the euclidean center and radius of \( H_1 \). Let \( \lambda(H_1) := (\gamma(H_1), \rho(H_1)) \in \mathbb{R}^n \times \mathbb{R}_{>0} \).

5. Two distinct hyperspheres \( H_1, H_2 \) of \( \mathbb{R}^n \) are in interior contact iff the Lorentz distance between \( \lambda(H_1) \) and \( \lambda(H_2) \) is zero.

6. The set \( \mathcal{C} := \{\lambda(H_1) \mid H_1 \subset I \text{ is a hypersphere}\} \) is a domain of \( \mathbb{R}^{n+1} \).

**Proof.** \( \mathcal{C} = \{x \in \mathbb{R}^n \times [0, \rho(H)] \mid d(x, \lambda(H)) < 0\} \) is open and connected.

7. The mapping \( \varphi := \lambda \circ g \circ \lambda^{-1} : \mathcal{C} \to \mathcal{C}' := \{\lambda(g(H_1)) \mid H_1 \subset I \text{ is a hypersphere}\} \) satisfies \( d(x, y) = 0 \) iff \( d(\varphi(x), \varphi(y)) = 0 \) for all \( x, y \in \mathcal{C} \).

**Proof.** From 5. and 4., for all distinct hyperspheres \( H_1, H_2 \subset I \),

\[
   d(\lambda(H_1), \lambda(H_2)) = 0 \iff H_1 \text{ and } H_2 \text{ are in interior contact}
\]

\[
   \iff g(H_1) \text{ and } g(H_2) \text{ are in interior contact}
\]

\[
   \iff d(\lambda(g(H_1)), \lambda(g(H_2))) = 0.
\]
8. From 7. and Theorem 1.3, \( \varphi \) is the restriction of a Lie transformation, i.e. there is an \((n + 3) \times (n + 3)\)-matrix \( A_2 =: (a_{ij})_{i,j=1,\ldots,n+3} \) as in Definition 1.2 b), such that \( \Re \sigma_2(y) = \Re (\sigma_2(x) A_2) \) for all \( x \in C \), \( y = \varphi(x) \).

**Definition 3.3.** A light line of \( \mathbb{R}^{n+1} \) is a line \( u + \mathbb{R}v \), \( u, v \in \mathbb{R}^{n+1} \), \( v \neq 0 \), where \( d(v, v) = 0 \).

9. \( f|_I \) is the restriction of a Möbius transformation.

**Proof.** Let \( x \in I \). Let \( l_1, l_2 \) be two distinct light lines which contain \((x, 0)\). Then \( \{ (x, 0) \} = l_1 \cap l_2 \subset \partial C \). The images \( \varphi(l_1 \cap C) \neq \emptyset \) are contained in uniquely determined light lines \( l'_i \), \( i = 1, 2 \). Since \( \varphi \) is continuous, \( \{ (g(x), 0) \} = l'_1 \cap l'_2 \) is contained in \( \partial C' \). Hence for all \( x \in I \), we have \( \Re \sigma_2((g(x), 0)) = \Re (\sigma_2((x, 0)) A_2) \) which implies

\[
\Re \sigma_1(g(x)) = \Re (\sigma_1(x) A_1),
\]

\[
\sigma_1(x) \cdot (a_{1,n+2}, \ldots, a_{n+1,n+2}, a_{n+3,n+2}) = 0
\]

where \( A_1 \) is the \((n+2) \times (n+2)\)-matrix obtained from \( A_2 \) by deleting the \((n+2)\)th row and \((n+2)\)th column. Equation \((3.2)\) is a quadratic equation in \( x = (x_1, \ldots, x_n) \) which holds for any \( x \in I \), and we obtain \( a_{1,n+2} = \ldots = a_{n+1,n+2} = a_{n+3,n+2} = 0 \). Together with \( A_2M_2A_2^T = M_2 \) we have \( A_1M_1A_1^T = M_1 \), where \( M_1 \) is chosen as in Definition 1.2 a). Equation \((3.1)\) implies that \( g \) is the restriction of a Möbius transformation. Hence also \( f|_I \) is the restriction of a Möbius transformation.

**Remark 3.4.** It is possible to prove Theorem 2.1 by Carathéodory’s theorem. If \( n = 3 \) and \( f : D \to \mathbb{R}^3 \) is injective and has the sphere preserving property, then we can apply Carathéodory’s theorem to any hypersphere \( H \subset D \) whose interior is contained in \( D \), after removing a point \( p \in H \) and \( f(p) \in f(H) \), to show that \( f \) is a Möbius transformation between \( H \) and its image \( f(H) \). This Möbius transformation is the restriction of the same Möbius transformation for all hyperspheres. Induction proves the result for all \( n \geq 2 \).

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**References**


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