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ASYMPTOTIC REGULARITY OF DAUBECHIES' SCALING FUNCTIONS

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ABSTRACT. Let ϕ_N, $N \geq 1$, be Daubechies' scaling function with symbol $(\frac{1+e^{-i\xi}}{2})^N Q_N(\xi)$, and let $s_p(\phi_N), 0 < p \leq \infty$, be the corresponding L^p Sobolev exponent. In this paper, we make a sharp estimation of $s_p(\phi_N)$, and we prove that there exists a constant C independent of N such that

$$N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2} - \frac{C}{N} \leq s_p(\phi_N) \leq N - \frac{\ln |Q_N(2\pi/3)|}{\ln 2}.$$

This answers a question of Cohen and Daubechies (*Rev. Mat. Iberoamericana*, 12(1996), 527-591) positively.

1. INTRODUCTION

For $N \geq 1$, let

$$P_N(t) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} t^k.$$

Then

$$(1-t)^N P_N(t) + t^N P_N(1-t) = 1$$

and P_N is the unique polynomial solution of the equation with degree not greater than $N-1$.

Let $Q_N(\xi)$ be a trigonometric polynomial with real coefficients satisfying

$$|Q_N(\xi)|^2 = P_N(\sin^2 \frac{\xi}{2}). \tag{1}$$

It is known that such Q_N exists by the Riesz Lemma, but Q_N is not unique. Set

$$H_N(\xi) = (\frac{1+e^{-i\xi}}{2})^N Q_N(\xi) = \frac{1}{2} \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}.$$

We are interested in the Q_N such that the solution ϕ_N of the refinement equation

$$\phi_N(x) = \sum_{k\in\mathbb{Z}} c_k \phi_N(2x-k) \tag{2}$$

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with $\int_{\mathbb{R}} \phi_N(x)dx = 1$ that generates an orthonormal basis of $L^2(\mathbb{R})$. The functions ϕ_N are the well known Daubechies' scaling functions [6]. For an integrable function f, we let $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$ be the Fourier transform of f. Then

$$\widehat{\phi_N}(\xi) = H_N(\frac{\xi}{2})\widehat{\phi_N}(\frac{\xi}{2}) \tag{3}$$

and

$$\widehat{\phi_N}(\xi) = \prod_{j=1}^{\infty} H_N(2^{-j}\xi). \tag{4}$$

The regularity of the scaling functions has central importance in the theory of wavelets. In [14] Volkmer proved that the Hölder index of ϕ_N is $(1-\frac{\ln 3}{2\ln 2})N+o(N)$ as N tends to infinity. Recently Bi, Dai and Sun ([1]) improved the estimation as

$$(1-\frac{\ln 3}{2\ln 2})N + \frac{\ln N}{4\ln 2} + O(1).$$

Another popular approach to the regularity is to use the Sobolev exponent. Recall that the Sobolev exponent $s_p(f), 0 < p < \infty$, is defined by

$$s_p(f) = \sup\ \{s :\ \int_{\mathbb{R}} |\hat{f}(\xi)|^p (1+|\xi|)^{ps} d\xi < \infty\},$$

and for $p = \infty$,

$$s_\infty(f) = \sup\ \{s : \hat{f}(\xi)(1+|\xi|)^s \ \text{ is a bounded function}\}.$$

There is considerable literature devoted to estimating the Sobolev exponent for scaling functions in general, for example, [8] and [13] for $s_2(f)$, [2] for $s_1(f)$, [10] and [9] for $s_p(f)$ with $1 \le p < \infty$, [12] for Triebel-Lizorkin space and Besov space, and [11] for L^p Lipschitz space. For Daubechies' scaling functions, Volkmer [15] proved that

$$N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2} - \frac{1}{2} \le s_2(\phi_N) \le N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2}.$$

Recently, Cohen and Daubechies ([3], [7]) computed $s_p(\phi_N)$ for $p = 1, 2, 4, 8$ and $N = 1, 2, \cdots, 19$, and found that the difference of $s_p(\phi_N)$ between different p becomes very small for N large. Based on this observation, they asked

Problem. Let ϕ_N be defined by (2). For $0 < p,\ q < \infty$, is it true that

$$\lim_{N\to\infty} (s_p(\phi_N) - s_q(\phi_N)) = 0?$$

In this paper, we answer this question affirmatively and generalize the estimation in [15] in part.

Theorem. *Let ϕ_N be defined by* (2). *For $0 < p < \infty$, there exists a constant C independent of N such that*

$$N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2} - \frac{C}{N} \le s_p(\phi_N) \le N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2},$$

and for $p = \infty$,

$$s_\infty(\phi_N) = N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2}.$$

N	$p=1$	$p=2$	$p=8$	$N-\frac{\ln\|Q_N(2\pi/3)\|}{\ln 2}$
2	0.521293	0.999820	1.310014	1.339036
3	0.979675	1.414947	1.631688	1.636040
4	1.391644	1.775305	1.912144	1.912537
5	1.767934	2.096541	2.174682	2.176608
6	2.116733	2.388060	2.431755	2.432246
7	2.441544	2.658569	2.680307	2.681743
8	2.746639	2.914556	2.925926	2.926549
9	3.035292	3.161380	3.165533	3.167644
10	3.309107	3.402546	3.405141	3.405724
11	3.572141	3.639569	3.638529	3.641301
12	3.825525	3.873991	3.871917	3.874766
13	4.071021	4.105802	4.105305	4.106422
14	4.311641	4.336042	4.335502	4.336511
15	4.547368	4.564708	4.562449	4.565229
16	4.780028	4.792323	4.792645	4.792735
17	5.010231	5.018884	5.016283	5.019164
18	5.238588	5.244390	5.243230	5.244627
19	5.464480	5.468841	5.466868	5.469221

In the table, we list the approximate value of the L^p Sobolev exponent $s_p(\phi_N)$. The first three columns $s_p(\phi_N), p = 1, 2, 8$, are obtained by Cohen and Daubechies in [3]. The last column $N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2}$ is the approximate value from the theorem. Note that the numerical data matches with the theorem.

2. Upper bound estimation

In this section, we will prove the upper bound estimate of $s_p(\phi_N)$.

Proposition 1. *Let ϕ_N be defined by* (2). *Then for $0 < p \le \infty$,*

$$s_p(\phi_N) \le N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2}. \tag{5}$$

Proof. It follows from (3) that

$$|\widehat{\phi_N}(\frac{2^k\pi}{3})| = 2^{-(k-1)N}|Q_N(\frac{2\pi}{3})|^{k-1}|\widehat{\phi_N}(\frac{2\pi}{3})|.$$

Hence (5) holds for $p = \infty$.

To prove the case for $0 < p < \infty$, we let $\tilde{\phi}_N$ be the compactly supported distribution defined by

$$\widehat{\tilde{\phi}_N}(\xi) = \prod_{j=1}^{\infty} Q_N(\xi/2^j).$$

Let $n_k = (4^k - 1)/3$; then by a similar method as used in Proposition 3 in [4], we obtain for any $\epsilon > 0$ there exists a constant C such that for $\xi \in [-\pi, \pi]$ and for sufficiently large k,

$$|\widehat{\tilde{\phi}_N}(\xi + 2n_k\pi)| \ge C|Q_N(\frac{2\pi}{3})|^{2k}4^{-k\epsilon}.$$

Since

$$\widehat{\phi_N}(\xi) = \prod_{j=1}^{\infty} \left(\frac{1+e^{-i2^{-j}\xi}}{2}\right)^N \widehat{\tilde{\phi}_N}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^N \widehat{\tilde{\phi}_N}(\xi),$$

there exists an integer k_0 such that for $\xi \in [\frac{5\pi}{9}, \frac{7\pi}{9}]$ and $k \geq k_0$,

$$|\widehat{\phi_N}(\xi + 2n_k\pi)| \geq C4^{-Nk-\epsilon k}|Q_N(\frac{2\pi}{3})|^{2k}.$$

Obviously

$$\int_{\mathbb{R}} |\widehat{\phi_N}(\xi)|^p(1+|\xi|)^{ps}d\xi < \infty$$

implies that

$$\int_{[\frac{5\pi}{9},\frac{7\pi}{9}]+2n_k\pi} |\widehat{\phi_N}(\xi)|^p(1+|\xi|)^{ps}d\xi$$

is bounded on k. Hence there exists a constant C such that $4^{(s-N-\epsilon)kp}|Q_N(\frac{2\pi}{3})|^{2kp} \leq C$ for all k. This implies that

$$s - N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2} - \epsilon \leq 0$$

and (5) follows from the definition of $s_p(\phi_N)$, $0 < p < \infty$. □

3. Lower bound estimation

In this section, we prove the lower bound estimate for $s_p(\phi_N)$.

Proposition 2. *Let ϕ_N be defined by (2). Then for $0 < p < \infty$ and for any integer $M \geq 2$ there exist a constant $1/2 < r < 1$ and an integer N_0 independent of p and M such that for $N \geq N_0$,*

$$s_p(\phi_N) \geq N - \frac{pM\ln|Q_N(2\pi/3)| + \ln(2+2^M r^{Np})}{pM\ln 2}.$$

Also

$$s_\infty(\phi_N) \geq N - \frac{\ln|Q_N(2\pi/3)|}{\ln 2}.$$

Obviously our main theorem follows from Propositions 1 and 2 by choosing the above M as the integral part of $-pN\ln r/\ln 2$. We need some lemmas to prove the proposition. The main estimate is Lemma 6, based on the accurate estimates of $Q_N(\xi)$ on $[0, \frac{2\pi}{3})$ and $Q_N(\xi)Q_N(2\xi)$ on $[\frac{2\pi}{3}, \pi]$. First we introduce an auxiliary function

$$g(\xi) = \begin{cases} (\cos\frac{\xi}{2})^{-2}, & |\xi| \leq \frac{\pi}{2}, \\ 4(\sin\frac{\xi}{2})^2, & \frac{\pi}{2} \leq |\xi| \leq \pi, \\ g(\xi - 2m\pi), & \xi \in 2m\pi + [-\pi, \pi]. \end{cases} \tag{6}$$

Lemma 3. *There exists a constant C independent of N and ξ such that*

$$C^{-1}N^{-C}g(\xi)^N \leq |Q_N(\xi)|^2 \leq g(\xi)^N. \tag{7}$$

Proof. The right inequality was proved by Cohen and Séré [5, Lemma 2.3]. It remains to prove the left inequality. Write

$$a_k(\xi) = \binom{N-1+k}{k}(\sin\frac{\xi}{2})^{2k}, \quad 0 \leq k \leq N-1.$$

Then

$$\frac{a_k(\xi)}{a_{k-1}(\xi)} = \frac{N+k-1}{k} \sin^2 \frac{\xi}{2}.$$

Let k_0 be the integral part of $(N-1)\tan^2 \frac{\xi}{2}$. Then by observing that

$$\frac{a_k(\xi)}{a_{k-1}(\xi)} \geq 1 \quad \text{if and only if} \quad k \leq (N-1)\tan^2 \frac{\xi}{2}$$

and that $|\tan \frac{\xi}{2}| \leq 1$ for $|\xi| \leq \frac{\pi}{2}$, we have

$$\max_{1\leq k\leq N-1} a_k(\xi) = a_{k_0}(\xi), \quad |\xi| \leq \pi/2.$$

By using the Stirling formula

$$k! = k^k e^{-k}\sqrt{2\pi k}(1+o(1)),$$

we have for $|\xi| \leq \pi/2$,

$$a_{k_0}(\xi) = \frac{(N+k_0-1)!}{k_0!(N-1)!}\big(\sin\frac{\xi}{2}\big)^{2k_0} = \frac{(N+k_0-1)^{N+k_0-1}}{k_0^{k_0}(N-1)^{N-1}}\big(\sin\frac{\xi}{2}\big)^{2k_0} B_N$$

where $C^{-1}N^{-C} \leq B_N \leq CN^C$. By substituting $-1 < k_0 - (N-1)\tan^2 \frac{\xi}{2} \leq 0$ into the above expression and simplifying, we have

$$a_{k_0}(\xi) = \tilde{B}_N\big(\cos\frac{\xi}{2}\big)^{-2N} = \tilde{B}_N\ g(\xi)^N, \qquad |\xi| \leq \pi/2,$$

where $(C')^{-1}N^{-C'} \leq \tilde{B}_N \leq C'N^{C'}$. This yields the left inequality of (7) for $|\xi| \leq \pi/2$.

For $\frac{\pi}{2} \leq |\xi| \leq \pi$, $\tan^2\frac{\xi}{2} \geq 1$ implies that

$$a_0(\xi) \leq a_1(\xi) \leq \cdots \leq a_{N-1}(\xi).$$

By using the Stirling formula again and making a similar estimation, we have

$$C^{-1}N^{-C}g(\xi)^N = C^{-1}N^{-C}(2\sin\frac{\xi}{2})^{2N} \leq a_{N-1}(\xi) \leq |Q_N(\xi)|^2, \qquad \frac{\pi}{2} \leq |\xi| \leq \pi,$$

which completes the proof. □

Lemma 4. *Let $g(\xi)$ be defined by* (6). *Then*

(8) $$0 \leq g(\xi)g(2\xi) \leq |g(\frac{2\pi}{3})|^2, \qquad |\xi| \in [\frac{2\pi}{3}, \pi],$$

and for $0 < \delta < \frac{\pi}{6}$ there exists $0 < r_1 < 1$ such that

(9) $$0 \leq g(\xi)g(2\xi) \leq r_1^2|g(\frac{2\pi}{3})|^2, \qquad |\xi| \in [\frac{2\pi}{3}+\delta, \pi].$$

Proof. Recall that $g(\xi)$ is an even periodic function, hence it suffices to prove (8) for $\xi \in [0,\pi]$. Note that

$$g(\xi)g(2\xi) = \begin{cases} 16\ \sin^2\frac{\xi}{2}\ \sin^2\xi, & \xi \in [\frac{2\pi}{3}, \frac{3\pi}{4}], \\ 4\sin^2\frac{\xi}{2}\ \cos^{-2}\xi, & \xi \in [\frac{3\pi}{4}, \pi]. \end{cases}$$

It is easy to check that the product is strictly decreasing on $[\frac{2\pi}{3}, \pi]$. Hence

$$0 \leq g(\xi)g(2\xi) \leq g(\frac{2\pi}{3})g(\frac{4\pi}{3}) = |g(\frac{2\pi}{3})|^2.$$

The second part follows from the strictly decreasing property. □

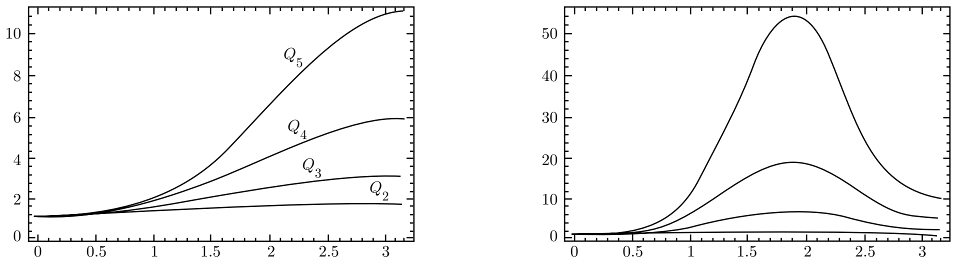

FIGURE 1.

Lemma 5. *For any integer $N \geq 1$,*

$$|Q_N(\xi)| \quad \leq \quad |Q_N(\frac{2\pi}{3})|, \qquad |\xi| \in [0, \frac{2\pi}{3}), \tag{10}$$

$$|Q_N(\xi)Q_N(2\xi)| \quad \leq \quad |Q_N(\frac{2\pi}{3})|^2, \qquad |\xi| \in [\frac{2\pi}{3}, \pi). \tag{11}$$

Furthermore for any $0 < \delta < \pi/6$, there exists $0 < r_2 < 1$ and an integer N_1 such that for $N > N_1$,

$$|Q_N(\xi)| \quad \leq \quad r_2^N |Q_N(\frac{2\pi}{3})|, \qquad |\xi| \in [0, \frac{2\pi}{3} - \delta), \tag{12}$$

$$|Q_N(\xi)Q_N(2\xi)| \quad \leq \quad r_2^N |Q_N(2\pi/3)|^2, \qquad |\xi| \in [\frac{2\pi}{3} + \delta, \pi]. \tag{13}$$

Proof. The first two inequalities were proved in [6, p. 222]. We use Lemma 3 to prove (12): for $|\xi| \in [0, \frac{2\pi}{3} - \delta]$, there exists $0 < r < 1$ such that

$$|Q_N(\xi)|^2 \leq g(\xi)^N \leq r^N g(\frac{2\pi}{3})^N \leq CN^C r^N |Q_N(\frac{2\pi}{3})|^2.$$

We pick r_2 so that $0 < r < r_2 < 1$. Hence (12) holds for N large enough. The proof of (13) is similar by using Lemma 4. □

In regard to the above lemma, we include the graphs of $Q_N(\xi)$ and $Q_N(\xi)Q_N(2\xi)$, $N = 2, 3, 4, 5$, for the convenience of the reader (see Figure 1).

For any $0 < \delta < \pi/6$ and $\xi \in R$, we define

$$I_k(\xi, \delta) = \{j: \ 1 \leq j \leq k, \ 2^j \xi \in \bigcup_{m \in \mathbb{Z}} [-\frac{2\pi}{3} + \delta, \frac{2\pi}{3} - \delta] + 2m\pi \ \}$$

and let $i_k(\xi, \delta)$ be the number of elements of $I_k(\xi, \delta)$.

Lemma 6. *Let N_1 be as in Lemma 5. Then there exists a constant C_N and a constant $0 < r_3 < 1$ depending on $0 < \delta < \pi/6$ only, such that for $k > 2$ and $N \geq N_1$,*

$$\prod_{j=1}^{k} |Q_N(2^j \xi)| \leq C_N \ r_3^{N i_k(\xi, \delta)} |Q_N(\frac{2\pi}{3})|^k. \tag{14}$$

Proof. We use $r_2(\delta)$ to denote the r_2 in Lemma 5, and choose $r_3(\delta)$ so that $r_2(\delta)$, $r_2(\delta/2) < r_3(\delta) < 1$. It is easy to see that by letting C_N be large enough, the lemma holds for $k = 1$ and $k = 2$. We assume that (14) holds for $k < l$ with $l \geq 3$. For $k = l$, we divide the proof into four cases:

(i) If $2\xi \in [-\frac{2\pi}{3}+\delta, \frac{2\pi}{3}-\delta]+2m\pi$, then $i_k(\xi,\delta) = i_{k-1}(2\xi,\delta)+1$. We can write

$$\prod_{j=1}^{k} |Q_N(2^j\xi)| = |Q_N(2\xi)| \prod_{j=1}^{k-1} |Q_N(2^j(2\xi))|,$$

and (14) follows from (12) with $r_2(\delta) < r_3(\delta) < 1$ and the induction hypothesis.

(ii) If $2\xi \in \big([-\frac{2\pi}{3}, -\frac{2\pi}{3}+\delta) \cup (\frac{2\pi}{3}-\delta, \frac{2\pi}{3}]\big) + 2m\pi$, then $i_k(\xi,\delta) = i_{k-1}(2\xi,\delta)$ and the same induction hypothesis together with (10) implies (14).

(iii) If $2\xi \in \big([-\frac{2\pi}{3}-\frac{\delta}{2}, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \frac{2\pi}{3}+\frac{\delta}{2}]\big) + 2m\pi$, then it follows that $2\xi, 4\xi \notin \bigcup_{m\in\mathbb{Z}}[-\frac{2\pi}{3}+\delta, \frac{2\pi}{3}-\delta]+2m\pi$, hence $i_k(\xi,\delta) = i_{k-2}(4\xi,\delta)$. Write

$$\prod_{j=1}^{k} |Q_N(2^j\xi)| = |Q_N(2\xi)Q_N(4\xi)| \prod_{j=1}^{k-2} |Q_N(2^j(4\xi))|$$

and (14) follows from (11).

(iv) If $2\xi \in \big([-\pi, -\frac{2\pi}{3}-\frac{\delta}{2}) \cup (\frac{2\pi}{3}+\frac{\delta}{2},\ \pi]\big) + 2m\pi$, then $i_k(\xi,\delta) \le i_{k-2}(4\xi,\delta)+1$. By using the above product, $r_2(\delta/2) < r_3(\delta) < 1$ and (13), we have

$$\prod_{j=1}^{k} |Q_N(2^j\xi)| \le r_2(\frac{\delta}{2})^N C_N r_3^{N i_{k-2}(4\xi,\delta)} |Q_N(\frac{2\pi}{3})|^k \le C_N r_3^{N i_k(\xi,\delta)} |Q_N(\frac{2\pi}{3})|^k.$$

The induction step follows from these four cases. $\square$

For any integer $M \ge 2$, $k \ge 1$ and $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_{kM})$ with $\epsilon_i = 0$ or 1, let $\alpha_{kM}(\epsilon)$ be the cardinality of the set

$$A_{kM}(\epsilon) = \{l:\ 1 \le l \le k,\ (\epsilon_{(l-1)M+1}, \cdots, \epsilon_{lM}) \text{ has two consecutive } 0 \text{ or } 1\}.$$

Then $\alpha_{kM}(\epsilon) = \sum_{l=0}^{k-1} \alpha_M(\epsilon^l)$ where $\epsilon^l = (\epsilon_{lM+1}, \cdots, \epsilon_{(l+1)M})$ and

(15)

$$\begin{aligned}\sum_{\epsilon=(\epsilon_1,\cdots,\epsilon_{kM})\in\{0,1\}^{kM}} r^{\alpha_{kM}(\epsilon)} &= \sum_{l=0}^{k-1} \sum_{\epsilon^l=(\epsilon_{Ml+1},\cdots,\epsilon_{M(l+1)})\in\{0,1\}^M} \prod_{j=0}^{k-1} r^{\alpha_M(\epsilon^j)} \\ &= \Big(\sum_{\epsilon=(\epsilon_1,\cdots,\epsilon_M)\in\{0,1\}^M} r^{\alpha_M(\epsilon)}\Big)^k = (2+(2^M-2)r)^k,\end{aligned}$$

where $r > 0$ and the last equality follows from the fact that $\alpha_M(\epsilon) = 1$ for any $\epsilon \in \{0,1\}^M$ except $\epsilon = (0,1,0,1,\cdots) \in \{0,1\}^M$ or $(1,0,1,0,\cdots) \in \{0,1\}^M$.

Lemma 7. *Let $0 < \delta < \pi/6$. For $\xi \in [\pi, 2\pi)$, write $\xi = 2\pi(\sum_{j=1}^{kM} \epsilon_j 2^{-j} + \eta)$ with $0 \le \eta < 2^{-kM}$. Then*

(16) $$\alpha_{kM}(\epsilon) - 1 \le i_{kM}(\xi,\delta).$$

Proof. Suppose $l \in A_{kM}(\epsilon)$ and $l \ge 2$. Then there exists an index $j \ge 2$ such that $(l-1)M+1 \le j \le lM-1$ and $\epsilon_j = \epsilon_{j+1}$. Hence

$$2^{j-1}\xi = 2m\pi + 2\pi(\frac{\epsilon_j}{2} + \frac{\epsilon_{j+1}}{4} + \eta')$$

for some integer m and $0 \le \eta' < 1/4$. For $\epsilon_j = \epsilon_{j+1} = 0$,

$$2\pi(\frac{\epsilon_j}{2} + \frac{\epsilon_{j+1}}{4} + \eta') \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

and for $\epsilon_j = \epsilon_{j+1} = 1$,

$$2\pi(\frac{\epsilon_j}{2} + \frac{\epsilon_{j+1}}{4} + \eta' - 1) \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

Hence $2^{j-1}\xi \in \bigcup_{m\in\mathbb{Z}}[-2\pi/3+\delta, 2\pi/3-\delta] + 2m\pi$, i.e., $j-1 \in I_{kM}(\xi,\delta)$. What we have just shown is that each $l \in A_{kM}(\epsilon)$ corresponds to at least one distinct $j \in I_{kM}(\xi,\delta)$ provided that $l \geq 2$. The lemma follows from this assertion. $\square$

Proof of Proposition 2. Recall that

$$\hat{\phi}_N(\xi) = \big(\frac{1-e^{-i\xi}}{i\xi}\big)^N \prod_{j=1}^{\infty} Q_N(\xi/2^j).$$

Let $r = r_3(\pi/6)$. Then for $\xi \in [2^{kM-1}\pi, 2^{kM}\pi]$ and $N \geq N_1$, Lemma 6 implies that

$$\begin{aligned} |\hat{\phi}_N(\xi)| &\leq C2^{-kMN} \prod_{j=1}^{kM-1} |Q_N(2^{j-kM}\xi)| \\ &\leq C'2^{-kMN} r^{Ni_{kM}(2^{-kM}\xi,\pi/6)} |Q_N(\frac{2\pi}{3})|^{kM}, \end{aligned}$$

where C' depends on N only. It now follows from (3), (16) and (15) that

$$\begin{aligned} &\int_{2^{kM-1}\pi}^{2^{(k+1)M-1}\pi} |\hat{\phi}_N(\xi)|^p d\xi = \sum_{l=0}^{M-1} \int_{2^{kM-1+l}\pi}^{2^{kM+l}\pi} |\hat{\phi}_N(\xi)|^p d\xi \leq 2^M \int_{2^{kM-1}\pi}^{2^{kM}\pi} |\hat{\phi}_N(\xi)|^p d\xi \\ &\quad \leq C'2^{-NkMp} |Q_N(\frac{2\pi}{3})|^{kMp} \int_{2^{kM-1}\pi}^{2^{kM}\pi} r^{Npi_{kM}(2^{-kM}\xi,\pi/6)} d\xi \\ &\quad \leq C''2^{-NkMp} |Q_N(\frac{2\pi}{3})|^{kMp} \sum_{\epsilon_j\in\{0,1\},1\leq j\leq kM} r^{Np\alpha_{kM}(\epsilon)} \\ &\quad \leq C''2^{-NkMp} |Q_N(\frac{2\pi}{3})|^{kMp} (2 + 2^M r^{Np})^k. \end{aligned}$$

This completes the proof. $\square$

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