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ON FOURTH-ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper is concerned with the existence and uniqueness of a solution for a class of fourth-order elliptic boundary value problems. The existence of a solution is proven by the method of upper and lower solutions without any monotone nondecreasing or nonincreasing property of the nonlinear function. Sufficient conditions for the uniqueness of a solution and some techniques for the construction of upper and lower solutions are given. All the existence and uniqueness results are directly applicable to fourth-order two-point boundary value problems.

1. INTRODUCTION

Fourth-order two-point boundary value problems have been investigated by many researchers in recent years, and various forms of the equation and boundary condition have been discussed (cf. [1]–[3], [5]–[9], [16], [19], [20]). Of special interest is the following fourth-order two-point boundary value problem:

$$\begin{aligned} &u^{(\mathrm{iv})} = f(x,u,u'') \qquad (0<x<1),\\ &\beta_0 u(0) - \alpha_0 u'(0) = h_1, \qquad \beta_1 u(1) + \alpha_1 u'(1) = h_1^*,\\ &\beta_0^* u''(0) - \alpha_0^* u'''(0) = h_2, \qquad \beta_1^* u''(1) + \alpha_1^* u'''(1) = h_2^*. \end{aligned} \tag{0.1}$$

The above problem arises from the study of static deflection of an elastic bending beam where u denotes the deflection of the beam and $f(x,u,u'')$ represents the loading force that may depend on the deflection and the curvature of the beam. In this paper we extend the problem (0.1) to a more general fourth-order elliptic boundary value problem in the form

$$\begin{aligned} &\Delta(a(x)\Delta u) = f(x,u,\Delta u) \qquad (x\in\Omega),\\ &\alpha_1(x)\frac{\partial u}{\partial \nu} + \beta_1(x)u = h_1(x), \quad \alpha_2(x)\frac{\partial (a\Delta u)}{\partial \nu} + \beta_2(x)(a\Delta u) = h_2(x) \qquad (x\in\partial\Omega), \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, Δ is the Laplace operator in Ω, and $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. We assume that $a(x)$ is a positive C^1-function on $\bar\Omega \equiv \Omega\cup\partial\Omega$, $f(x,u,v)$ is a C^α-function of its arguments, and for each $i=1,2$, α_i, β_i and h_i are $C^{1+\alpha}$-functions on $\partial\Omega$ with

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either $\alpha_i = 0$, $\beta_i = 1$ (Dirichlet condition) or $\alpha_i > 0$, $\beta_i \geq 0$ (Neumann and Robin condition). As in problem (0.1) the consideration of (1.1) also has an implication in the static deflection of an elastic plate.

Literature dealing with the two-point boundary value problem (0.1) is extensive, and much of the discussion is devoted to the Dirichlet boundary condition

$$(0.1)_b \qquad u(0) = h_1, \quad u(1) = h_1^*, \qquad u''(0) = h_2, \quad u''(1) = h_2^*.$$

An earlier existence result is given in [1] where the function $f(x,u,v)$ is assumed uniformly bounded. This result was extended by Yang [20] who assumed that $f(x,u,v)$ satisfies the growth condition

$$|f(x,u,v)| \leq a|u| + b|v| + c \quad \text{for } u, v \in \mathbb{R},$$

where a, b and c are positive constants with $a/\pi^4 + b/\pi^2 < 1$. In the special case $f(x,u,v) = -c(x)u + d(x)$, Usmani [19] showed that problem (0.1), $(0.1)_b$ has a unique solution if $\inf[c(x)] < \pi^4$. More recently, Ma, Zhang and Fu [9] obtained an existence result for (0.1), $(0.1)_b$ by the method of upper and lower solutions. However, they require that $f(\cdot,u,v)$ be nondecreasing in u and nonincreasing in v. Additional works dealing with the existence problem of fourth-order two-point boundary value problems can be found in [2, 3], [5]-[9], [16].

In the case of the elliptic boundary value problem (1.1) the existence problem was investigated by many researchers in the framework of coupled system of second-order equations (e.g. see [12, 13, 15, 18]). Cosner and Schaefer [4] studied a comparison principle for a class of fourth-order elliptic operators, while Tarantello [17] showed the existence of a negative solution for the problem

$$(1.2) \qquad \begin{array}{ll} \Delta^2 u + c\Delta u = b[(u+1)^+ - 1] & \text{in } \Omega, \\ u = 0, \ \Delta u = 0 & \text{on } \partial\Omega, \end{array}$$

where $w^+ \equiv \max\{w, 0\}$, and b and c are positive constants with $c < \lambda_0$ and $b \geq \lambda_0(\lambda_0 - c)$ (see Eq. (2.6)). More recently, Micheletti and Pistoia [10, 11] investigated the existence of multiple solutions for problem (1.2) and a more general nonlinear function $f(x,u)$ using a variational approach.

The purpose of this paper is to show the existence and uniqueness of a solution to the problem (1.1), including the two-point boundary value problem (0.1), by the method of upper and lower solutions without any monotone condition on $f(x,u,v)$. When $f(x,u,v)$ does have a monotone nondecreasing property in u (but not necessarily nonincreasing in v) we can construct a sequence which converges monotonically to a maximal solution or a minimal solution, depending on whether the initial iteration is an upper solution or a lower solution (see Theorem 1). It is also shown that under some additional conditions the monotone sequences converge to a unique solution of (1.1) when $f(x,u,v)$ is either nondecreasing or nonincreasing in u (see Theorems 2 and 3). The above existence-uniqueness results, including some techniques for the construction of upper and lower solutions, are stated in section 2. Proofs of these results are given in section 3.

2. The main results

The existence and uniqueness of a solution to problem (1.1) are obtained in relation to a pair of upper and lower solutions which are defined in the following definition. For notational convenience we set

$$B_i[w] \equiv \alpha_i \partial w/\partial \nu + \beta_i w, \qquad i = 1, 2.$$

Definition 2.1. A pair of functions $\tilde{u}$, $\hat{u} \in C^4(\Omega) \cap C^{2+\alpha}(\bar{\Omega})$ are called coupled upper and lower solutions of (1.1) if $\tilde{u} \geq \hat{u}, \Delta\tilde{u} \leq \Delta\hat{u}$ and

$$\begin{aligned} &\Delta(a\Delta\tilde{u}) \geq f(x,u,\Delta\tilde{u}), \quad \Delta(a\Delta\hat{u}) \leq f(x,u,\Delta\hat{u}) \quad \text{for } \hat{u} \leq u \leq \tilde{u}, \\ &B_1[\tilde{u}] \geq h_1 \geq B_1[\hat{u}], \qquad B_2[a\Delta\tilde{u}] \leq h_2 \leq B_2[a\Delta\hat{u}]. \end{aligned} \tag{2.1}$$

For a given pair of coupled upper and lower solutions $\tilde{u}$, $\hat{u}$ we define

$$\langle \hat{u}, \tilde{u} \rangle \equiv \{u \in C^2(\bar{\Omega});\ \hat{u} \leq u \leq \tilde{u},\ \Delta\tilde{u} \leq \Delta u \leq \Delta\hat{u}\} \tag{2.2}$$

and make the following basic hypothesis.

(H_1): $f(x,u,v)$ is Hölder continuous in x and satisfies the Lipschitz condition

$$\begin{aligned} &|f(x,u_1,v_1) - f(x,u_2,v_2)| \leq K(|u_1-u_2| + |v_1-v_2|) \qquad (x \in \bar{\Omega}), \\ &\qquad\qquad \text{for } \hat{u} \leq u_j \leq \tilde{u},\ \Delta\tilde{u} \leq v_j \leq \Delta\hat{u} \qquad (j=1,2), \end{aligned} \tag{2.3}$$

where K is a positive constant.

For the existence of maximal and minimal solutions we also assume that $f(\cdot,u,v)$ possesses the following monotone nondecreasing property in u:

(H_2):

$$f(x,u_1,v) \leq f(x,u_2,v) \quad \text{for } \hat{u} \leq u_1 \leq u_2 \leq \tilde{u},\ \Delta\tilde{u} \leq v \leq \Delta\hat{u}. \tag{2.4}$$

Under this condition the differential inequalities for $\tilde{u}$ and $\hat{u}$ in (2.1) become

$$\Delta(a\Delta\tilde{u}) \geq f(x,\tilde{u},\Delta\tilde{u}), \qquad \Delta(a\Delta\hat{u}) \leq f(x,\hat{u},\Delta\hat{u}). \tag{2.5}$$

Theorem 1. *Let $\tilde{u}$, $\hat{u}$ be a pair of coupled upper and lower solutions, and let Hypothesis (H_1) hold. Then there exists at least one solution $u^* \in \langle \hat{u}, \tilde{u} \rangle$ to* (1.1). *If, in addition, (H_2) holds, then problem* (1.1) *has a maximal solution $\bar{u}$ and a minimal solution $\underline{u}$ such that $\hat{u} \leq \underline{u} \leq \bar{u} \leq \tilde{u}$. In fact, there exist sequences $\{\bar{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ with $\bar{u}^{(0)} = \tilde{u}, \underline{u}^{(0)} = \hat{u}$ such that $\{\bar{u}^{(k)}\}$ converges monotonically from above to $\bar{u}$ and $\{\underline{u}^{(k)}\}$ converges monotonically from below to $\underline{u}$.*

We recall that the maximal and minimal solutions in Theorem 1 are in the sense that if u is a solution in $\langle \hat{u}, \tilde{u} \rangle$, then $\underline{u} \leq u \leq \bar{u}$. Hence problem (1.1) has a unique solution in $\langle \hat{u}, \tilde{u} \rangle$ if $\bar{u} = \underline{u}$. To ensure this we assume that $f(\cdot,u,v)$ is a C^1-function and $B_1 = B_2 \equiv B$ (that is, $\alpha_1 = \alpha_2 \equiv \alpha$, $\beta_1 = \beta_2 \equiv \beta$).

Let λ_0 be the smallest eigenvalue and ϕ the corresponding positive eigenfunction of the eigenvalue problem

$$\Delta\phi'' + \lambda\phi = 0 \quad \text{in } \Omega, \qquad B[\phi] = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

It is well-known that $\lambda_0 > 0$ when $\beta(x) \not\equiv 0$, and $\lambda_0 = 0$ when $\beta(x) \equiv 0$. Define

$$\begin{aligned} &\bar{a} \equiv \max_{(x\in\bar{\Omega})}[a(x)], \qquad \underline{a} \equiv \min_{(x\in\bar{\Omega})}[a(x)], \\ &M_1 \equiv \max_{(x,u,v)\in Q}|f_u(x,u,v)|, \qquad m_1 = \min_{(x,u,v)\in Q}|f_u(x,u,v)|, \\ &M_2 \equiv \max_{(x,u,v)\in Q}[-a^{-1}(x)f_v(x,u,v)], \qquad m_2 = \min_{(x,u,v)\in Q}[-a^{-1}(x)f_v(x,u,v)], \end{aligned} \tag{2.7}$$

where $Q \equiv \{(x,u,v);\ x \in \bar{\Omega},\ \hat{u} \leq u \leq \tilde{u},\ \Delta\tilde{u} \leq v \leq \Delta\hat{u}\}$. Then we have the following uniqueness result.

Theorem 2. *Let the conditions in Theorem 1 be satisfied, and let $B_1 = B_2 \equiv B$ and $f(\cdot, u, v)$ be a C^1-function of (u, v). If either*

$$\lambda_0(\lambda_0 - M_2) > M_1/\underline{a} \quad or \quad \lambda_0(\lambda_0 - m_2) < m_1/\bar{a}, \tag{2.8}$$

then problem (1.1) has a unique solution $u^* \in \langle \hat{u}, \tilde{u} \rangle$.

The existence-uniqueness result in Theorem 2 remains true if the monotone nondecreasing property (2.4) is replaced by the following monotone nonincreasing property.

$(H_2)'$:

$$f(x, u_1, v) \geq f(x, u_2, v) \quad \text{for } \hat{u} \leq u_1 \leq u_2 \leq \tilde{u},\ \Delta\tilde{u} \leq v \leq \Delta\hat{u}. \tag{2.9}$$

In this situation the differential inequalities in (2.1) for $\tilde{u}$ and $\hat{u}$ are reduced to

$$\Delta(a\Delta\tilde{u}) \geq f(x, \hat{u}, \Delta\tilde{u}), \qquad \Delta(a\Delta\hat{u}) \leq f(x, \tilde{u}, \Delta\hat{u}), \tag{2.10}$$

and we have the following analogous result.

Theorem 3. *Let the conditions in Theorem 2 hold except that Hypothesis (H_2) be replaced by $(H_2)'$. Then problem (1.1) has a unique solution $u^* \in \langle \hat{u}, \tilde{u} \rangle$. Moreover, there exist sequences $\{\bar{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ which converge monotonically to u^* as $k \to \infty$.*

Remark 2.1. It is obvious that all the conclusions in Theorems 1 to 3 are directly applicable to the two-point boundary problem (0.1) where $\alpha_i \geq 0, \beta_i \geq 0$ with $\alpha_i + \beta_i > 0$, and $\alpha_i^* \geq 0$, $\beta_i^* \geq 0$ with $\alpha_i^* + \beta_i^* > 0$ $(i = 0, 1)$.

Construction of upper and lower solutions. It is seen from the above theorems that the main condition for the existence problem of (1.1) is the existence of a pair of coupled upper and lower solutions. In the following we give some sufficient conditions on $f(\cdot, u, v)$ and the boundary condition for the construction of these functions.

(a) Suppose $\beta_1(x) \not\equiv 0$, $\beta_2(x) \not\equiv 0$, and there exists a constant $M > 0$ such that

$$|f(x, u, v)| \leq M \quad \text{for } u, v \in \mathbb{R}. \tag{2.11}$$

Then for any $h_1(x)$ and $h_2(x)$, each of the equations

$$\Delta(a\Delta\overline{U}) = M \quad \text{and} \quad \Delta(a\Delta\underline{U}) = -M$$

under the same boundary condition as that in (1.1) has a unique solution $\overline{U}$ and $\underline{U}$, respectively, and $\overline{U} \geq \underline{U}$ and $\Delta\overline{U} \leq \Delta\underline{U}$. It is easy to verify that the pair $\tilde{u} = \overline{U}$ and $\hat{u} = \underline{U}$ are coupled upper and lower solutions of (1.1). This construction ensures the existence of a solution that extends a result of [1] to the more general problem (1.1).

(b) Suppose $h_2(x) = 0$ and there exist constants c_1, c_2 with $c_1 \geq c_2 \geq 0$ such that

$$\begin{gathered} c_1\beta_i(x) \geq h_i(x) \geq c_2\beta_i(x) \qquad (i = 1, 2), \\ f(x, u, 0) = 0 \quad \text{for } c_2 \leq u \leq c_1 \qquad (x \in \Omega). \end{gathered} \tag{2.12}$$

Then $\tilde{u} = c_1$ and $\hat{u} = c_2$ are coupled upper and lower solutions.

(c) Suppose $B_1 = B_2, h_1(x) = h_2(x)$, and there exists a constant $M \geq 0$ such that

$$f(x, u, 0) \geq 0, \quad f(x, u, -\lambda_0 v) \leq \lambda_0^2 v \quad \text{for } u, v \in [0, M], \tag{2.13}$$

where λ_0 is the smallest eigenvalue of (2.6). It is easy to verify that the pair $\tilde{u} = M\phi$ and $\hat{u} = 0$ are coupled upper and lower solutions, where ϕ is the normalized positive eigenfunction corresponding to λ_0.

(d) Consider the two-point boundary value problem (0.1) with the boundary condition $(0.1)_b$. Assume that $h_1(x) \geq 0$, $h_2(x) \leq 0$ and there exist positive constants a, b and c with $a/\pi^4 + b/\pi^2 < 1$ such that

$$\begin{aligned} f(x,0,0) \geq 0, \quad f_u(x,u,v) \geq 0, \quad f(x,u,-v) &\leq au + bv + c \\ \text{for } u \geq 0,\ v \geq 0 \qquad &(0 \leq x \leq 1). \end{aligned} \tag{2.14}$$

It is easy to verify that by choosing a sufficiently small $\delta > 0$ and a sufficiently large $\rho > 0$ the pair $\tilde{u} = \rho \sin \gamma(x+\delta)$ and $\hat{u} = 0$ are upper and lower solutions, where $\gamma = \pi(1+2\delta)$. If, in addition,

$$f_u(x,u,v) \leq a, \quad f_v(x,u,v) \leq b \quad \text{for } u \geq 0,\ v \geq 0\ (0 \leq x \leq 1), \tag{2.15}$$

then condition (2.8) is satisfied with $M_1 = a$, $M_2 = b$, and by Theorem 2 problem (0.1), $(0.1)_b$ has a unique nonnegative solution u^*. By the maximum principle, u^* is the unique positive solution if either $f(x,0,0) \neq 0$ or $h_1 + h_1^* \neq 0$ or $h_2 + h_2^* \neq 0$ (cf. [12, 14]).

(e) Consider the problem (1.2) where $c \leq \lambda_0$ and $b \geq 0$. Denote by $\phi^*(x)$ the positive eigenfunction corresponding to the smallest eigenvalue of (2.6) in a larger domain $\Omega^* \supset \bar{\Omega}$. This implies that for some constant $\varepsilon_0 > 0$, $\phi^*(x) \geq \varepsilon_0$ on $\bar{\Omega}$. It is easy to verify that for a sufficiently small constant $\delta > 0$ and a sufficiently large constant $\rho > 0$, the pair $\tilde{u} = -\delta\phi$ and $\hat{u} = -\rho\phi^*$ are coupled upper and lower solutions of (1.2) if

$$b \geq \lambda_0(\lambda_0 - c). \tag{2.16}$$

This implies that under condition (2.16) problem (1.2) has at least one negative solution, a result obtained in [17].

On the other hand, if $b \leq \lambda_0(\lambda_0 - c)$, then for any constant $\rho_0 > 0$ the pair $\tilde{u} = \rho_0\phi$ and $\hat{u} = 0$ are coupled upper and lower solutions. Since $f_u(u,v) = b$, $f_v(u,v) = -c$ for all $u \geq 0$ and $v \in \mathbb{R}$, where $f(u,v) = b[(u+1)^+ - 1] - cv$, we see that $M_1 = m_1 = b$ and $M_2 = m_2 = |c|$, and therefore condition (2.8) holds if $\lambda_0(\lambda_0 - |c|) \neq b$. By the uniqueness result of Theorem 2, $u^* = 0$ is the only nonnegative solution of (1.2) if

$$b < \lambda_0(\lambda_0 - |c|). \tag{2.17}$$

It is interesting to note that if $b = \lambda_0(\lambda_0 - c)$, then problem (1.2) has an infinite number of solutions (positive and negative) in the form $u = \sigma\phi$, where $\sigma \geq -1$ is an arbitrary constant.

3. Proof of the theorems

Let $C^{m+\alpha}(\bar{\Omega})$ be the set of functions in $C^m(\bar{\Omega})$ that are Hölder continuous in $\bar{\Omega}$ with exponent $\alpha \in (0,1)$, where m is a nonnegative integer. The set of vector functions $\mathbf{u} \equiv (u,v)$ with u, v in $C^{m+\alpha}(\bar{\Omega})$ is denoted by $\mathcal{C}^{m+\alpha}(\bar{\Omega})$. Let $\tilde{u}, \hat{u}$ be a pair of coupled upper and lower solutions, and let $\tilde{\mathbf{u}} \equiv (\tilde{u}, -\Delta\tilde{u})$, $\hat{\mathbf{u}} \equiv (\hat{u}, -\Delta\hat{u})$. Define

$$\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \equiv \{(u,v) \in \mathcal{C}^{2+\alpha}(\bar{\Omega});\ (\hat{u}, -\Delta\hat{u}) \leq (u,v) \leq (\tilde{u}, -\Delta\tilde{u})\}. \tag{3.1}$$

By letting $v = -a\Delta u$ we may write (1.1) in the equivalent form

$$
\begin{aligned}
-a\Delta u &= v, \qquad B_1[u] = h_1, \\
-\Delta v &= f(x, u, -v/a), \qquad B_2[v] = -h_2.
\end{aligned} \tag{3.2}
$$

To prove the existence and uniqueness of a solution to (1.1) it suffices to show the same for (3.2).

Proof of Theorem 1. Let $\tilde{v} = -a\Delta\tilde{u}$, $\hat{v} = -a\Delta\hat{u}$. It is easy to see from (2.1) that the pair $(\tilde{u}, \tilde{v})$, $(\hat{u}, \hat{v})$ satisfy the relation

$$
\begin{aligned}
&\begin{rcases}
-a\Delta\tilde{u} \geq v, \quad -\Delta\tilde{v} \geq f(x, u, -\tilde{v}/a) \\
-a\Delta\hat{v} \leq v, \quad -\Delta\hat{v} \leq f(x, u, -\hat{v}/a) \quad \text{for } \hat{u} \leq u \leq \tilde{u},\ \hat{v} \leq v \leq \tilde{v}
\end{rcases} \quad (x \in \Omega), \\
&\qquad B_1[\tilde{u}] \geq h_1(x) \geq B_1[\hat{u}], \quad B_2[\tilde{v}] \geq -h_2(x) \geq B_2[\hat{v}] \qquad (x \in \partial\Omega).
\end{aligned}
$$

This implies that the pair $(\tilde{u}, \tilde{v})$, $(\hat{u}, \hat{v})$ are "generalized" coupled upper and lower solutions of (3.2) in the sense of [12]. The existence of a solution to (3.2) follows from Theorem 8.10.5 of [12] (see also [15, 18]). When condition (2.4) is satisfied the vector function

$$
\mathbf{f}(x, u, v) \equiv (v,\ f(x, u, -v/a)) \tag{3.3}
$$

is quasimonotone nondecreasing in $\langle\hat{u}, \tilde{u}\rangle$. The existence of the maximal solution $\bar{u}$ and the minimal solution $\underline{u}$ and the convergence of the monotone sequences $\{\bar{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$ follow from Theorem 8.4.1 of [12]. □

Proof of Theorem 2. It suffices to show that $\bar{u} = \underline{u}$ where $\bar{u}$ and $\underline{u}$ are the respective maximal and minimal solutions of (1.1). Let $w = \bar{u} - \underline{u}$, $z = \bar{v} - \underline{v}$, where $\bar{v} = -a\Delta\bar{u}$, $\underline{v} = -a\Delta\underline{u}$. By (3.2) and the maximal and minimal property of $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$, we have $w \geq 0$, $z \geq 0$ and

$$
\begin{aligned}
-a\Delta w = z, \quad -\Delta z = f(x, \bar{u}, -\bar{v}/a) - f(x, \underline{u}, -\underline{v}/a) \qquad (x \in \Omega), \\
B_1[w] = B_2[z] = 0 \qquad (x \in \partial\Omega).
\end{aligned} \tag{3.4}
$$

Multiplying the differential equations by ϕ, integrating over Ω, and applying the mean-value theorem lead to

$$
\begin{aligned}
-\int_\Omega \phi\Delta w dx &= \int_\Omega (\phi z/a) dx, \\
-\int_\Omega \phi\Delta z dx &= \int_\Omega \phi[f_u(x, \xi_1, \xi_2)w - f_v(x, \xi_1, \xi_2)z/a] dx,
\end{aligned}
$$

where $\xi_1 \equiv \xi_1(x)$ and $\xi_2 \equiv \xi_2(x)$ are some intermediate values. By the Green's theorem and using (2.6), (2.7) and the boundary condition in (3.4) we obtain the relation

$$
\begin{gathered}
\lambda_0 \int_\Omega \phi w dx = \int_\Omega (\phi z/a) dx, \\
\int_\Omega (M_1\phi w + M_2\phi z) dx \geq \lambda_0 \int_\Omega \phi z dx \geq \int_\Omega (m_1\phi w + m_2\phi z) dx.
\end{gathered}
$$

This leads to the relation

$$
\begin{gathered}
\underline{a}^{-1} \int_\Omega \phi z dx \geq \int_\Omega (\phi z/a) dx = \lambda_0 \int_\Omega \phi w dx \geq \bar{a}^{-1} \int_\Omega \phi z dx, \\
M_1(\lambda_0 - M_2)^{-1} \int_\Omega \phi w dx \geq \int_\Omega \phi z dx \geq m_1(\lambda_0 - m_2)^{-1} \int_\Omega \phi w dx
\end{gathered}
$$

which yields

$$M_1[\underline{a}\lambda_0(\lambda_0 - M_2)]^{-1} \int_\Omega \phi w dx \geq \int_\Omega \phi w dx \geq m_1[\bar{a}\lambda_0(\lambda_0 - m_2)]^{-1} \int_\Omega \phi w dx.$$

Hence under either one of the conditions in (2.8) the above relation can hold only when $w = 0$. This shows that $\bar{u} = \underline{u}$ and is the unique solution of (1.1). □

Proof of Theorem 3. When Hypothesis (H_2) is replaced by $(H_2)'$ the function $\mathbf{f}(x, u, v)$ in (3.3) is mixed quasimonotone in $\langle \hat{u}, \tilde{u} \rangle$. By Theorem 8.10.1 of [12] there exist sequences $\{\bar{u}^{(k)}, \bar{v}^{(k)}\}$, $\{\underline{u}^{(k)}, \underline{v}^{(k)}\}$ which converge to a pair of quasisolutions $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ that satisfy the relation

$$\begin{aligned} -a\Delta\bar{u} = \bar{v}, \qquad & -\Delta\bar{v} - f(x, \underline{u}, -\bar{v}/a) \\ -\Delta\underline{u} = \underline{v}, \qquad & -\Delta\underline{v} = f(x, \bar{u}, -\underline{v}/a) \end{aligned} \qquad (x \in \Omega),$$

$$B_1[\bar{u}] = B_1[\underline{u}] = h_1(x), \quad B_2[\bar{v}] = B_2[\underline{v}] = -h_2(x) \qquad (x \in \partial\Omega).$$

To guarantee that this pair of quasisolutions are true solutions and yield a unique solution it suffices to show $(\bar{u}, \bar{v}) - (\underline{u}, \underline{v})$ (cf. [12]). However, since the pair $w \equiv \bar{u} - \underline{u}$ and $z \equiv \bar{v} - \underline{v}$ satisfy the relation (3.4) except with the equation for z replaced by

$$-\Delta z = f(x, \underline{u}, -\bar{v}/a) - f(x, \bar{u}, -\underline{v}/a),$$

we conclude from the argument in the proof of Theorem 2 that $\bar{u} = \underline{u}$ and $\bar{v} = \underline{v}$. This proves the theorem. □

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