PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 4, Pages 981–985
S 0002-9939(99)05312-5
Article electronically published on August 5, 1999

CONTINUED FRACTIONS WITH BOUNDED PARTIAL QUOTIENTS

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(Communicated by David E. Rohrlich)

ABSTRACT. This paper gives the exact bound of the continued fraction expansion of $\frac{a\theta+b}{c\theta+d}$ when θ has bounded partial quotients and $h\colon x \mapsto \frac{ax+b}{cx+d}$ is a Möbius transformation where all entries are integers.

1. INTRODUCTION

The article deals with the following problem: given a real number θ, it follows from the results of Raney [Ra] that if a, b, c, d are integers with $ad - bc \neq 0$, then $\frac{a\theta+b}{c\theta+d}$ has bounded partial quotients if θ does (see also [Ha] and [Sh1]).

In [La-Sh], the following result is obtained: let θ be a real number with continued fraction expansion $\theta = [a_0; a_1, a_2, \dots]$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a matrix with integer entries and nonzero determinant. Put $D = |\det(M)|$ and $\frac{a\theta+b}{c\theta+d} = [a_0^*; a_1^*, a_2^*, \dots]$. If θ has bounded partial quotients with $a_j \leq K$ for all sufficiently large j, then $a_j^* \leq D(K+2)$ for all sufficiently large j. The proof uses the homogeneous Diophantine approximation constant $L_\infty(\theta) = \limsup_{q\to\infty}(q\|q\theta\|)^{-1}$ (see also [Cu-Me]).

In this paper, the exact bound is given. Put $y_K = [\overline{K, 1}] = \frac{K+\sqrt{K^2+4K}}{2}$. Then

$$a_j^* \leq D - 1 + [Dy_K] \tag{1}$$

for all sufficiently large j. For instance, when $D = 1000$ and $K = 2$, the bound is 3731 instead of 4000 given in [La-Sh], and in the well-known case of the multiplication by 2, the bound is $2K + 2$ instead of $2K + 4$. The proof uses a former result [St1] and transducers described in [Li-St] for computing the continued fraction expansion of $\frac{a\theta+b}{c\theta+d}$. This expansion is given by a word where all letters are nonnegative integers which is transformed into a word where all letters are positive integers except perhaps the first one.

In Section 2, the transducer is described in detail. In Section 3, we prove that the bound is optimal and is attained when $\theta = [\overline{1, K}] = \frac{K+\sqrt{K^2+4K}}{2K}$ and $M = \left(\begin{smallmatrix} D & 0 \\ D-1 & 1 \end{smallmatrix}\right)$. Inequality (1) is a consequence of three lemmas: the first one gives arithmetical properties of the transducer and the other lemmas provide information when (0) is in the output of the transducer.

Received by the editors June 5, 1998.
1991 *Mathematics Subject Classification.* Primary 11A55.
Key words and phrases. Continued fractions, bounded partial quotients, Möbius transformation, quadratic number, transducer.
The author thanks P. Liardet who pointed out this problem.

2. Image by a Möbius transformation

We recall basic definitions and facts given in [Li-St]: Let $\mathcal{M}_{2,\mathbf{N}}$ be the set of all matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ $(a, b, c, d \in \mathbf{N})$ such that $ad - bc \neq 0$. M is said to be in $\mathcal{D}_2$ when $a \geq c$ and $b \geq d$, in $\mathcal{D}'_2$ when $a \leq c$ and $b \leq d$ and in $\mathcal{E}_2$ when $(a-c)(b-d) < 0$. $\{\mathcal{D}_2, \mathcal{D}'_2, \mathcal{E}_2\}$ is a partition of $\mathcal{M}_{2,\mathbf{N}}$. Every matrix of $\mathcal{E}_2$ verifies $\max(a, b, c, d) \leq |\det M| = D$. For all matrices $M \in \mathcal{D}_2 \cup \mathcal{D}'_2$, there exists a unique factorization

$$(F) \qquad M = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} M'$$

such that $c_0 \in \mathbf{N}$, $c_1, \dots, c_n \in \mathbf{N}\backslash\{0\}$ and $M' \in \mathcal{E}_2$. This factorization will be denoted by $M = \Pi_{c_0 c_1 \cdots c_n} M'$. Moreover, if $n \geq 1$, then $\frac{p_{n-1}}{q_{n-1}} = [c_0; c_1, \dots, c_{n-1}]$ is a common convergent of $\frac{a}{c}$ and $\frac{b}{d}$. Now all computations for the continued fraction expansion of $\frac{a\theta+b}{c\theta+d}$ can be reduced to the case where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{E}_2$ and $\theta > 1$. Put $\theta = [a_0; a_1, a_2, \dots]$ and $D = |\det(M)|$. The continued fraction expansion of $\frac{a\theta+b}{c\theta+d}$ is given by the finite state transducer $\mathcal{T}_D = (\mathcal{C}, \mathcal{B}, \mathcal{A}, \Phi, \Psi)$ where:

- The input alphabet is $\mathcal{C} = \mathbf{N}\backslash\{0\}$.
- The space of states $\mathcal{B}$ is the finite set of all matrices $B \in \mathcal{E}_2$ such that $|\det(B)| = D$ and the initial state is $B_0 = M$.
- The output alphabet is $\mathcal{A} = \mathbf{N}$ and the monoid generated by $\mathcal{A}$ is denoted by $\mathcal{A}^*$.
- $\Phi = \{\phi_c\colon c \in \mathcal{C}\}$ and $\Psi = \{\psi_c\colon c \in \mathcal{C}\}$ are two families of maps ($\phi_c\colon \mathcal{B} \to \mathcal{B}$ and $\psi_c\colon \mathcal{B} \to \mathcal{A}^*$) defined as follows: for all B in $\mathcal{B}$ and for all c in $\mathcal{C}$, if $B\left(\begin{smallmatrix} c & 1 \\ 1 & 0 \end{smallmatrix}\right) = B' \in \mathcal{E}_2$, then $\phi_c(B) = B'$ and $\psi_c(B) = \wedge$ (the empty word). Otherwise, by (F), $B\left(\begin{smallmatrix} c & 1 \\ 1 & 0 \end{smallmatrix}\right) = \Pi_{c_0 c_1 \cdots c_n} B'$; then $\phi_c(B) = B'$ and $\psi_c(B) = c_0 c_1 \cdots c_n$.

Now, with any input word $a_0 a_1 \cdots a_k \in \mathcal{C}^{k+1}$, we associate a sequence of states $B_1 = \phi_{a_0}(M)$ and $B_{i+1} = \phi_{a_i}(B_i)$, $i = 1, \dots, k$, and we define

$$[\Psi, \Phi]_{a_0 a_1 \cdots a_k} = \psi_{a_0}(M)\psi_{a_1}(B_1) \cdots \psi_{a_k}(B_k)$$

which is a word in $\mathcal{A}^*$. Moreover (00) is never a factor of this word.

Finally, let μ be the contraction map which transforms a word in $\mathcal{A}^*$ into a word where all letters are positive integers (except perhaps the first one), replacing from left to right factors $a(0)b$ by the letter $a+b$. If

$$\mu \circ [\Psi, \Phi]_{a_0 a_1 \cdots a_n} = c_0 c_1 \cdots c_k,$$

then $\frac{a\theta+b}{c\theta+d} = [c_0; c_1, \dots, c_{k-1}, \dots]$ and the partial quotient following c_{k-1} is $\geq c_k$.

3. Real numbers with bounded partial quotients

In Lemma 1, we prove that the bound given in (1) can be attained.

Lemma 1. *Let K be a positive integer and x_K the quadratic number defined by $x_K = [\overline{1, K}] = \frac{K+\sqrt{K^2+4K}}{2K}$. Put $M = \left(\begin{smallmatrix} D & 0 \\ D-1 & 1 \end{smallmatrix}\right)$ and let h be the Möbius transformation associated with M. Then*

$$h(x) = [1; D-1+[Dy_K], \dots].$$

Proof. By applying the transducer $\mathcal{T}_D, \phi_1(M) = \left(\begin{smallmatrix} 0 & 1 \\ D & 0 \end{smallmatrix}\right)$ and $\psi_1(M) = 1(D-1)$. Let $\left(\frac{p_n}{q_n}\right)$ be the convergents of $y_K = [\overline{K, 1}] = \frac{K+\sqrt{K^2+4K}}{2}$. Then $[\left(\begin{smallmatrix} K & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)]^n =$

$\left(\begin{smallmatrix} p_{2n-1} & p_{2n-2} \\ q_{2n-1} & q_{2n-2} \end{smallmatrix}\right)$ for all integers $n \geq 1$. When n tends to infinity $D\frac{p_{2n-1}}{q_{2n-1}}$ and $D\frac{p_{2n-2}}{q_{2n-2}}$ have the same limit Dy_K. Hence, there exists an integer n such that

$$\phi_1(M)\begin{pmatrix} p_{2n-1} & p_{2n-2} \\ q_{2n-1} & q_{2n-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} [Dy_K] & 1 \\ 1 & 0 \end{pmatrix} N, \qquad N \in \mathcal{D}_2.$$

Finally $\mu \circ [\Psi, \Phi]_{\overline{1K}^n} = 1(D - 1 + [Dy_K]) \cdots$.

Our main result is the following:

Theorem. *Let $\theta = [a_0; a_1, a_2, \dots]$ be a positive real number such that $a_j \leq K$ for all integers j, and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a matrix in $\mathcal{E}_2$ such that $|\det(M)| = D$. Then $\frac{a\theta+b}{c\theta+d} = [a_0^*; a_1^*, a_2^*, \dots]$ has bounded partial quotients and $a_j^* \leq D - 1 + [Dy_K]$ for all integers j.*

The proof requires three technical lemmas. Lemma 2 gives more information about the transducer.

Lemma 2. *Let k be a positive integer and $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ a matrix in $\mathcal{E}_2$. Let $[M] = [c_0; c_1, \dots, c_r, 1]$ be the continued fraction expansion of $\frac{\delta-\beta}{\alpha-\gamma}$ which contains at least two partial quotients and whose last partial quotient is 1. Then*

(i) *If $k > c_0$ and $\alpha > \gamma$, the first letter of $\psi_k(M)$ is ≥ 1.*
(ii) *If $k > c_0$ and $\alpha < \gamma$, the first letter of $\psi_k(M)$ is 0.*
(iii) *If $k = c_0$, $\psi_k(M) = \wedge$ and $[\phi_k(M)] = [c_1; \dots, c_r, 1]$.*
(iv) *If $k < c_0$, $\psi_k(M) = \wedge$ and $[\phi_k(M)] = [0; c_0 - k, c_1, \dots, c_r, 1]$.*

Proof. Suppose $k > c_0$ and $\alpha > \gamma$. $M\Pi_k = \left(\begin{smallmatrix} \alpha k+\beta & \alpha \\ \gamma k+\delta & \gamma \end{smallmatrix}\right)$. Since $c_0 < \frac{\delta-\beta}{\alpha-\gamma} \leq c_0 + 1 \leq k$, then $(\alpha k + \beta) - (\gamma k + \delta) \geq 0$, $M\Pi_k \in \mathcal{D}_2$ and (i) is verified. Similarly, $M\Pi_k \in \mathcal{D}_2'$ when $k > c_0$ and $\alpha < \gamma$ and (ii) is verified. Now if $k \leq c_0$, then $k < \frac{\delta-\beta}{\alpha-\gamma}$ and $[(\alpha k + \beta) - (\gamma k + \delta)](\alpha - \gamma) < 0$. Hence $[M\Pi_k] \in \mathcal{E}_2$. Moreover $[M\Pi_k]$ is the continued fraction expansion of $\frac{\alpha-\gamma}{(\delta-\beta)-k(\alpha-\gamma)}$ which is the image of $[M]$ by the Möbius transformation associated with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -k \end{smallmatrix}\right)$. Then we obtain $[M\Pi_k] = [c_1; \dots, c_r, 1]$ when $k = c_0$ and since

$$\begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_0 - k & 1 \\ 1 & 0 \end{pmatrix}, \qquad [M\Pi_k] = [0; c_0 - k, c_1, \dots, c_r, 1]$$

when $k < c_0$.

The next lemmas give more information when we get (0) in the output of the transducer.

Lemma 3. *Let θ and M be defined as in the Theorem. Put $[\Psi, \Phi]_{a_0 a_1 \cdots a_r} = c_0 c_1 \cdots c_p$ and $\mu \circ [\Psi, \Phi]_{a_0 a_1 \cdots a_r} = c_0' c_1' \cdots c_n'$. Then $c_i \leq DK$ $(i = 1, 2, \dots, p)$, $c_0' \leq [Dy_K]$ and $c_1' \leq [Dy_K]$ when $c_0' = 0$.*

Proof. Put $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. If $\gamma = 0$, then $\delta > \gamma$ and by Lemma 2, $\psi_{a_0}(M) = \wedge$ or $\psi_{a_0}(M) = c_0$ with $c_0 = [\frac{\alpha a_0+\beta}{\delta}] \leq DK$. Similarly, $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \gamma & \delta \\ 0 & \beta \end{smallmatrix}\right)$ when $\alpha = 0$. Then $\psi_{a_0}(M) = \wedge$ or $(0)c_1$ with $c_1 \leq DK$. Suppose now that $\alpha\gamma \neq 0$ and $\psi_{a_0}(M) \neq \wedge$. Then all letters of $\psi_{a_0}(M)$ are partial quotients of $\frac{\alpha}{\gamma}$ except perhaps the last one. Hence, all these letters are $\leq D - 1$. Finally, if this last letter is a partial quotient of $\frac{\alpha a_0+\beta}{\gamma a_0+\delta}$ and is not a partial quotient of $\frac{\alpha}{\gamma}$, it is clearly $< K$.

Now, it is obvious that a positive number θ with bounded partial quotients $\leq K$ is such that $\frac{1}{y_K} \leq \theta \leq y_K$. Let h be the Möbius transformation associated with

M. A simple computation leads to $\frac{1}{Dy_K} \leq h(\theta) \leq Dy_K$. Then $c'_0 \leq [Dy_K]$ and $c'_1 \leq [Dy_K]$ when $c'_0 = 0$.

Lemma 4. *As in Lemma 3, put* $[\Psi, \Phi]_{a_0 a_1 \cdots a_r} = c_0 c_1 \cdots c_p$. *Suppose that* $ab(0)c$ *is a factor of* $c_0 c_1 \cdots c_p$ *such that* $c \geq D$. *Then* $b \leq D-1$ *and* $a \neq 0$.

Proof. Suppose that $b \geq D$. Then, there exists a matrix $M = \left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{smallmatrix}\right) \in \mathcal{E}_2$ and a positive integer $k \leq K$ such that the last letter of $\psi_k(M')$ is b. From the proof of Lemma 3, either $\alpha'\gamma' = 0$, or b is a convergent of $\frac{\alpha' k + \beta'}{\gamma' k + \delta'}$ and is not a convergent of $\frac{\alpha'}{\gamma'}$. In both cases $M'' = \phi_k(M')$ is a matrix of the form $\left(\begin{smallmatrix} \alpha'' & 0 \\ \gamma'' & \delta'' \end{smallmatrix}\right)$. Hence $\alpha'' \neq 0$ and for all words $k_1 k_2 \cdots k_j$ in $\mathcal{C}^*$, the first entry of $M'' \Pi_{k_1 k_2 \cdots k_j}$ is $\neq 0$. Then, from the proof of Lemma 3, the letters following b cannot be $(0)c$ with $c \geq D$. Therefore $b \leq D - 1$.

Suppose now that $\psi_k(M') = (0)b$. Then the first entry of $M'' = \phi_k(M')$ is $\alpha' k + \beta' \neq 0$. As before, $(0)b$ cannot be followed by $(0)c$ with $c \geq D$.

Now, we are ready to give the proof of the Theorem. Put $[\Psi,\Phi]_{a_0 a_1 \cdots a_r} = c_0 c_1 \cdots c_p$. By Lemma 3, $c_i \leq DK$ $(i = 0, 1, \ldots, p)$. Suppose that there exists a factor $k_1(0)k_2(0)\cdots(0)k_s$ in the word $c_0 c_1 \cdots c_p$. If $k_2 \geq D$, then by Lemma 3 one has $k_1 \leq D-1$, $\sum_{i=2}^{i=s} k_i \leq [Dy_K]$ and $\sum_{i=1}^{i=s} k_i \leq D - 1 + [Dy_K]$. Moreover, the letter preceding k_1 is $\neq 0$.

Suppose now that $D \leq k_1 \leq DK$. From the proof of Lemma 4, there exists a matrix $M'' = B_m \in \mathcal{E}_2$ of the form $\left(\begin{smallmatrix} \alpha'' & 0 \\ \gamma'' & \delta'' \end{smallmatrix}\right)$ such that $(0)k_2 \cdots (0)k_s \cdots = \psi_{a_m}(M'')\psi_{a_m+1}(B_{m+1}) \cdots \psi_{a_r}(B_r)$. Let h'' be the Möbius transformation associated with M''. For all numbers $\theta > 1$ with partial quotients $\leq K$, it is clear that $h''(\theta) \geq \frac{x_K}{D}$ with $x_K = [\overline{1, K}]$. Then $\sum_{i=2}^{i=s} k_i \leq [\frac{D}{x_K}]$. Note that $DK + [\frac{D}{x_K}] = [Dy_K]$. From Lemma 4, if $ab(0)k_1$ is a factor of $c_0 c_1 \cdots c_p$, then $b \leq D - 1$ and $a \neq 0$. Finally, all letters of $\mu \circ [\Psi, \Phi]_{a_0 a_1 \cdots a_r}$ are $\leq D - 1 + [Dy_K]$.

Now, we can give a generalization of the above Theorem.

Corollary. *Let* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be a matrix with entries in* $\mathbf{Z}$ *such that* $|\det(M)| = D \geq 1$ *and* θ *a real number with partial quotients* $a_j \leq K$ *for all sufficiently large* j. *Put* $\frac{a\theta + b}{c\theta + d} = [a_0^*; a_1^*, a_2^*, \ldots]$. *Then* $a_j^* \leq D - 1 + [Dy_K]$ *for all sufficiently large* j.

Proof. It is clear that there exist two integers i and k such that:

- $[a_k^*; a_{k+1}^*, a_{k+2}^*, \ldots] = \frac{a'\theta' + b'}{c'\theta' + d'}$.
- $\theta' = [a_i; a_{i+1}, a_{i+2}, \ldots]$ with $a_j \leq K$ for all $j \geq i$.
- $M' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathcal{E}_2$ and $|\det(M')| = D$.

Put $K' = \max_{0 \leq j \leq k-1} a_j^*$. Then, by the Theorem, $a_j^* \leq \max(K', D - 1 + [Dy_K])$ for all integers j.

References

[Cu-Me] T. W. Cusick and M. Mendès France, *The Lagrange spectrum of a set*, Acta Arith. **34** (1979), 287–293. MR **80i:**10038a

[Ha] M. Hall, *On the sum and product of continued fractions*, Annals of Math. **48** (1947), 966–993. MR **9:**226b

[La-Sh] J. C. Lagarias and J. O. Shallit, *Linear Fractional Transformations of Continued Fractions with Bounded Partial Quotients*, Journal de théorie des nombres de Bordeaux **9** (1997), 267–279. CMP 98:11

[Li-St] P. Liardet and P. Stambul, *Algebraic Computations with Continued Fractions*, Journal of Number Theory **73** (1998), 92–121. CMP 99:04

[Ra] G. N. Raney, *On continued fractions and finite automata*, Math. Annalen **206** (1973), 265–283. MR **49:**4922

[Sh1] J. O. Shallit, *Real numbers with bounded partial quotients: a survey*, Enseign. Math. **38** (1992), 151–187.

[St1] P. Stambul, *Contribution à l'étude des propriétés arithmétiques des fractions continuées*, Thèse de l'Université de Provence (1994).

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