

UNIONS OF JOHN DOMAINS

JUSSI VÄISÄLÄ

(Communicated by Christopher D. Sogge)

ABSTRACT. We study the question: When is the union of John domains a John domain?

1. INTRODUCTION

1.1. John domains in the euclidean n -space R^n were introduced by F. John [Jo] in connection with his work on elasticity. The term is due to O. Martio and J. Sarvas [MS]. Roughly speaking, a domain is a John domain if it is possible to travel from one point of the domain to another without going too close to the boundary.

Several equivalent characterizations for John domains can be found in [NV]. In the present paper we shall adopt the definition based on diameter cigars. Let $\gamma \subset R^n$ be an arc with endpoints a and b ; we shall write this as $\gamma : a \curvearrowright b$. For $x \in \gamma$ we set

$$\zeta(x, \gamma) = d(\gamma[a, x]) \wedge d(\gamma[x, b]).$$

Here $\gamma[a, x]$ is the subarc of γ between a and x , $d(\cdot)$ denotes diameter, and $r \wedge s = \min(r, s)$. For $c \geq 1$, we say that γ is a c -John arc in a domain $D \subset R^n$ if $\gamma \subset D$ and if

$$\zeta(x, \gamma) \leq cd(x, \partial D)$$

for all $x \in \gamma$, where $d(\cdot, \cdot)$ denotes distance with $d(x, \emptyset) = \infty$. A domain D is a c -John domain if each pair of distinct points in D can be joined by a c -John arc.

Suppose that D_1 and D_2 are c -John domains in R^n with $D_1 \cap D_2 \neq \emptyset$. The union $D = D_1 \cup D_2$ need not be a John domain. For example, let H and H' be the upper and lower half spaces of R^n , respectively. Then H and $H' \cup B^n$ are John domains but their union is not. If one of the domains is bounded, then D is a c' -John domain for some c' by 3.1 or by 3.4, but c' can be arbitrarily large. In the main result 3.4 we show that if the volume of $D_1 \cap D_2$ is comparable to the volume of one of the domains D_1, D_2 , then we obtain an upper bound for c' . In this result, one cannot replace volume by diameter, except for the special case where the domains are simply connected domains in the plane with connected intersection.

I thank Ritva Hurri-Syrjänen for calling my attention to this problem.

1.2. *Notation.* For $x \in R^n$ and $r > 0$, we let $B(x, r)$ and $\bar{B}(x, r)$ denote the open and closed balls in R^n with center x and radius r , and we write $B(r) =$

Received by the editors June 1, 1998.

2000 *Mathematics Subject Classification.* Primary 30C65.

Key words and phrases. John domain.

$B(0, r)$, $\bar{B}(r) = \bar{B}(0, r)$, $B^n = B(0, 1)$. For a measurable set $A \subset R^n$, $|A|$ is the Lebesgue measure of A , and we set $\Omega_n = |B^n|$. Notation like ab/xyz means $(ab)/(xyz)$.

2. PRELIMINARIES

In this section we give various auxiliary results needed in the proof of the main result 3.4.

2.1. Lemma. *Let A be a bounded set in R^n and let $r > 0$. Then there is a finite sequence of points z_1, \dots, z_p in A such that the balls $B_j = B(z_j, r)$ cover A and such that each point in R^n belongs to at most N_n balls B_j , where N_n is a positive integer depending only on n .*

Proof. Divide R^n into a lattice of closed cubes of side $s = r/2\sqrt{n}$. For each cube Q meeting A choose a point $z_Q \in A \cap Q$. Since $Q \subset B(z_Q, r)$, the balls $B(z_Q, r)$ cover A . If the balls $B(z_Q, r)$ and $B(z_R, r)$ intersect, then $d(Q, R) < 2r = 4s\sqrt{n}$, and the lemma follows.

2.2. Plumpness. An open set $U \subset R^n$ is c -plump, $c \geq 1$, if for each $x \in U$ and $r > 0$ with $U \setminus B(x, r) \neq \emptyset$, there is $z \in \bar{B}(x, r)$ with $B(z, r/c) \subset U$.

The following result is obvious:

2.3. Lemma. *A c -John domain is $2c$ -plump. If U is c -plump, then $\Omega_n d(U)^n \leq c^n |U|$.*

2.4. Notation. Let U be an open proper subset of R^n . For $x \in U$ we write

$$\delta(x) = d(x, \partial U).$$

For $t > 0$ we set

$$U(t) = \{x \in U : \delta(x) < t\}, \quad \partial_i U(t) = \{x \in U : \delta(x) = t\}.$$

Then $\partial U(t) = \partial U \cup \partial_i U(t)$. Since $\partial_i U(t)$ cannot have points of density, we get:

2.5. Lemma. $|\partial_i U(t)| = 0$ for all $t > 0$.

2.6. Lemma. *If $U \subset R^n$ is open and c -plump and if $0 < t < d(U)/2c$, then $\partial U \subset \partial_i U(t) + \bar{B}(ct)$.*

Proof. Let $\varepsilon > 0$. It suffices to show that $\partial U \subset \partial_i U(t) + B(ct + \varepsilon)$. Let $a \in \partial U$. Choose $x \in U$ with $|x - a| < \varepsilon$. Since $ct < d(U)/2$, $U \setminus B(x, ct) \neq \emptyset$. Since U is c -plump, there is $z \in \bar{B}(x, ct)$ with $\delta(z) \geq t$. Pick a point y in the line segment $[a, z]$ with $\delta(y) = t$. Then $y \in \partial_i U(t)$, and $|a - y| \leq |a - z| \leq |a - x| + |x - z| < \varepsilon + ct$. Hence $a \in \partial_i U(t) + B(ct + \varepsilon)$.

2.7. Lemma. *Suppose that $U \subset R^n$ is bounded, open and c -plump, and that $0 < t < d(U)/2c$. Then $|U(t)| \leq 8^n c^n N_n |V(t)|$, where $V(t) = \{x \in U : t/2 < \delta(x) < t\}$ and N_n is given by 2.1.*

Proof. The idea is from [MV, (2.8)]. We remark that on the last line of p. 19 of [MV], one should replace $B^n(x, t - s + ct)$ by $B^n(x, 2t - s + ct)$.

Set $s = 3t/4$, $A = \partial_i U(s)$. By 2.1 we can find a finite sequence of points z_1, \dots, z_p in A such that the balls $B(z_j, t/4)$ cover A and such that each point in R^n is covered at most N_n times.

By 2.6 we have $\partial U \subset A + \bar{B}(cs)$, and hence $U(t) \subset A + B(cs + t)$. Consequently, $U(t)$ is covered by the balls $B(z_j, t/4 + cs + t) \subset B(z_j, 2ct)$, $1 \leq j \leq p$. Thus $|U(t)| \leq p\Omega_n(2ct)^n$. On the other hand, since $B(z_j, t/4) \subset V(t)$, we have $N_n|V(t)| \geq p\Omega_n(t/4)^n$, and the lemma follows.

2.8. Theorem. *Suppose that $U \subset R^n$ is open and c -plump with $d(U) = 1$. Then $|U(t)| \leq \mu(t, c, n) \rightarrow 0$ as $t \rightarrow 0$.*

Proof. Since $d(U) = 1$, we have $|U(t)| \leq |U| \leq \Omega_n$ for all t . We may thus assume that $t \leq 1/3c < d(U)/2c$. By 2.7 and 2.5 we obtain

$$|U(t/2)| = |U(t) - |V(t)| \leq \lambda|U(t)|, \quad \lambda = \lambda(c, n) = 1 - 1/8^n c^n N_n < 1.$$

Iterating this result gives $|U(2^{-k}t)| \leq \lambda^k|U(t)|$ for all positive integers k . Hence $|U(2^{-k}/3c)| \leq \lambda^k\Omega_n$ for all k , and the theorem follows.

2.9. Remark. The proof shows that one can choose $\mu(t, c, n) = \beta t^\alpha$ with $\alpha = \alpha(c, n) > 0$ and $\beta = \beta(n) > 0$.

2.10. Corollary. *Let U be as in 2.8. Suppose that $A \subset U$ with $|A| \geq q > 0$. Then there is $z \in A$ with $\delta(z) \geq r(c, q, n) > 0$.*

Proof. This holds with any r such that $\mu(r, c, n) < q$.

3. MAIN RESULTS

3.1. Theorem. *Suppose that D_1 and D_2 are c -John domains in R^n , and that $z \in D_1 \cap D_2$ and $r > 0$ with $B(z, r) \subset D_1 \cup D_2$, $d(D_1) \wedge d(D_2) \leq c_0r$. Then $D_1 \cup D_2$ is a c' -John domain with $c' = 2c(c_0 + 1)$.*

Proof. We may assume that $d(D_1) = 1 \leq d(D_2)$. Then $r \geq 1/c_0$. Let $a \in D_1$, $b \in D_2$. Choose c -John arcs $\alpha : a \curvearrowright z$ and $\beta : b \curvearrowright z$ in D_1 and D_2 , respectively. The continuum $\alpha \cup \beta$ contains an arc $\gamma : a \curvearrowright b$. It suffices to show that γ is a c' -John arc in $D = D_1 \cup D_2$.

For $x \in \alpha$ and $y \in \beta$ set

$$\zeta_1(x) = d(\alpha[a, x]) \wedge d(\alpha[x, z] \cup \beta), \quad \zeta_2(y) = d(\alpha \cup \beta[z, y]) \wedge d(\beta[y, b]).$$

It suffices to show that

$$(3.2) \quad \zeta_1(x) \leq c'd(x, \partial D),$$

and

$$(3.3) \quad \zeta_2(y) \leq c'd(y, \partial D)$$

for all $x \in \alpha$ and $y \in \beta$.

Choose a point $a_1 \in \alpha$ dividing α to subarcs of equal diameter. Let a_2 be the last point of α with $\alpha[a_2, z] \subset \bar{B}(z, r/2)$. Let b_1 and b_2 be the corresponding points in β .

Let $x \in \alpha$. To prove (3.2) we consider three cases.

Case 1. $x \in \alpha[a, a_1]$. Now $\zeta_1(x) = \zeta(x, \alpha) \leq cd(x, \partial D_1) \leq cd(x, \partial D)$.

Case 2. $x \in \alpha[a_2, z]$. Now $\zeta_1(x) \leq d(\alpha) \leq d(D_1) = 1$ and $d(x, \partial D) \geq r/2 \geq 1/2c_0 > 1/c'$.

Case 3. $a_2 \in \alpha[a_1, z]$ and $x \in \alpha[a_1, a_2]$. This case may be empty. We have $\zeta_1(x) \leq d(D_1) = 1$, and $\zeta(x, \alpha) \geq r/2$. Hence

$$\zeta_1(x) \leq 2\zeta(x, \alpha)/r \leq 2cc_0d(x, \partial D_1) \leq 2cc_0d(x, \partial D).$$

Next let $y \in \beta$. If $y \in \beta[b, b_1]$, we can argue as in Case 1 above. Suppose that $y \in \beta[b_2, z]$. Now $d(y, \partial D) \geq r/2$, and

$$\zeta_2(y) \leq d(\alpha \cup \beta[z, y]) \leq 1 + r \leq 2(c_0 + 1)d(y, \partial D).$$

Finally, assume that $b_2 \in \beta[b_1, z]$ and that $y \in \beta[b_1, b_2]$. This case may again be empty. Now

$$\zeta_2(y) \leq d(\alpha) + d(\beta[z, y]) \leq 1 + \zeta(y, \beta) \leq 1 + cd(y, \partial D_2) \leq 1 + cd(y, \partial D).$$

Since $r/2 \leq \zeta(y, \beta) \leq cd(y, \partial D)$, we obtain $1 \leq 2cc_0d(y, \partial D)$, and (3.3) follows.

3.4. Theorem. *Suppose that $D_1, D_2 \subset R^n$ are c -John domains with $|D_1| \wedge |D_2| \leq c_0|D_1 \cap D_2| < \infty$. Then $D_1 \cup D_2$ is a c' -John domain with $c' = c'(c, c_0, n)$.*

Proof. By Lemma 2.3, the domains D_k are $2c$ -plump, and at least one of them is bounded. Hence we may assume that $d(D_1) = 1 \leq d(D_2)$. By 2.3 this gives $|D_k| \geq \Omega_n/(2c)^n$, and hence $|D_1 \cap D_2| \geq \Omega_n/c_0(2c)^n$. By 2.10 we can find $z \in D_1 \cap D_2$ such that $d(z, \partial D_1) \geq r > 0$, where r depends only on (c, c_0, n) . Since $B(z, r) \subset D_1 \subset D_1 \cup D_2$, the theorem follows from 3.1.

Theorem 3.4 can be applied to prove the John property of more general unions. As an example we give the following result:

3.5. Theorem. *Suppose that D_0, D_1, \dots is a finite or infinite sequence of c -John domains in R^n . Suppose also that $|D_j| \leq c_0|D_0 \cap D_j|$ for all $j \geq 1$. Then the domain $D = \bigcup\{D_j : j \geq 0\}$ is a c_1 -John domain with $c_1 = c_1(c, c_0, n)$.*

Proof. Set $G_j = D_0 \cup D_j$ for $j \geq 1$. Since $|D_0| \wedge |D_j| \leq |D_j| \leq c_0|D_0 \cap D_j|$, G_j is a c' -John domain with $c' = c'(c_0, c, n) \geq c$ by 3.4.

Assume that $1 \leq j < k$ and that $a \in D_j, b \in D_k$. It suffices to find $c_1 \geq 1$ and a c_1 -John arc $\gamma : a \curvearrowright b$ in D . For this, it suffices to show that $G_j \cup D_k$ is a c_1 -John domain. This follows from 3.4, since

$$|G_j| \wedge |D_k| \leq |D_k| \leq c_0|D_0 \cap D_k| \leq c_0|G_j \cap D_k|,$$

and hence the theorem holds with $c_1 = c'(c'(c, c_0, n), c_0, n)$.

3.6. Diameter conditions. One may ask whether the measure condition of 3.4 can be replaced by the diameter condition

$$d(D_1) \wedge d(D_2) \leq c_0d(D_1 \cap D_2) < \infty.$$

It is easy to see, however, that in this case, $D = D_1 \cup D_2$ need not be a John domain with any reasonable constant. An example where $V = D_1 \cap D_2$ is connected is given by two cubes in R^3 which intersect in a narrow cylinder V . In R^2 , an example with connected V is not quite so simple. Let Q be the open unit square $\{(x, y) : 0 < x < 1, 0 < y < 1\}$. Let k be a positive integer, and set $D_1 = Q \setminus \{(2^{-k}, j2^{-k}) : 1 \leq j \leq 2^k - 1\}$. Then D_1 is a c -John domain with c independent of k . Let D_2 be the half plane $\{(x, y) : x < 2^{-k-1}\}$. Then $V = D_1 \cap D_2$ is connected and $d(V) > 1$, but the John constant of $D_1 \cup D_2$ tends to ∞ as $k \rightarrow \infty$.

In the last example above, D_1 is not simply connected. We next show that for simply connected planar domains, the diameter condition gives an upper bound for the John constant of D . The proof makes use of some basic results in plane topology, which will be considered in Section 4.

3.7. Theorem. *Suppose that D_1 and D_2 are simply connected c -John domains in R^2 , that $D_1 \cap D_2$ is connected, and that $d(D_1) \wedge d(D_2) \leq c_0 d(D_1 \cap D_2) < \infty$. Then $D_1 \cup D_2$ is a c' -John domain with $c' = c'(c, c_0)$.*

Proof. Write $D = D_1 \cup D_2$, $V = D_1 \cap D_2$. We may assume that $d(D_1) = 1 \leq d(D_2)$; then $d(V) \geq 1/c_0$. Set $t = (16cc_0)^{-1}$. If $d(x, \partial D_k) \geq t$ for some $x \in V$ and for some $k = 1, 2$, the theorem follows from 3.1. Hence we may assume that

$$(3.8) \quad d(x, \partial D_k) < t$$

for all $x \in V$ and for $k = 1, 2$.

Fix $a, b \in V$ with $|a - b| > 1/2c_0$. For $k = 1, 2$, choose a c -John arc $\alpha_k : a \curvearrowright b$ in D_k . If $x \in \alpha_k$ and $|x - a| \wedge |x - b| \geq ct$, then $d(x, \partial D_k) \geq t$, and hence $x \notin V$ by (3.8). Consequently,

$$(3.9) \quad \alpha_k \cap V \subset B(a, ct) \cup B(b, ct)$$

for $k = 1, 2$.

Since $2ct = 1/8c_0 < 1/2c_0 < |a - b|$, the balls $B(a, ct)$ and $B(b, ct)$ are disjoint. Since $\alpha_1 \cap \alpha_2 \subset V$, it follows from (3.9) that there are points $a' \in \alpha_1 \cap \alpha_2 \cap B(a, ct)$ and $b' \in \alpha_1 \cap \alpha_2 \cap B(b, ct)$ such that the arcs $\alpha'_k = \alpha_k[a', b']$ meet only at the common endpoints a', b' . Thus $J = \alpha'_1 \cup \alpha'_2$ is a Jordan curve in D . Since D is simply connected by 4.2, we have $i(J) \subset D$.

Let L be the line through $(a + b)/2$, perpendicular to $[a, b]$. Then L separates a' and b' in R^2 . By 4.4, there are points $x_k \in \alpha'_k \cap L$ such that the open line segment $]x_1, x_2[$ lies in $i(J)$. By (3.9) we have $x_2 \notin D_1$ and $x_1 \notin D_2$. Since V is connected, we can choose an arc $\gamma : a' \curvearrowright b'$ in V . Since γ and α'_1 lie in the simply connected domain D_1 , we have $\gamma \sim \alpha'_1$ in $R^2 \setminus \{x_2\}$, and similarly $\gamma \sim \alpha'_2$ in $R^2 \setminus \{x_1\}$. Hence we can apply Lemma 4.3 with the substitution $(a, b, \alpha_1, \alpha_2, \beta) \mapsto (a', b', \alpha'_1, \alpha'_2, [x_1, x_2])$ to find a point $z \in \gamma \cap]x_1, x_2[\subset i(J)$.

It suffices to show that $d(z, J) \geq t$, because then $B(z, t) \subset D$, and the theorem follows from 3.1. Assume that $x \in J$ with $|x - z| < t$. Then

$$|x - a| \geq |z - a| - |x - z| > |a - b|/2 - t > 1/4c_0 - 1/16cc_0 > 1/8c_0,$$

and similarly $|x - b| > 1/8c_0$. We may assume that $x \in \alpha_1$. Since α_1 is a c -John arc in D_1 , we have $d(x, \partial D_1) > 1/8cc_0 = 2t$. Since $z \in \gamma \subset V$, (3.8) gives $d(z, \partial D_1) < t$, and we obtain the contradiction

$$t > |z - x| \geq d(x, \partial D_1) - d(z, \partial D_1) > 2t - t = t.$$

4. SOME TOPOLOGY OF THE PLANE

4.1. General remarks. The purpose of this section is to prove some basic results of plane topology, needed in the proof of 3.7. We recall that a domain $D \subset R^2$ is simply connected if and only if the singular homology group $H_1(D)$ is trivial, and also if and only if the complement of D in the extended plane is connected.

Suppose that $A \subset R^2$ and that $\alpha_1, \alpha_2 : a \curvearrowright b$ are arcs in A . Choose homeomorphisms $\alpha_k^* : [0, 1] \rightarrow \alpha_k$ with $\alpha_k^*(0) = a$, $\alpha_k^*(1) = b$, $k = 1, 2$. If α_1^* is path homotopic to α_2^* in A , we say that α_1 is homotopic to α_2 in A and write $\alpha_1 \sim \alpha_2$ in A .

For an arc $\alpha : a \curvearrowright b$, we let $\hat{\alpha} = \alpha \setminus \{a, b\}$ denote the set of its interior points. If $J \subset R^2$ is a Jordan curve, the bounded component of $R^2 \setminus J$ is written as $i(J)$.

4.2. Lemma. *Suppose that D_1 and D_2 are simply connected domains in R^2 and that $D_1 \cap D_2$ is nonempty and connected. Then $D_1 \cup D_2$ is simply connected.*

Proof. This is well known and follows from the exact Mayer-Vietoris sequence $0 = H_1(D_1) \oplus H_1(D_2) \rightarrow H_1(D_1 \cup D_2) \rightarrow \tilde{H}_0(D_1 \cap D_2) = 0$.

4.3. Lemma. *Suppose that $\alpha_1, \alpha_2 : a \curvearrowright b$ are arcs in R^2 with $\alpha_1 \cap \alpha_2 = \{a, b\}$, and that $\beta : x_1 \curvearrowright x_2$ is an arc with $x_k \in \hat{\alpha}_k$ and $\hat{\beta} \subset i(\alpha_1 \cup \alpha_2)$. Suppose also that $\gamma : a \curvearrowright b$ is an arc in $R^2 \setminus \{x_1, x_2\}$ such that $\gamma \sim \alpha_1$ in $R^2 \setminus \{x_2\}$ and $\gamma \sim \alpha_2$ in $R^2 \setminus \{x_1\}$. Then γ meets β .*

Proof. Assume that $\gamma \cap \beta = \emptyset$. Fix a point $z \in \hat{\beta}$. Since the inclusion $R^2 \setminus \beta[z, x_2] \hookrightarrow R^2 \setminus \{x_2\}$ is a homotopy equivalence, we have $\gamma \sim \alpha_1$ in $R^2 \setminus \beta[z, x_2]$, and hence in $R^2 \setminus \{z\}$. Similarly, $\gamma \sim \alpha_2$ in $R^2 \setminus \{z\}$, and hence $\alpha_1 \sim \alpha_2$ in $R^2 \setminus \{z\}$. Since $z \in i(\alpha_1 \cup \alpha_2)$, this is well known to be a contradiction.

4.4. Lemma. *Let $J \subset R^2$ be a Jordan curve, and let a, b be points in J dividing J into subarcs α_1, α_2 . Let $L \subset R^2$ be a line separating a and b in R^2 . Then there are $x_1 \in \alpha_1 \cap L$ and $x_2 \in \alpha_2 \cap L$ such that $]x_1, x_2[\subset i(J)$.*

Proof. Choose an endcut γ_a of $D = i(J)$ with endpoints a and $a' \in D$ such that $\gamma_a \cap L = \emptyset$. Similarly choose an endcut $\gamma_b : b' \curvearrowright b$. Since $L \cap D$ is closed in D and separates the points a' and b' in D , there is a component F of $L \cap D$ separating a' and b' in D ; see [Ne, V.14.3]. Now $F =]x_1, x_2[$ with some $x_1, x_2 \in \partial D$, and $]x_1, x_2[$ is a crosscut of D . From [Ne, V.11.8] it follows that x_1 and x_2 lie in different components of $J \setminus \{a, b\}$.

REFERENCES

- [Jo] F. John, *Rotation and strain*, Comm. Pure. Appl. Math. **14** (1961), 391–413. MR **25**:1672
- [MS] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Math. **4** (1978), 383–401. MR **81i**:30039
- [MV] O. Martio and M. Vuorinen, *Whitney cubes, p -capacity, and Minkowski content*, Exposition. Math. **5** (1987), 17–40. MR **88e**:28004
- [NV] R. Näkki and J. Väisälä, *John disks*, Exposition. Math. **9** (1991), 3–43. MR **92i**:30021
- [Ne] M.H.A. Newman, *Elements of topology of plane sets of points*, Cambridge University Press, 1961.

MATEMATIKKAN LAITOS, HELSINGIN YLIOPISTO, PL 4, 00014 HELSINKI, FINLAND
E-mail address: jussi.vaisala@helsinki.fi