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A GENERALIZATION OF THE LEFSCHETZ FIXED POINT THEOREM AND DETECTION OF CHAOS

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Abstract. We consider the problem of existence of fixed points of a continuous map $f : X \to X$ in (possibly) noninvariant subsets. A pair (C, E) of subsets of X induces a map $f^\dagger : C/E \to C/E$ given by $f^\dagger([x]) = [f(x)]$ if $x, f(x) \in C \setminus E$ and $f^\dagger([x]) = [E]$ elsewhere. The following generalization of the Lefschetz fixed point theorem is proved: *If X is metrizable, C and E are compact ANRs, and $f^\dagger$ is continuous, then f has a fixed point in $\overline{C \setminus E}$ provided the Lefschetz number of $\widetilde{H}^*(f^\dagger)$ is nonzero.* Actually, we prove an extension of that theorem to the case of a composition of maps. We apply it to a result on the existence of an invariant set of a homeomorphism such that the dynamics restricted to that set is chaotic.

1. Introduction

Let $f : X \to X$ be a continuous map of a metrizable space X. The Lefschetz fixed point theorem asserts the existence of a fixed point of f provided X is a compact absolute neighborhood retract (ANR) and the Lefschetz number of f is nonzero. In [Sr], an extension of that theorem to the case of a not necessarily invariant subset $C \subset X$ (i.e. possibly $f(C) \not\subset C$) was considered. It was assumed that C is a compact ANR, $E \subset C$ is another compact ANR, and

$$C \cap f(E) \subset E, \quad C \cap \overline{f(C) \setminus C} \subset E.$$

Under that hypotheses, [Sr, Th. 9] states that there exists a fixed point of f in $\overline{C \setminus E}$ if the Lefschetz number of some endomorphism of cohomologies of (C, E) is nonzero.

Here we improve slightly that result. The above condition on a compact pair (C, E) of ANRs is replaced by the condition considered by Robbin and Salamon in the paper [RS]: f induces the map $f^\dagger : C/E \to C/E$ defined as $f^\dagger([x]) = [f(x)]$ if both x and $f(x)$ belong to $C \setminus E$, and $f^\dagger([x]) = [E]$ elsewhere, and, following [RS], we assume that it is continuous. Theorem 1 gives the existence of the above fixed point provided the Lefschetz number of the reduced Alexander-Spanier cohomologies $\widetilde{H}^*(f^\dagger)$ of $f^\dagger$ is nonzero. In Proposition 2, we show that the condition on continuity of $f^\dagger$ is less restrictive than the one from [Sr].

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In order to obtain a result on chaotic dynamics we extend Theorem 1 by replacing f by a composition of maps. In Theorem 2 we consider the composition $f_{k-1}\circ \ldots \circ f_0$ of continuous maps of metrizable spaces, where the last map f_{k-1} goes to the domain of f_0. We assume the existence of pairs $(C_0, E_0), \ldots, (C_{k-1}, E_{k-1})$ of compact ANRs, where each C_j is contained in the domain of f_j, such that the maps $f_j^\dagger : C_j/E_j \to C_{j+1}/E_{j+1}$, given by $f_j^\dagger([x]) = [f_j(x)]$ if $x \in C_j \setminus E_j$ and $f_j(x) \in C_{j+1} \setminus E_{j+1}$, and $f_j^\dagger([x]) = [E_{j+1}]$ elsewhere, are continuous. Theorem 2 asserts the existence of a fixed point $x_0 \in \overline{C_0 \setminus E_0}$ such that $f_j \circ \ldots \circ f_0(x_0) \in C_j$ for $j = 1, \ldots, k-2$, provided the Lefschetz number of $\widetilde{H}^*(f_{k-1}^\dagger \circ \ldots \circ f_0^\dagger)$ is nonzero. That theorem does not follow from Theorem 1 because, in general, $f_{k-1}^\dagger \circ \ldots \circ f_0^\dagger$ is not equal to $(f_{k-1} \circ \ldots \circ f_0)^\dagger$. In the last section we present a proof of Theorem 2.

We apply Theorem 2 in a proof of a result on the existence of a kind of chaotic behavior of a discrete dynamical system. By chaos for a homeomorphism f we mean the existence of a compact invariant set which is semiconjugated to the shift on r symbols and, moreover, the counterimage of any periodic point of the shift contains a periodic point of f. The above definition of chaos was introduced by Zgliczyński in [Z1, Z2]; similar conditions were considered in [CKM, MM1, MM2]. In Theorem 3, we conclude the existence of chaos in the above sense under the assumption of the existence of disjoint pairs $(C_1, E_1), \ldots, (C_r, E_r)$ of compact ANRs such that all C_i/E_i have the same cohomologies as a sphere of a fixed dimension and every induced map $f^\dagger$ between the spaces C_i/E_i generates an isomorphism in cohomologies. Such an assumption is satisfied in the case of topological horseshoes (generalizing the Smale's horseshoe) considered by Zgliczyński, and our result generalizes [Z1, Th. 4.1] and [Z2, Th. 1]. The conclusion of Theorem 3 is similar to that of [Sz1, Cor. 5.1] and [Sz2, Th. 2.2]. Unlike in the mentioned papers, we do not assume that the compact invariant sets in which we look for chaotic dynamics are isolated.

Before we precisely state the main results, we fix the notation used in the paper. Let (X, A) be a topological pair. If $A \neq \emptyset$, then X/A denotes the quotient space obtained by collapsing of A to a point; that point in X/A is denoted by $[A]$. In the case $A = \emptyset$, X/A is defined as the disjoint union of X and a point $*$ outside of X, and $[A] := *$. In both cases, by $[x]$ we denote the equivalence class of $x \in X$ in X/A.

In this paper H^* (resp. $\widetilde{H}^*$) denotes the Alexander-Spanier (resp. the reduced Alexander-Spanier) cohomology functor with coefficients in the field of rational numbers $\mathbb{Q}$. If (X, A) is a compact pair, then there are natural isomorphisms

$$H^*(X, A) \cong H^*(X/A, [A]) \cong \widetilde{H}^*(X/A)$$

by the strong excision property (see [Sp, Th. 6.6.5]). If (X, A) is a pair of compact ANRs (absolute neighborhood retracts), then $H^*(X, A)$ (hence also $\widetilde{H}^*(X/A)$) are of *finite type*, i.e. they are finitely generated graded vector spaces. If $f : (X, A) \to (X, A)$ is a continuous map and $H^*(X, A)$ is of finite type, then the *Lefschetz number* of f is defined as

$$\Lambda(f) := \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H^i(f)$$

where $H^i(f) : H^i(X,A) \to H^i(X,A)$ is the induced endomorphism. Similarly, if $g : X \to X$ is continuous and $\widetilde{H}^*(X)$ is of finite type, then we define

$$\widetilde{\Lambda}(g) := \sum_{i=0}^{\infty} (-1)^i \mathrm{tr}\, \widetilde{H}^i(g).$$

By the invariance of the Lefschetz number under conjugacy, if (X,A) is a pair of compact ANRs, then

$$\Lambda(f) = \Lambda(f'') = \widetilde{\Lambda}(f'),$$

where the maps $f' : X/A \to X/A$ and $f'' : (X/A,[A]) \to (X/A,[A])$ are induced by $f : (X,A) \to (X,A)$.

The present text is a modified version of the author's unpublished reports cited in [Gi] and [Sz1].

2. Generalizations of the Lefschetz theorem

Let (C,E) be a pair of subsets of X and let $f : X \to X$. The map $f^\dagger : C/E \to C/E$ given by

$$f^\dagger([x]) := \begin{cases} [f(x)] & \text{if } x \in C \setminus E,\ f(x) \in C \setminus E, \\ [E] & \text{if } x \in E \text{ or } f(x) \in (X \setminus C) \cup E \end{cases}$$

was introduced by Robbin and Salamon in [RS]. (They call a compact pair (C,E) an *index pair* for a discrete-time flow f provided $f^\dagger$ is continuous and $C \setminus E$ is an isolating neighborhood; compare [RS, Def. 5.1].)

The following result generalizes the Lefschetz fixed point theorem:

Theorem 1. *If $f : X \to X$ is a continuous map of a metrizable space X, (C,E) is a pair of compact ANRs in X, the map $f^\dagger : C/E \to C/E$ is continuous, and $\widetilde{\Lambda}(f^\dagger) \neq 0$, then there exists an $x \in \overline{C \setminus E}$ such that $f(x) = x$.*

The above theorem was essentially given as [Sr, Th. 9], where the condition on continuity of $f^\dagger$ was replaced by a more restrictive one; see Section 1. We will compare both conditions in Proposition 2. We do not present a separate proof of Theorem 1 since it is a particular case of a more general Theorem 2 below. In order to state it, we extend the definition of $f^\dagger$.

Let $f : X \to X'$ be a continuous map, and let (C,E) and (C',E') be two pairs of subsets of X and X', respectively. f induces the map $f^\dagger : C/E \to C'/E'$ given by

$$f^\dagger([x]) := \begin{cases} [f(x)] & \text{if } x \in C \setminus E,\ f(x) \in C' \setminus E', \\ [E'] & \text{if } x \in E \text{ or } f(x) \in (X' \setminus C') \cup E'. \end{cases}$$

We say that (C,E) *fits to* (C',E') *with respect to* f if $f^\dagger$ is continuous.

Let k be a positive integer and let j denote an element of $\mathbb{Z}_k$. Let $f_j : X_j \to X_{j+1}$ be a continuous function between metrizable spaces and let (C_j,E_j) be a pair of compact ANRs contained in X_j.

Theorem 2. *Assume that (C_j,E_j) fits to (C_{j+1},E_{j+1}) with respect to f_j. If $\widetilde{\Lambda}(f_0^\dagger \circ \ldots \circ f_{k-1}^\dagger) \neq 0$, then there exist $x_0 \in \overline{C_0 \setminus E_0}$ and $x_j \in C_j$, $j = 1, \ldots, k-1$,* *such that*

$$f_j(x_j) = x_{j+1}.$$

In particular, x_0 *is a fixed point of the composition* $f_{k-1} \circ \ldots \circ f_0$.

A proof of Theorem 2 is postponed to Section 5.

3. On the continuity of $f^\dagger$

Let $f : X \to X'$ be continuous and let (C, E) in X and (C', E') in X' be two pairs of compact sets. We present two criteria which guarantee continuity of the map $f^\dagger$. First of them is essentially the same as [RS, Th. 4.3]:

Proposition 1. $f^\dagger : C/E \to C'/E'$ *is continuous if and only if*

- **(a)** *for every* $x_0 \in E \cap f^{-1}(C' \setminus E')$ *there exists a neighborhood* A *of* x_0 *in* X *such that* $f(A \cap (C \setminus E)) \subset X' \setminus C'$,
- **(b)** *for every* $x_0 \in (C \setminus E) \cap f^{-1}(\partial C' \setminus E')$ *there exists a neighborhood* B *of* x_0 *in* X *such that* $f(B \cap (C \setminus E)) \subset C'$.

Proof. The same as in [RS].

The second result refers to the condition considered in [Sr]:

Proposition 2. *If*

$$f(E) \cap C' \subset E', \quad C' \cap \overline{f(C) \setminus C'} \subset E',$$

then $f^\dagger : C/E \to C'/E'$ *is continuous.*

Proof. We use Proposition 1. Since there are no points x such that $x \in E$ and $f(x) \in C' \setminus E'$, the condition (a) is satisfied.

Let $x_0 \in C \setminus E$ and $f(x_0) \in \partial C' \setminus E'$. Assume that for every B, a neighborhood of x_0, $f(B \cap (C \setminus E)) \not\subset C'$. Then

$$f(B) \cap f(C \setminus E) \cap (X' \setminus C') \neq \emptyset.$$

It follows by the continuity of f that $f(x_0) \in \overline{f(C \setminus E) \cap (X' \setminus C')}$, hence $f(x_0) \in C' \cap \overline{f(C) \setminus C'} \subset E'$. But this contradicts the condition $f(x_0) \in \setminus E'$, hence the proof is finished.

4. A result on the existence of chaos

We apply Theorem 2 to a result on chaotic dynamics. Let r be a positive integer, let

$$\Sigma_r := \{1, \ldots, r\}^{\mathbb{Z}}$$

denote the set of bi-infinite sequences of numbers $1, \ldots, r$, and let $\sigma : \Sigma_r \to \Sigma_r$ denote the shift map.

Theorem 3. *Let* $f : X \to X$ *be a homeomorphism and for* $p = 1, \ldots, r$ *let* (C_p, E_p) *be a pair of compact ANRs contained in* X*. Assume that for every* $p, q = 1, \ldots, r$ (C_p, E_p) *fits to* (C_q, E_q) *with respect to* f *and* $C_p \cap C_q = \emptyset$ *provided* $p \neq q$*. If for some* n *and every* p

$$\widetilde{H}^n(C_p/E_p) = \mathbb{Q}, \quad \widetilde{H}^i(C_p/E_p) = 0 \quad \forall i \neq n$$

and each homomorphism $\widetilde{H}^*(f^\dagger) : \widetilde{H}^*(C_p/E_p) \to \widetilde{H}^*(C_q/E_q)$ *is an isomorphism, then there exist a compact set* $S \subset C_1 \cup \ldots \cup C_r$ *invariant with respect to* f *and* *a continuous surjective map* $\phi : S \to \Sigma_r$ *such that* $\sigma \circ \phi = \phi \circ f$. *Moreover, the counterimage of each* k*-periodic sequence in* Σ_r *contains at least one* k*-periodic point of* f.

The above theorem asserts the existence of a kind of chaotic behavior. We do not get all conditions of the classical definition of chaos given by Devaney in [De], nevertheless the semiconjugacy to the shift guarantees positive entropy of f in S. If we do not assume that f is a homeomorphism, then, in general, in the above theorem one should replace Σ_r by the set of sequences $\{1, \dots, r\}^{\{0,1,\dots\}}$.

As we mentioned in the Introduction, similar results were already written in the papers [Sz1, Sz2, Z1, Z2]. In all those results it is assumed that the considered sets are isolating neighborhoods — we drop that assumption here.

Proof. Put $C = C_1 \cup \ldots \cup C_r$ and

$$S = \bigcap_{i=-\infty}^{\infty} f^i(C).$$

For $x \in S$ put $\phi(x) = \{a_i\}_{i=-\infty}^{\infty} \in \Sigma_r$, where $a_i \in \{1, \dots, r\}$ are such that

$$f^n(x) \in C_{a_n}.$$

Let $a = \{a_i\}$ be a k-periodic sequence in Σ_r. Denote by

$$f_i^\dagger : C_{a_i}/E_{a_i} \to C_{a_{i+1}}/E_{a_{i+1}}$$

the map induced by f. Since all $\widetilde{H}^*(f_i^\dagger)$ are isomorphisms and the cohomologies are 1-dimensional, $\widetilde{\Lambda}(f_{k-1}^\dagger \circ \ldots \circ f_0^\dagger)$ is nonzero. By Theorem 2, there exists a k-periodic point $x \in C_{a_0}$ such that $\phi(x) = a$. Since periodic points are dense in Σ_r, the map ϕ is surjective, hence the result is proved.

As an illustration of Theorem 3 we consider a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which transforms a rectangle into a horseshoe intersecting the rectangle in the way drawn in Figure 1.

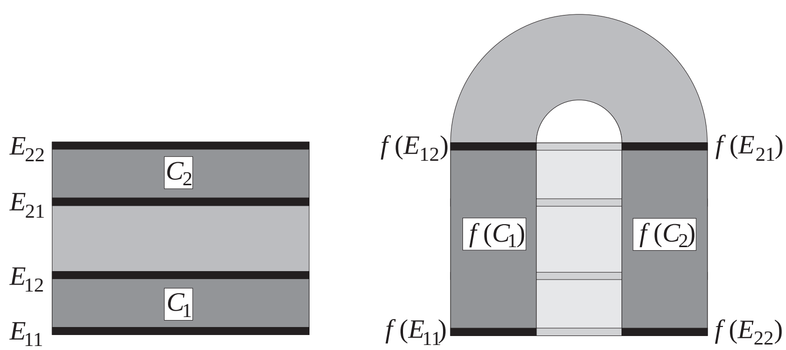

FIGURE 1. The horseshoe-map $f : \mathbb{R}^2 \to \mathbb{R}^2$. The sets $E_i := E_{i1} \cup E_{i2}$, $i = 1, 2$, and their images are marked in black.

C_1 and C_2 form the lower and, respectively, the upper part of the rectangle; E_i consists of the top and the bottom of C_i. The rectangle is not an isolating neighborhood; in particular, its lower left corner is a fixed point of f. By Proposition 2, Theorem 3 applies to f with $r = 2$ and $n = 1$.

5. A proof of Theorem 2

Let (C,E) be a topological pair, let D be a topological space and let $e : E \to D$ be a continuous map. Define the adjunction

$$M := C \cup_e D$$

as the quotient space $D \times 0 \cup C \times 1/\sim$ where $\sim$ is the minimal equivalence relation which identifies $(x,1) \in E \times 1$ with $(e(x),0) \in D \times 0$. For $z \in C$ or $z \in D$ and $i \in \{0,1\}$, let $[z,i]$ denote the equivalence class of (z,i) in the relation $\sim$. Define the following subsets of the adjunction:

$$\begin{aligned} C^* &:= \{[x,1] \in M : x \in C\}, \\ D^* &:= \{[y,0] \in M : y \in D\}, \\ E^* &:= \{[x,1] \in M : x \in E\} = \{[e(x),0] \in M : x \in E\} = C^* \cap D^*. \end{aligned}$$

In the sequel we assume that $f : X \to X'$ is a continuous map between metrizable spaces and (C,E) and (C',E') are pairs of compact sets in X and X', respectively. Let D' be another topological space and let $e' : f(C) \cup E' \to D'$ be an embedding, i.e. a continuous injective map. Let $M' := C' \cup_{e'|_{E'}} D'$. Define a map $h : C \to M'$ by

$$h(x) := \begin{cases} [f(x),1] & \text{if } x \in C \setminus E,\ f(x) \in C' \setminus E', \\ [e'(f(x)),0] & \text{if } x \in E \text{ or } f(x) \in (X' \setminus C') \cup E'. \end{cases}$$

Proposition 3. *The map h is continuous if and only if $f^\dagger$ is continuous.*

Proof. The map $b : M' \to C'/E'$ defined by $b([y,1]) := [y]$ and $b([z,0]) := [E']$ is continuous. Let $q : C \to C/E$ denote the quotient map. It follows that $f^\dagger \circ q = b \circ h$, hence the continuity of h implies the continuity of $f^\dagger$.

Assume now that $f^\dagger$ is continuous. Let $x_0 \in C$. Assume first that $h(x_0) \in {C'}^* \setminus {E'}^* = M' \setminus {D'}^*$. Then $x_0 \in C \setminus E$. Since the map $x \mapsto [x]$ is a homeomorphism $C \setminus E$ to $C/E \setminus [E]$, the map $[y] \mapsto [y,1]$ is a homeomorphism $C'/E' \setminus [E']$ to ${C'}^* \setminus {E'}^*$, $C \setminus E$ is open in C and ${C'}^* \setminus {E'}^*$ is open in M', the continuity of $f^\dagger$ at $[x_0]$ implies the continuity of h at x_0.

Assume that $h(x_0) \in {E'}^*$. Let Z be an open neighborhood of $h(x_0)$ in M'. Then

$$Z = \{[y,1] \in M' : y \in U\} \cup \{[z,0] \in M' : z \in V\}$$

for some open sets $U \subset C'$ and $V \subset D'$. Let U' and V' be open sets in X', $U = U' \cap D'$ and $e'(V' \cap f(C)) = V \cap e'(f(C))$. Since $h(x_0) \in {E'}^*$, $f(x_0) \in U' \cap V'$. There exists a W, a neighborhood of x_0 in C, such that $f(W) \subset U' \cap V'$. It follows that $h(W) \subset Z$, and hence h is continuous at x_0.

Finally, assume that $h(x_0) \in {D'}^* \setminus {E'}^* = M' \setminus {C'}^*$. Then $f(x_0) \in X' \setminus C'$. Let again Z be a neighborhood of x_0 in M'. We can assume that $Z := \{[z,0] \in M' : z \in V\}$ where V is an open set in $D' \setminus e'(E')$. Let V' be an open set in X' such that $e'(V' \cap f(C)) = V \cap e'(f(C))$. Then $f(x_0) \in V'$ and we can assume that $V' \subset X' \setminus C'$. Choose a neighborhood W of x_0 in C such that $f(W) \subset V'$. Then $h(W) \subset Z$ and the proof is finished.

Let D be a compact space, let $e : E \to D$ be an embedding, and let $M := C \cup_e D$. Assume that there is a continuous map $\Phi : D \to D'$ such that $\Phi(e(x)) = e'(f(x))$

for each $x \in E$. We define a map $g : M \to M'$ as follows:

$$g([x,1]) := \begin{cases} [f(x),1] & \text{if } x \in C \setminus E,\ f(x) \in C' \setminus E', \\ [e'(f(x)),0] & \text{if } x \in E \text{ or } f(x) \in (X' \setminus C') \cup E', \end{cases}$$
$$g([y,0]) := [\Phi(y),0] \text{ if } y \in D.$$

Proposition 4. *The map g is continuous if and only if $f^\dagger$ is continuous.*

Proof. By Proposition 3, we should prove that g is continuous if and only if h is continuous. Assume that g is continuous. Since $h = g \circ i$, where $i : C \ni x \to [x,1] \in M$, h is also continuous.

On the other hand, assume that h is continuous. It follows that $g|_{C^*}$ is continuous. Since the continuity of $g|_{D^*}$ is a consequence of the continuity of Φ, the result follows.

Let $r : M \to C/E$ denote the map given by $[x,1] \mapsto [x]$ if $x \in C$ and $[y,0] \mapsto [E]$ if $y \in D$. In the same way define the map $r' : M' \mapsto C'/E'$.

Proposition 5. *The following diagram is commutative:*

$$\begin{array}{ccc} (M, D^*) & \xrightarrow{\quad g \quad} & (M', {D'}^*) \\ \Big\downarrow r & & \Big\downarrow r' \\ (C/E, [E]) & \xrightarrow{\quad f^\dagger \quad} & (C'/E', [E']) \end{array}$$

Proof. Immediately from definitions.

Proof of Theorem 2. Let $j \in \mathbb{Z}_k$. The set $E_j \cup f_{j-1}(C_{j-1})$ is compact and metrizable, hence, by the Kuratowski Theorem, it can be embedded in a Banach space. In order to simplify the notation, assume just that $E_j \cup f_{j-1}(C_{j-1}) \subset V_j$, where V_j is a Banach space. By the Mazur Theorem, $\text{Conv}(E_j \cup f_{j-1}(C_{j-1}))$, its closed convex hull in V_j, is a compact AR (absolute retract). Define

$$D_j := \text{Conv}(E_j \cup f_{j-1}(C_{j-1})) \times S^1,$$
$$e_j : E_j \cup f_{j-1}(C_{j-1}) \ni x \to (x,1) \in D_j.$$

The map e_j is an embedding. Let

$$M_j := C_j \cup_{e_j|_{E_j}} D_j$$

be the adjunction. Since D_j, C_j, and E_j are compact ANRs, M_j is also a compact ANR. In order to prove the theorem we define a map $g_j : M_j \to M_{j+1}$ of the form considered in Proposition 4. To this aim we introduce two auxiliary functions:

$$\phi_j : \text{Conv}(E_j \cup f_{j-1}(C_{j-1})) \to [0,1],$$

a Urysohn Lemma continuous function such that $\phi_j^{-1}(0) = E_j$, and

$$\Psi_j : \text{Conv}(E_j \cup f_{j-1}(C_{j-1})) \to \text{Conv}(E_{j+1} \cup f_j(C_j)),$$

a function given by the Tietze–Dugundji Theorem such that $\Psi_j(x) = f_j(x)$ for $x \in E_j$. Now we are ready to define the map g_j. Let $x \in C_j$, $z \in \mathrm{Conv}(E_j \cup f_{j-1}(C_{j-1}))$, and $\alpha \in S^1$. Define

$$g_j([x,1]) := \begin{cases} [f(x),1] & \text{if } x \in C_j \setminus E_j,\ f(x) \in C_{j+1} \setminus E_{j+1}, \\ [(f(x),1),0] & \text{if } x \in E_j \text{ or } f(x) \in (X_{j+1} \setminus C_{j+1}) \cup E_{j+1}, \end{cases}$$
$$g_j([(z,\alpha),0]) := [(\Psi_j(z), \alpha \exp(\pi i \phi_j(z)/k)),0].$$

Since $f_j^\dagger$ is continuous, g_j is also continuous by Proposition 4. As a consequence of the formula for g_j we get the following assertion: if $[w,i] \in M_0$ is such that

$$g_{k-1} \circ \ldots \circ g_0([w,i]) = [w,i],$$

then necessarily

$$w \in C_0,\ i = 1,\ (f_{k-1} \circ \ldots \circ f_0)(w) = w,$$
$$(f_j \circ \ldots \circ f_0)(w) \in C_{j+1},$$

for every $j = 0, \ldots, k-1$. Thus, in order to prove Theorem 2 it suffices to show that the composition $g_{k-1} \circ \ldots \circ g_0$ has a fixed point in $\overline{C_0^* \setminus E_0^*}$.

Let $r_j : (M_j, D_j^*) \to (C_j/E_j, [E_j])$ be given by $[x,1] \mapsto [x]$ in C_j^* and $[y,0] \mapsto [E_j]$ in D_j^*. By the strong excision property of Alexander-Spanier cohomologies, each map $H^*(r_j)$ is an isomorphism. It follows by Proposition 5 that the following diagram in cohomologies is commutative:

$$\begin{array}{ccc} H^*(C_0/E_0,[E_0]) & \xrightarrow{\ H^*(f_{k-1}^\dagger \circ \ldots \circ f_0^\dagger)\ } & H^*(C_0/E_0,[E_0]) \\ \cong \Big\downarrow H^*(r_0) & & \cong \Big\downarrow H^*(r_0) \\ H^*(M_0,D_0^*) & \xrightarrow{\ H^*(g_{k-1} \circ \ldots \circ g_0)\ } & H^*(M_0,D_0^*) \end{array}$$

Thus $\widetilde{H}^*(f_{k-1}^\dagger \circ \ldots \circ f_0^\dagger)$ and $H^*(g_{k-1} \circ \ldots \circ g_0)$ are conjugated, and hence their Lefschetz numbers coincide; in particular

$$\Lambda(g_{k-1} \circ \ldots \circ g_0) \neq 0.$$

It follows by the Bowszyc fixed point theorem ([Bo, Theorem (4.5)]) that $g_{k-1} \circ \ldots \circ g_0$ has a fixed point in $\overline{M_0 \setminus D_0^*} = \overline{C_0^* \setminus E_0^*}$, hence the result is proved.

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