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BOUNDEDNESS OF INTEGRAL OPERATORS ON GENERALIZED MORREY SPACES AND ITS APPLICATION TO SCHRÖDINGER OPERATORS

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Abstract. In this paper, we study boundedness of integral operators on generalized Morrey spaces and its application to estimates in Morrey spaces for the Schrödinger operator $L_2 = -\Delta + V(x) + W(x)$ with nonnegative $V \in (RH)_\infty$ (reverse Hölder class) and small perturbed potentials W.

1. Introduction

In this paper, we study boundedness of integral operators with the kernel of Riesz type on generalized Morrey spaces (Theorems 2.1, 2.2) and its application to the study of mapping properties of the Schrödinger operator $L_2 = -\Delta + V(x) + W(x)$ with nonnegative $V \in (RH)_\infty$ (reverse Hölder class) and small perturbed potentials W (Theorems 1.2, 1.3, 3.1, 3.2). We say $V \in (RH)_\infty$ if there exists a constant C such that

$$\sup_{y\in B(x,r)} |V(y)| \le C\frac{1}{|B(x,r)|}\int_{B(x,r)} |V(y)|\,dy \tag{1.1}$$

for every $x \in \mathbf{R}^n$ and $r > 0$, where $B(x,r) = \{y \in \mathbf{R}^n; |y-x| < r\}$. As a typical example, for any polynomial $P(x)$ and $\alpha > 0$, $V(x) = |P(x)|^\alpha$ belongs to this class. For $L_1 = -\Delta + V, V \in (RH)_\infty, V \not\equiv 0$, we already know nice estimates for the fundamental solution $\Gamma(x,y)$ for L_1 due to Z. Shen [Sh1]: for every $k > 0$ there exists a constant C_k such that

$$|\Gamma(x,y)| \le \frac{C_k}{(1+m(x,V)|x-y|)^k |x-y|^{n-2}}, \tag{1.2}$$

$$|\nabla_x \Gamma(x,y)| + |\nabla_y \Gamma(x,y)| \le \frac{C_k}{(1+m(x,V)|x-y|)^k |x-y|^{n-1}}, \tag{1.3}$$

where $m(x,V)$ is defined by

$$\frac{1}{m(x,V)} = \sup\{r > 0; \frac{r^2}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy \le 1\}. \tag{1.4}$$

We note that $V \in (RH)_\infty$ satisfies $V(x) \le Cm(x,V)^2$ for some constant C. For further properties of $m(x,V)$, we refer to [Sh1]. The estimates (1.2)-(1.3) enable

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us to obtain the boundedness of L_1^{-1}, the integral operator with the kernel $\Gamma(x,y)$, on various spaces. In particular, we can prove that there exists a constant C such that

$$|m(x,V)^2(L_1^{-1}f)(x)| + |m(x,V)\nabla L_1^{-1}f(x)| \le CM(|f|)(x)$$

for $f \in L_c^\infty(\mathbf{R}^n)$, where $M(|f|)$ is the Hardy-Littlewood maximal function of f and $L_c^\infty(\mathbf{R}^n) = \{f \in L^\infty(\mathbf{R}^n); \textit{supp } f \textit{ is compact}\}$ (see [KS] and Lemma 3.1). The main purpose of this paper is to show estimates of L_2^{-1} (see Definition 1.2) for certain perturbed potentials W (not necessarily nonnegative) on various spaces. We should remark that, if we consider an example $W(x) = -\epsilon/|x|^2$, we cannot expect such nice estimates for the fundamental solution to L_2. Our method relies on the estimates for L_1 and a simple perturbation argument. To develop a perturbation theory, we consider integral operators T and S_M defined by

$$Tf(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n/s}}\,dy \tag{1.5}$$

and

$$S_M f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n/s}(1+|x-y|)^M}\,dy \tag{1.6}$$

for some $s > 1$ and $M > 0$. Many authors studied the boundedness of T on various spaces, e.g., on L^p spaces, weighted L^p spaces, and classical Morrey spaces. Among them, we note the recent result due to Olsen, which covers all previous results, except for weighted L^p spaces (see, e.g., [Ad], [CF], [Ta]). We recall that a function f is said to belong to the classical Morrey space $M_p^r, 1 \le p \le r$, if

$$\|f\|_{M_p^r} \equiv \sup_{Q \subset \mathbf{R}^n} |Q|^{1/r-1/p}\left(\int_Q |f(x)|^p\,dx\right)^{1/p} < +\infty \tag{1.7}$$

holds, where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$. For a function $W(x)$, we denote by $W\cdot T$ the product of W and T, where W is the multiplication operator by $W(x)$. The following theorem is due to Olsen [Ol], although Olsen stated the estimate only in the three-dimensional case.

Theorem 1.1 ([Ol]). *Let $n \ge 3$ and $1/v + 1/s = 1$. Assume $1 < p \le r, p < u \le v, 1/r + 1/s > 1$, and $W \in M_u^v$. Then there exists a constant C such that*

$$\|W \cdot Tf\|_{M_p^r} \le C\|W\|_{M_u^v}\|f\|_{M_p^r}, \quad f \in M_p^r.$$

We assume that V satisfies the following conditions: there exists a positive constant m_0 such that

$$V(x) \ge 0,\ V \in (RH)_\infty,\ m(x,V) \ge m_0. \tag{1.8}$$

For example, $V(x) = |P(x)|^\alpha$ satisfies (1.8) for every polynomials $P(x)$ and $\alpha > 0$ (see [Sh1]).

Definition 1.1. For a Banach space Y $(\subset L^p_{loc}(\mathbf{R}^n)$ for some $1 < p < +\infty$) we consider the Banach space $\mathcal{X}_Y$ associated with Y:

$$\mathcal{X}_Y = \{u \in Y; m(x,V)^2 u, m(x,V)\nabla u, \nabla^2 u \in Y\}$$

with its norm $\|u\|_{\mathcal{X}_Y} = \|m(x,V)^2 u\|_Y + \|m(x,V)\nabla u\|_Y + \|\nabla^2 u\|_Y$.

Let $L_1 = -\Delta + V$ and $L_2 = L_1 + W$. Now, we state the estimate in the classical Morrey spaces for L_2.

Theorem 1.2. *Assume $n \geq 3, 1 < p \leq r < n/2, p < u \leq n/2$, and (1.8) for V. Let $Y = M_p^r$ and suppose $W \in M_u^{n/2}$ and $\|W\|_{M_u^{n/2}}$ is sufficiently small. Then there exists a constant C such that*

$$\|Wu\|_{M_p^r} + \|m(x,V)^2 u\|_{M_p^r} + \|m(x,V)\nabla u\|_{M_p^r} + \|\nabla^2 u\|_{M_p^r} \leq C\|L_2 u\|_{M_p^r}, \quad u \in C_0^\infty(\mathbf{R}^n). \tag{1.9}$$

Here C depends on $\|W\|_{M_u^{n/2}}$ and $\|L_1^{-1}\|_{Y\to\mathcal{X}_Y}$.

For the definition of $\|L_1^{-1}\|_{Y\to\mathcal{X}_Y}$, see section 3.

Definition 1.2. (L_2^{-1}) Under the assumption that WL_1^{-1} is bounded on Y and W is small in the sense that $\|WL_1^{-1}\|_{Y\to Y} < 1$, we define the operator L_2^{-1} from Y to $\mathcal{X}_Y$ by $L_2^{-1} = L_1^{-1}(1 + WL_1^{-1})^{-1}$. Note that, when L_1 is an isomorphism from $\mathcal{X}_Y$ to Y, $u = L_2^{-1} f \in \mathcal{X}_Y$ represents a unique solution to $L_2 u = f$ for $f \in Y$.

Since we will see that L_1 gives an isomorphism from $\mathcal{X}$ to Y for the case $Y = L^p(\mathbf{R}^n)$ with $1 < p < +\infty$, we have the following theorem.

Theorem 1.3. (1) *Assume $n \geq 3, 1 < p < n/2, p < u \leq n/2$, and (1.8) for V. Suppose $W \in M_u^{n/2}$ and $\|W\|_{M_u^{n/2}}$ is sufficiently small. Then there exists a constant C such that*

$$\|WL_2^{-1}f\|_{L^p} + \|m(x,V)^2 L_2^{-1} f\|_{L^p} + \|m(x,V)\nabla L_2^{-1} f\|_{L^p} + \|\nabla^2 L_2^{-1} f\|_{L^p} \leq C\|f\|_{L^p}, \quad f \in L^p. \tag{1.10}$$

(2) *Under the additional assumption $|W|^{1/2} \in M_v^n$ for some $v \leq n$, there exists a constant C such that*

$$\||W|^{1/2}\nabla L_2^{-1} f\|_{L^p} \leq C\|f\|_{L^p}, \quad f \in L^p.$$

Example 1.1. Let $W(x) = \epsilon/|x|^2$ with $|\epsilon| \leq \epsilon_0$ for sufficiently small $\epsilon_0 > 0$. Then $W \in M_u^{n/2}$ for every $u < n/2$ and $\|W\|_{M_u^{n/2}} \leq \epsilon_0 C$ for some constant C which depends only on n. Hence we can apply Theorem 1.2 and Theorem 1.3 for every $1 < p \leq r < n/2$ and $1 < p < n/2$, respectively. Note that W does not belong to any L^u spaces.

For a general weight function $\Phi(x,r) \geq 0$, the generalized Morrey space L_p^Φ with $1 \leq p < +\infty$ is defined as follows:

$$L_p^\Phi = \{f \in L_{loc}^p(\mathbf{R}^n); \sup_{x\in\mathbf{R}^n, r>0} \frac{1}{\Phi(x,r)} \int_{B(x,r)} |f(y)|^p\, dy \equiv \|f\|_{p,\Phi}^p < +\infty\}. \tag{1.11}$$

In [Mi] and [Na], the authors studied the boundedness of the Hardy-Littlewood maximal function and singular integral operators on those spaces under a certain conditions on Φ ((2.1) and (2.2) in section 2). We should mention that for the restricted case $p/r \leq u/v$, the estimate in Theorem 1.1 has been proved in [CF] and its proof is rather simple compared with the one in [Ol]. Since this type of estimate is useful even for the restricted case for our application, we shall generalize it to the generalized Morrey spaces in section 2 under a certain additional condition (2.3) on $\Phi(x,r)$ (Theorem 2.1). Furthermore, we improve the estimate in Theorem 2.1 for the intergral operator S_M (Theorem 2.2). In section 3 we apply it to the Schrödinger operator and show a similar estimate as in Theorem 1.2 on the generalized Morrey space (Theorems 3.1, 3.2). The proof of Theorem 1.2 is also given in a similar way as in Theorem 3.1.

2. Boundedness of integral operators on generalized Morrey spaces

In this section, we show boundedness of the integral operators T and S_M on generalized Morrey spaces. We assume that the weight function $\Phi(x,r)$ satisfies the following conditions: there exist a positive constant C and nonnegative constants λ, δ with $\lambda+\delta < n(1-p/s')$ such that

$$C^{-1} \le \frac{\Phi(x,t)}{\Phi(x,r)} \le C \quad (r \le t \le 2r), \tag{2.1}$$

$$\int_r^\infty \frac{\Phi(x,t)}{t^{n-(n/s')p}} \frac{dt}{t} \le C \frac{\Phi(x,r)}{r^{n(1-p/s')}}, \tag{2.2}$$

$$\Phi(x,r) \le C r^\lambda (1+r^\delta) \tag{2.3}$$

for every $x \in \mathbf{R}^n$ and $r>0$, where $s' = s/(s-1)$.

Example 2.1. It is easy to see that $\Phi(x,r) = r^\lambda \log(2+r)$ with $0 \le \lambda < n(1-p/s')$ satisfies the conditions above for every $\delta > 0$. Note that L_p^Φ does not belong to any classical Morrey spaces.

We state our main results for boundedness of $W \cdot T$ on generalized Morrey spaces.

Theorem 2.1. *Assume $n \ge 3, 1 < s, 1 < p < s'$, and (2.1)–(2.3) for $\Phi(x,r)$. Let $\lambda^* = n/(n-\lambda)$, $(\lambda+\delta)^* = n/(n-(\lambda+\delta))$ and $W \in L^\Phi_{s'/\lambda^*} \cap L^\Phi_{s'/(\lambda+\delta)^*}$. Then there exists a constant C such that*

$$\|W \cdot Tf\|_{p,\Phi} \le C(\|W\|_{s'/\lambda^*,\Phi} + \|W\|_{s'/(\lambda+\delta)^*,\Phi})\|f\|_{p,\Phi}, \quad f \in L_p^\Phi. \tag{2.4}$$

Remark 2.1. Note that, for the case $\Phi(x,t) = t^{n(1-p/r)}$ we have $L_p^\Phi = M_p^r$. Hence Theorem 2.1 in the case $\delta = 0$ is a generalization of [CF, Theorem 2].

For the integral operator S_M, we can improve the estimate in Theorem 2.1.

Theorem 2.2. *Assume $n \ge 3, 1 < s, 1 < p < s'$, and (2.1)-(2.3) for $\Phi(x,r)$. Let $W \in L^\Phi_{s'}$ and suppose $M \ge (\lambda+\delta)/p$. Then there exists a constant C such that*

$$\|W \cdot S_M f\|_{p,\Phi} \le C\|W\|_{s',\Phi}\|f\|_{p,\Phi}, \quad f \in L_p^\Phi. \tag{2.5}$$

In [Na, Theorem 3], without the assumption (2.3), Nakai proved a boundedness of T from L_p^Φ to $L_q^{\Phi^{q/p}}$, where $1/q = 1/p - 1/s'$ which is a generalization of [CF, Corollary, p. 277].

Theorem 2.3 ([Na])**.** *Suppose $1 < p < s'$ and $\Phi(x,r)$ satisfies the conditions (2.1) and (2.2). Let $1/q = 1/p - 1/s'$. Then there exists a constant C such that*

$$\|Tf\|_{q,\Phi^{q/p}} \le C\|f\|_{p,\Phi}.$$

We note Hölder's inequality on the generalized Morrey spaces.

Lemma 2.1. *Let $\Phi_1(x,r)$ and $\Phi_2(x,r)$ be weight functions. Suppose $p,q,t \in (0,+\infty)$ satisfy $1/t = 1/p + 1/q$ and let $\Psi(x,r) = \Phi_1(x,r)^{t/p}\Phi_2(x,r)^{t/q}$. Then we have*

$$\|fg\|_{t,\Psi} \le \|f\|_{p,\Phi_1}\|g\|_{q,\Phi_2}, \quad f \in L_p^{\Phi_1}, g \in L_q^{\Phi_2}.$$

Proof. For each $x \in \mathbf{R}^n$ and $r > 0$, we have

$$\begin{aligned}\left(\int_{B(x,r)} |fg|^t\,dy\right)^{1/t} &\le \left(\int_{B(x,r)} |f|^p\,dy\right)^{1/p}\left(\int_{B(x,r)} |g|^q\,dy\right)^{1/q}\\ &\le \|f\|_{p,\Phi_1}\Phi_1(x,r)^{1/p}\|g\|_{q,\Phi_2}\Phi_2(x,r)^{1/q}\\ &= \|f\|_{p,\Phi_1}\|g\|_{q,\Phi_2}\Psi(x,r)^{1/t}.\end{aligned}$$

This implies the desired inequality. □

We need the boundedness of $W \cdot T$ on the same generalized Morrey space to apply a perturbation argument to the operator $L_2 = L_1 + W$. From this point of view, W must belong to some L^q space if we apply Theorem 2.3.

Corollary 2.1. *Suppose* $1 < p < s'$ *and* $\Phi(x,r)$ *satisfies the conditions* (2.1) *and* (2.2). *Let* $W \in L^{s'}(\mathbf{R}^n)$. *Then there exists a constant* C *such that*

$$\|W \cdot Tf\|_{p,\Phi} \le C\|W\|_{L^{s'}}\|f\|_{p,\Phi}.$$

Proof. It is an easy consequence of Theorem 2.3 and Lemma 2.1. □

Corollary 2.1 should be compared with Theorem 2.2 (or 2.1). The difference between Theorem 2.2 and Corollary 2.1 can be seen in Example 3.1, for example. To prove Theorem 2.1 we need the following lemma.

Lemma 2.2 ([Na]). *Under the assumptions* (2.1)-(2.2), *there exist positive constants* C *and* μ *such that*

$$\int_r^{+\infty} \frac{\Phi(x,t)}{t^{n(1-p/s')-\mu}}\frac{dt}{t} \le C\frac{\Phi(x,r)}{r^{n(1-p/s')-\mu}}$$

holds for every $r > 0$.

Proof of Theorem 2.1. We may assume $f \in L_p^\Phi \cap L_c^\infty$ without loss of generality in the proof. Actually, we show there exists a constant C which does not depend on f such that

(2.6)
$$\int_{B(y,r)} |W(x)(T|f|)(x)|^p\,dy \le C\Phi(y,r)(\|W\|_{s'/\lambda^*,\Phi} + \|W\|_{s'/(\lambda+\delta)^*,\Phi})^p\|f\|_{p,\Phi}^p$$

for every $y \in \mathbf{R}^n$ and $r > 0$. For general $f \in L_p^\Phi$, applying (2.6) to $f_R(x) = \chi_{\{|x|\le R\}\cap\{x;|f(x)|\le R\}}(x)|f|(x)$ which belongs to $L_c^\infty \cap L_p^\Phi$, we have

(2.7)
$$\int_{B(y,r)} |W(x)(T|f_R|)(x)|^p\,dy \le C\Phi(y,r)(\|W\|_{s'/\lambda^*,\Phi} + \|W\|_{s'/(\lambda+\delta)^*,\Phi})^p\|f\|_{p,\Phi}^p$$

for every $y \in \mathbf{R}^n$ and $r > 0$. Taking the limit $R \to \infty$, by the monotone convergence theorem, we obtain (2.6) for general $f \in L_p^\Phi$ and complete the proof.

(Step 1): We follow the argument in [CF]. We write, for $\epsilon > 0$ which will be determined later,

$$Tf(x) = \int_{\{|x-y|<\epsilon\}} \frac{|f(y)|}{|x-y|^{n/s}}\,dy + \int_{\{|x-y|\ge\epsilon\}} \frac{|f(y)|}{|x-y|^{n/s}}\,dy \equiv I_1 + I_2.$$

We can estimate I_1 by

(2.8)
$$|I_1| \leq \sum_{k=-\infty}^{-1} \int_{\{2^k\epsilon \leq |x-y| < 2^{k+1}\epsilon\}} \frac{|f(y)|}{|x-y|^{n/s}}\, dy \leq C\epsilon^{n(1-1/s)} Mf(x)$$

for some constant C. Now, we take σ to satisfy $n(1-\frac{p}{s'}) - \mu < \sigma < pn(\frac{1}{s} - \frac{1}{p'})$, where $\mu > 0$ is the constant in Lemma 2.2. Since $n/s = \sigma/p + \{-(\sigma/p - n/s)\}$, Hölder's inequality yields

(2.9)
$$|I_2| \leq \left(\int_{\{|x-y|\geq\epsilon\}} \frac{|f(y)|^p}{|x-y|^\sigma}\, dy\right)^{1/p} \left(\int_{\{|x-y|\geq\epsilon\}} |x-y|^{(\sigma/p - n/s)p'}\, dy\right)^{1/p'} \equiv I_3 I_4.$$

For I_3, we have by (2.1)

$$\begin{aligned} I_3 &\leq \left(\sum_{k=0}^{+\infty} \int_{\{2^k\epsilon \leq |x-y| < 2^{k+1}\epsilon\}} \frac{|f(y)|^p}{(2^k\epsilon)^\sigma}\, dy\right)^{1/p} \\ &\leq \left(\sum_{k=0}^{+\infty} \frac{C\Phi(x, 2^k\epsilon)}{(2^k\epsilon)^\sigma}\right)^{1/p} \|f\|_{p,\Phi}. \end{aligned} \tag{2.10}$$

We obtain

(2.11)
$$\begin{aligned} \sum_{k=0}^{+\infty} \frac{\Phi(x, 2^k\epsilon)}{(2^k\epsilon)^\sigma} &= \sum_{k=0}^{+\infty} \frac{(2^k\epsilon)^{n(1-p/s')-\mu}}{(2^k\epsilon)^\sigma} \frac{C\Phi(x, 2^k\epsilon)}{(2^k\epsilon)^{n(1-p/s')-\mu}} \\ &\leq C\sum_{k=0}^{+\infty} \frac{(2^k\epsilon)^{n(1-p/s')-\mu}}{(2^k\epsilon)^\sigma} \int_{2^k\epsilon}^{2^{k+1}\epsilon} \frac{\Phi(x,t)}{t^{n(1-p/s')-\mu}} \frac{dt}{t} \\ &\leq C\frac{\epsilon^{n(1-p/s')-\mu}}{\epsilon^\sigma} \int_\epsilon^\infty \frac{\Phi(x,t)}{t^{n(1-p/s')-\mu}} \frac{dt}{t} \end{aligned}$$

for some constant C. Here, since $\sigma > n(1-p/s') - \mu$, we used $(2^k)^{n(1-p/s')-\mu-\sigma} \leq 1$ for every $k \geq 0$. By Lemma 2.2, we can conclude

$$\sum_{k=0}^{+\infty} \frac{\Phi(x, 2^k\epsilon)}{(2^k\epsilon)^\sigma} \leq C\frac{\Phi(x,\epsilon)}{\epsilon^\sigma}.$$

On the other hand, it is easy to obtain

$$I_4 \leq C\epsilon^{n(1/p' - 1/s) + \sigma/p}. \tag{2.12}$$

Hence, it follows that

$$I_2 \leq C\Phi(x,\epsilon)^{1/p} \epsilon^{n(1-1/p-1/s)} \|f\|_{p,\Phi}.$$

Therefore, we obtain that

(2.13)
$$(Tf)(x) \leq C\left\{\epsilon^{n(1-1/s)} Mf(x) + \Phi(x,\epsilon)^{1/p} \epsilon^{n(1-1/p-1/s)} \|f\|_{p,\Phi}\right\}$$

holds for some constant C.

(STEP 2): By the condition (2.3), we have

$$(Tf)(x) \le C\left\{\epsilon^{n(1-1/s)}Mf(x) + (\epsilon^{\lambda/p} + \epsilon^{(\lambda+\delta)/p})\epsilon^{n(1-1/p-1/s)}\|f\|_{p,\Phi}\right\}.$$

Let $A = M(|f|)(x)$ and $B = \|f\|_{p,\Phi}$. We consider two cases.

Case 1. For the case $B/A \le 1$, i.e. $\|f\|_{p,\Phi} \le M(|f|)(x)$, we choose $\epsilon = (B/A)^{p/(n-\lambda)}$. Since $\epsilon \le 1$, it follows that

$$\begin{aligned}(Tf)(x) &\le C\left\{(\frac{B}{A})^{\frac{np(1-1/s)}{n-\lambda}}A + (\frac{B}{A})^{\frac{p}{n-\lambda}(\lambda/p+n(1-1/p-1/s))}B\right\}\\ &\le CM(|f|)(x)^{1-p\lambda^*/s'}\|f\|_{p,\Phi}^{p\lambda^*/s'}.\end{aligned}$$

Case 2. For the case $B/A \ge 1$, i.e. $\|f\|_{p,\Phi} \ge M(|f|)(x)$, we choose

$$\epsilon = (B/A)^{p/(n-(\lambda+\delta))}.$$

In this case, it follows in a similar way that

$$(Tf)(x) \le CM(|f|)(x)^{1-p(\lambda+\delta)^*/s'}\|f\|_{p,\Phi}^{p(\lambda+\delta)^*/s'}.$$

Hence, we obtain

(2.14)
$$\begin{aligned}(Tf)(x) &\le CM(|f|)(x)^{1-p\lambda^*/s'}\|f\|_{p,\Phi}^{p\lambda^*/s'} + CM(|f|)(x)^{1-p(\lambda+\delta)^*/s'}\|f\|_{p,\Phi}^{p(\lambda+\delta)^*/s'}\\ &\equiv J_1(x) + J_2(x).\end{aligned}$$

Define $a \in (1,\infty)$ by $1/a = 1 - p\lambda^*/s'$. By using Hölder's inequality, we have

$$\begin{aligned}&\left(\int_{B(y,r)}|W(x)J_1(x)|^p\,dy\right)^{1/p}\\ &\le C\left(\int_{B(y,r)}|Mf(x)|^{p/a}|W(x)|^p\,dy\right)^{1/p}\|f\|_{p,\Phi}^{1/a'}\\ &\le C\left(\int_{B(y,r)}|Mf(x)|^p\,dy\right)^{1/ap}\left(\int_{B(y,r)}|W(x)|^{s'/\lambda^*}\,dy\right)^{\lambda^*/s'}\|f\|_{p,\Phi}^{1/a'}\\ &\le C\Phi(y,r)^{1/p}\|W\|_{s'/\lambda^*,\Phi}\|f\|_{p,\Phi}.\end{aligned}$$

In the last inequality we used the fact that $\|Mf\|_{p,\Phi} \le C\|f\|_{p,\Phi}$ holds under the assumptions (2.1) and (2.2) (see [Na]). This implies

$$\|WJ_1\|_{p,\Phi} \le C\|W\|_{s'/\lambda^*,\Phi}\|f\|_{p,\Phi}.$$

In a similar way we have

$$\|WJ_2\|_{p,\Phi} \le C\|W\|_{s'/(\lambda+\delta)^*,\Phi}\|f\|_{p,\Phi}.$$

This complete the proof of Theorem 2.1. □

Proof of Theorem 2.2. We modify the proof of Theorem 2.1 in the following. We can estimate I_2 by I_3' and I_4, where

$$I_{3'} = \left(\int_{\{|x-y|\ge\epsilon\}}\frac{|f(y)|^p}{|x-y|^\sigma(1+|x-y|)^{Mp}}\,dy\right)^{1/p}.$$

We can estimate $I_{3'}$ in two different ways, namely $I_{3'} \le I_3$ and

$$I_{3'} \le \left(\sum_{k=0}^{\infty} \int_{\{2^k\epsilon \le |x-y| < 2^{k+1}\epsilon\}} \frac{|f(y)|^p}{|x-y|^{\sigma+Mp}}\, dy \right)^{1/p} \le C \frac{\Phi(x,r)^{1/p}}{\epsilon^M} \epsilon^{n(1-1/p-1/s)} \|f\|_{p,\Phi}. \tag{2.15}$$

Therefore, we obtain

(2.16)

$$(Tf)(x) \le C\epsilon^{n(1-1/s)} M(f)(x) + C\epsilon^{\lambda/p} \min(1+\epsilon^{\delta/p}, \frac{1+\epsilon^{\delta/p}}{\epsilon^M}) \epsilon^{n(1-1/p-1/s)} \|f\|_{p,\Phi}$$
$$\le C\epsilon^{n(1-1/s)} M(f)(x) + C\epsilon^{n(1-1/p-1/s)} \|f\|_{p,\Phi}.$$

Here we used the boundedness of $\epsilon^{\lambda/p} \min(1+\epsilon^{\delta/p}, \frac{1+\epsilon^{\delta/p}}{\epsilon^M})$ under the assumption $M \ge (\lambda+\delta)/p$. Thus, this yields

$$(Tf)(x) \le C(Mf)(x)^{1-p/s'} \|f\|_{p,\Phi}^{p/s'} \tag{2.17}$$

which completes the proof of Theorem 2.2. □

3. Application to Schrödinger operators

We consider Schrödinger operators $L_1 = -\Delta + V(x)$ and $L_2 = L_1 + W(x)$ on $\mathbf{R}^n, n \ge 3$. We study estimates in Morrey spaces and boundedness of L_2^{-1} for certain small perturbed potentials W by using estimates for the fundamental solution for L_1^{-1}.

Lemma 3.1 ([KS]). *Suppose $V(x) \ge 0,\ V \in (RH)_\infty$. Then there exist constants $C_j, j = 1, 2$, such that*

$$|m(x,V)^2 L_1^{-1} f(x)| \le C_1 M(f)(x), |m(x,V) \nabla L_1^{-1} f(x)| \le C_2 M(f)(x),$$

for $f \in L_c^\infty(\mathbf{R}^n)$.

Let Y ($\subset L^p_{loc}(\mathbf{R}^n)$ for some $1 < p < +\infty$) be a Banach space, and let $\mathcal{X}_Y$ be the Banach space associated with Y in Definition 1.1.

Corollary 3.1. *(1) Suppose $V(x) \ge 0,\ V \in (RH)_\infty$ and that there exists a constant C such that $\|M(f)\|_Y + \|\nabla^2(-\Delta)^{-1} f\|_Y \le C\|f\|_Y$ for $f \in Y \cap L_c^\infty$. Then there exists a constant C_1 such that*

$$\|L_1^{-1} f\|_{\mathcal{X}_Y} \le C_1 \|f\|_Y, \quad f \in Y \cap L_c^\infty.$$

(2) Under the same assumptions, there exists a constant C_2 such that

$$\|u\|_{\mathcal{X}_Y} \le C_2 \|L_1 u\|_Y, \quad u \in C_0^\infty(\mathbf{R}^n).$$

Proof. Lemma 3.1 and the assumption yield

$$\|m(x,V)^2 L_1^{-1} f\|_Y + \|m(x,V) \nabla L_1^{-1} f\|_Y + \|\Delta L_1^{-1} f\|_Y \le C\|f\|_Y.$$

Since $\nabla^2 L_1^{-1} = \nabla^2(-\Delta)^{-1}(-\Delta) L_1^{-1}$, we obtain (1). Since $L_1 u = f \in L_c^\infty \cap Y$ for $u \in C_0^\infty(\mathbf{R}^n)$ and $u = L_1^{-1} f$, (2) is a consequence of (1). □

We can show in the same way as in the proof of Theorem 2.1 that

$$\|m(x,V)^2 L_1^{-1} f\|_Y + \|m(x,V)\nabla L_1^{-1} f\|_Y \le C\|f\|_Y$$

holds for $f \in Y$. We denote by $\|L_1^{-1}\|_{Y\to\mathcal{X}_Y}$ the infimum of the constant C_2 in Corollary 3.1. Since it is known that the assumption in Corollary 3.1 holds for $Y = L^p(\mathbf{R}^n)$, $L^p(\mathbf{R}^n; w\,dx), 1 < p < +\infty$, with A_p-weight $w \ge 0$, the Morrey spaces $M_p^r, 1 < p \le r < \infty$, generalized Morrey spaces L_p^Φ (see, e.g., [CF], [GR], [Mi], [Na]), we can apply Corollary 3.1 to these Banach spaces Y. Now we state our main result in this section.

Theorem 3.1. *Assume $n \ge 3, 1 < p < n/2, 0 \le \lambda + \delta < n(1-2p/n)$, (1.8) for V, and that $\Phi(x,r)$ satisfies the conditions (2.1)–(2.3). Let $Y = L_p^\Phi$ and $W \in L_{n/2}^\Phi$. Suppose $\|W\|_{n/2,\Phi}$ is sufficiently small. Then there exists a constant C such that*

$$\|Wu\|_Y + \|u\|_{\mathcal{X}_Y} \le C\|L_2 u\|_Y, \quad u \in C_0^\infty(\mathbf{R}^n). \tag{3.1}$$

Here C depends on $\|W\|_{L_{n/2,\Phi}}$ and $\|L_1^{-1}\|_{Y\to\mathcal{X}_Y}$.

Proof. It is known that the fundamental solution $\Gamma(x,y)$ for L_1 satisfies (1.2) and (1.3) and hence

$$0 \le \Gamma(x,y) \le \frac{C_M}{|x-y|^{n-2}(1+|x-y|)^M}$$

for every $M > 0$ under the assumption (1.8) on V. Thus WL_1^{-1} is a bounded operator on Y under the assumption by Theorem 2.2 with $s = n/(n-2)$ and hence $(1+WL_1^{-1})^{-1}$ is also bounded on Y because of the smallness of $\|W\|_{n/2,\Phi}$ and can be written as Neumann series. For $u \in C_0^\infty(\mathbf{R}^n)$, $L_2 u = f$ is equivalent to $L_1 u = (1+WL_1^{-1})^{-1} f$. Then, the desired estimate is an easy consequence of Theorem 2.2 with $s = n/(n-2)$ and Corollary 3.1 (2). □

Proof of Theorem 1.2. If we combine Theorem 1.1 with $s = n/(n-2)$ and the estimate $|\Gamma(x,y)| \le C|x-y|^{2-n}$, we can prove Theorem 1.2 in the same way as in the proof of Theorem 3.1. □

Under somewhat weaker assumptions that V satisfies $V(x) \ge 0, V \not\equiv 0$, and $V \in (RH)_\infty$, if we apply Theorem 2.1 instead of Theorem 2.2, we can obtain a similar estimate as in Theorem 3.1. If we apply Corollary 2.1, we have

Theorem 3.2. *Assume $n \ge 3, 1 < p < n/2$, $V(x) \ge 0$, $V(x) \not\equiv 0, V \in (RH)_\infty$, and $\Phi(x,r)$ satisfies the conditions (2.1) and (2.2). Let $Y = L_p^\Phi$, $W \in L^{n/2}(\mathbf{R}^n)$. Suppose $\|W\|_{L^{n/2}}$ is sufficiently small. Then there exists a constant C such that*

$$\|Wu\|_Y + \|u\|_{\mathcal{X}_Y} \le C\|L_2 u\|_Y, \quad u \in C_0^\infty(\mathbf{R}^n).$$

Example 3.1. We compare Theorem 3.1 with Theorem 3.2 in the case $\Phi(x,r) = \log(2+r)$, for example. Hence, $\Phi(x,r)$ satisfies (2.1)-(2.3) with $\lambda = 0$ and small $\delta > 0$. Note that $L^{n/2}(\mathbf{R}^n) \subset L_{n/2}^\Phi$. Thus Theorem 3.1 is stronger than Theorem 3.2 in this case. For the case $\Phi(x,r) = r^\lambda \log(2+r)$ with $0 < \lambda < n(1-2p/n)$ and small $\delta > 0$, Theorem 3.1 and Theorem 3.2 complement each other.

Proof of Theorem 1.3. By Corollary 3.1 (2) there exists a constant C such that

$$\|m(x,V)^2 u\|_{L^p} + \|m(x,V)\nabla u\|_{L^p} + \|\nabla^2 u\|_{L^p} \le C\|L_1 u\|_{L^p}$$

for every $u \in C_0^\infty(\mathbf{R}^n)$. On the other hand, Kato's inequality implies the operator L_1 with the domain $D(L_1) = C_0^\infty(\mathbf{R}^n)$ is essentially m-accretive in L^p with $1 <$

$p < +\infty$ (see, e.g., [He], [Ok], [Ku]). Combining these results, we obtain that the closure of L_1 is an isomorphism from $\mathcal{X}_Y$ to $Y = L^p(\mathbf{R}^n)$ (see also [Gu]). Hence, $L_2^{-1}f = L_1^{-1}(1 + WL_1^{-1})^{-1}f$ represents a unique solution of $L_2u = f$ for $f \in L^p(\mathbf{R}^n)$. Theorem 1.3(1) is a consequence of Theorem 1.1 with $s = n/2$ and Theorem 1.2. Under the assumptions on V, we have

$$|\nabla_x \Gamma(x, y)| \leq \frac{C_M}{(1 + |x - y|)^M |x - y|^{n-1}}$$

for every $M > 0$. Hence Theorem 1.1 with $s = n$ yields the boundedness of $|W|^{1/2}\nabla L_1^{-1}$ on $L^p(\mathbf{R}^n)$. Thus it follows that

$$\begin{aligned} \||W|^{1/2}\nabla L_2^{-1}f\|_{L^p} &= \||W|^{1/2}\nabla L_1^{-1}(1 + WL_1^{-1})^{-1}f\|_{L^p} \\ &\leq C\|(1 + WL_1^{-1})^{-1}f\|_{L^p} \leq C\|f\|_{L^p}. \end{aligned}$$

Theorem 1.3(2) is a consequence of Theorem 1.1 with $s = n$. □

We should mention that our method can also be applied to other operators, e.g., uniformly elliptic operators and magnetic Schrödinger operators ([KS], [Sh2]).

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