THE RESIDUES OF THE RESOLVENT ON DAMEK-RICCI SPACES

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Abstract. We determine the poles and residues of the resolvent kernel of the Laplacian on a Damek-Ricci space S. We show that all poles are simple and the residues define convolution operators of finite rank. This generalizes a result of Guillopé-Zworski for the real hyperbolic n-space. If S corresponds to a symmetric space of negative curvature $G=K$, the image of each residue is a $g_c$-module with a specific highest weight. We compute the dimension by the Weyl dimension formula.

1. Preliminaries

In this section we will recall some basic notions on $H$-type groups and their canonical solvable extensions, following mainly [2] (see also [1]).

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle , \rangle$ such that $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$. If $\mathfrak{n}$ is abelian, we shall use the convention that $\mathfrak{v} = 0$ and $\mathfrak{n} = \mathfrak{z}$.

Define a linear mapping $J : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ by

$$\langle J_{Z} X, Y \rangle = \langle Z, [X, Y] \rangle$$

(note that $J_{Z}$ is skew-symmetric). Now $\mathfrak{n}$ is said to be an $H$-type algebra if for any $Z_1, Z_2 \in \mathfrak{z}$,

$$J_{Z_1}J_{Z_2} + J_{Z_2}J_{Z_1} = -2 \langle Z_1, Z_2 \rangle.$$  

The corresponding $H$-type group is the simply connected Lie group $N$ with Lie algebra $\mathfrak{n}$, endowed with the left-invariant metric induced by the inner product $\langle , , \rangle$ on $\mathfrak{n}$.

Consider the solvable extension, $S = AN$, the semidirect product of $A = \mathbb{R}^+$ and $N$, where each $t \in A$ acts on $N$ by $(x, z) \mapsto (tx, tz)$.

Let $\mathfrak{s}, \mathfrak{a}$ denote respectively the Lie algebras of $S, A$. Then $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{a} = RH$, where $ad H$ is the derivation of $\mathfrak{n}$ such that $ad H|_{\mathfrak{v}} = \frac{1}{2} I$ and $ad H|_{\mathfrak{z}} = I$. Also, $\mathfrak{s}$ carries the inner product extending the one on $\mathfrak{n}$ such that $\langle H, \mathfrak{n} \rangle = 0$; $S$ carries the induced left-invariant riemannian structure. Furthermore, let $q = \dim \mathfrak{z}$, $p = \dim \mathfrak{v}$, $n = \dim \mathfrak{s} = p + q + 1$ and $Q = \frac{1}{2}(p + 2q)$.

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Using coordinates from $\mathfrak{v} \oplus \mathfrak{j} \oplus \mathbb{R}^+$, the product on $S$ is expressed as

$$(X, Z, a)(X', Z', a') = (X + a\hat{a}X', Z + aZ' + \frac{1}{2}a^2[X, X'], aa').$$

The volume element of the induced left-invariant riemannian metric on $S$ is the left Haar measure

$$dm = a^{-Q-1}dXdZda.$$ 

We will use the fact that $S$ can be realized as the unit ball in $S$:

$$B(\hat{a}) = \{(X, Z, u) : |X|^2 + |Z|^2 + u^2 = 1\}$$

via a Cayley type transform $\tilde{C} : S \to B(\hat{a})$ (see [2], Section 4).

In $B(\hat{a})$ the geodesics through the origin are the diameters and the geodesic distance to the origin $r = d(\hat{p}, 0) = \log \frac{1+|\hat{p}|}{1-|\hat{p}|}$, thus $|\hat{p}| = \tanh (r/2)$, with $\hat{p} = \tilde{C}(p)$, if $p \in S$. Furthermore $\cosh \left(\frac{r}{2}\right)^{-2} = \frac{1}{1+\frac{1}{4}|X|^2+|Z|^2}$ and the image of the left Haar measure on $S$ via $\tilde{C}^{-1}$ is $d\mu = J(r)\, d\sigma dr$, where $r, \sigma$ are the radian coordinates on $B$, $r^2 = |X|^2 + |Z|^2 + u^2$ and $J(r) = 2^q \sinh(r/2)^p \sinh(r)^q$ (see [2], Section 4).

The symmetric spaces of negative curvature are a main subclass of the Damek-Ricci spaces. Let $G$ be a connected, noncompact, semisimple Lie group of real rank one. Let $K$ be a maximal compact subgroup of $G$ and let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras. If $G = NA$ is an Iwasawa decomposition of $G$, then $N$ is an $H$-type group and $S = NA \approx G/K$ is a solvable Lie group in the class introduced above. Indeed, if $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of $A$ and $N$ respectively, $\mathfrak{n}$ splits $\mathfrak{n} = \mathfrak{g}_{\alpha/2} \oplus \mathfrak{g}_\alpha$, where $\mathfrak{g}_{\alpha/2}$, $j = 1/2, 1$, denote the $j\alpha$-root spaces of $\mathfrak{a}$. In the notation above we have $\mathfrak{n}_\alpha = \mathfrak{j}$, $\mathfrak{n}_{\alpha/2} = \mathfrak{v}$, $\mathfrak{a} = \mathbb{R}H_\alpha$, with $H_\alpha \in \mathfrak{a}$ such that $\alpha(H_\alpha) = 1$. If on $S = NA$ we use the $G$-invariant metric induced by $2(p+q)^{-1}B$ (the Killing form of $\mathfrak{g}$), then $S$ is isometric to a Damek-Ricci space. We note that, because of our convention, if $\mathfrak{n}$ is abelian, then $p = 0$, $q = \dim \mathfrak{n}$.

2. The resolvent of the Laplacian on $S$

Damek-Ricci spaces have strong similarities with symmetric spaces of negative curvature, in particular they are harmonic spaces. On $S$ there is a radialization operator $\pi$ which corresponds to the standard operator in the case of the ball model of $S$ (see [2], p. 230). If $f \in C^\infty_c(S)$, $p \in S$ and $\hat{p} = \tilde{C}(p)$ then

$$\pi f(p) := \int_{S^{p+q}} \hat{f}(||\hat{p}||\sigma) \, d\sigma,$$

where $\hat{f} := f \circ \tilde{C}^{-1}$. In the symmetric case, if $f \in C^\infty(NA)$, then $\pi f(x) = \int_L \hat{f}(kx) \, dk$, where $\hat{f}$ denotes the right $K$-invariant extension of $f$.

If $\{Z_i\}$, $\{V_j\}$ are orthonormal bases of $\mathfrak{j}$ and $\mathfrak{v}$ respectively, the Laplace-Beltrami operator is given by $L = \sum_i Z_i^2 + \sum_j V_j^2 + H^2 - QH$; $L$ generates the algebra of left-invariant differential operators on $S$ which commute with $\pi$ (see [2], Theorem 5.2).

If $f$ is a smooth radial function on $S - \{e\}$, we will often abuse notation by writing $f(r) = f(x)$, where $r = d(x, e)$. The action of $L$ on radial functions is given by

$$Lf(r) = \frac{d^2}{dr^2}f(r) + \frac{1}{2}(p \coth(r/2) + 2q \coth(r)) \frac{d}{dr}f(r).$$
In the symmetric case, if $n$ is not abelian and we set $r = 2t$, then $L f(t)$ corresponds to $\frac{1}{4} C f(a_t)$, $C$ the Casimir element; [6], Section 1 (1). If $n$ is abelian, then $L$ corresponds to $C$.

A spherical function $\psi$ on $S$ is a radial eigenfunction of $L$ such that $\psi(e) = 1$. This generalizes the corresponding notion in the symmetric case and one has the following characterization (2).

**Proposition 2.1.** Let $\nu \in C$. The function $\phi_\nu = \pi(a^{\nu+Q/2})$ is a spherical function with eigenvalue $\lambda(\nu) = \nu^2 - Q^2/4$. Any spherical function on $S$ is of this form.

As in the symmetric case, we can express $\phi_\nu$ by a hypergeometric function as follows. By letting $z = -\sinh(r/2)$, the equation

$$
\left\{ \frac{d^2}{dr^2} + \frac{1}{2} (p \coth(r/2) + 2q \coth(r)) \frac{d}{dr} - \lambda(\nu) \right\} f_\nu(r) = 0
$$

transforms into the hypergeometric equation with parameters $a = Q/2 - \nu$, $b = Q/2 + \nu$, and $c = n/2$. Since $\phi_\nu(e) = 1$, it follows that

$$
\phi_\nu(r) = F \left( -\nu + Q/2, \nu + Q/2, \frac{n}{2} - \sinh(r/2)^2 \right).
$$

Furthermore, if $\Re \nu > 0$, the asymptotic behavior of $\phi_\nu(r)$, as $r \to \infty$, is given by (see [2], p. 239)

$$
\phi_\nu(r) \sim c(\nu) e^{r(\nu+Q/2)}, \quad \text{where} \quad c(\nu) = \frac{2^{-2\nu+Q} \Gamma(\nu/2) \Gamma(2\nu)}{\Gamma(\nu+Q/2) \Gamma(\nu+Q/4)}.
$$

Here $c(\nu)$ coincides with Harish Chandra’s $c$-function in the symmetric case. The Plancherel measure, $\mu(\nu) = (c(\nu)c(-\nu))^{-1}$, can be written $\mu(\nu) = c_o p(\nu) D(\nu)$, $c_o$ a constant and $p(\nu)$ the polynomial given by

$$
\prod_{j=0}^{q-1} \left( -\nu^2 + ((2j+1)^2/4) \right) \prod_{j=0}^{q-1} \left( -\nu^2 + ((j^2/4) \right), \quad q, \frac{q}{2} \text{ even},
$$

$$
-\prod_{j=1}^{q/4} (-\nu^2 + j^2)^2 \nu^3, \quad q = 1, \frac{q}{2} \text{ odd},
$$

$$
-\prod_{j=0}^{q-1} \left( -\nu^2 + ((2j+1)^2/4) \right) \prod_{j=0}^{q-1} \left( -\nu^2 + ((2j+1)^2/4) \right) \nu, \quad q \text{ odd, } \frac{q}{2} \text{ even},
$$

and $D(\nu)$ equals respectively 1, $\cot(\pi \nu)$, and $\tan(\pi \nu)$ (1).

**Remark.** We note that $p$ is always even, since $v$ is a module over the Clifford algebra of $\mathfrak{g}$. If $p = 0$, then $X \approx H^{q+1}$, $G \simeq SO(q+1, 1)$ and in this case $D(\nu)$ equals 1 or $\tan(\pi \nu)$ depending on whether $q$ is even or odd.

In [6], the resolvent of the Laplacian $R(\lambda(\nu))$ was studied on symmetric (and locally symmetric spaces) of negative curvature. In the symmetric case, it is given for $\Re \nu > p$ by convolution with a smooth radial function $Q_\nu$ on $S - \{e\}$ which is an eigenfunction of $L$ with eigenvalue $\lambda(\nu)$, and which has a meromorphic continuation to $C$. As we shall now see, these properties remain valid for any $S$ as above. Many arguments in [6] can be adapted, so we shall omit several proofs. On the other hand, we shall show how to obtain $Q_\nu$ by using a series solution. We thank N. Wallach for useful discussions on this point, which helped us to simplify the original argument.
If \( b \in \mathbb{R} \) and \( \delta > 0 \), let \( S_{b, \delta} = \{ \nu : \Re \nu > b, |\nu + j| > \delta, \forall j \in \mathbb{Z}, b \leq j \} \). That is, \( S_{b, \delta} = \{ \nu : \Re \nu > b \} \), if \( b \geq 0 \), and \( S_{b, \delta} \) is a half plane with finitely many discs removed, centered at \(-1, -2, \ldots, -k\), with \(-k \geq b\), if \( b < 0 \).

**Theorem 2.2.** If \( \nu \in \mathbb{C}, 2\nu \notin -\mathbb{N} \), then there exists a radial function \( Q_{\nu} \in C^\infty(S - \{e\}) \) with the following properties:

(a) \((L - \lambda(\nu))Q_{\nu} = 0\). For each \( x \in S \), \( Q_{\nu}(x) \) is holomorphic for \( \nu \notin -\frac{1}{2}\mathbb{N} \) and \( \nu \in \frac{1}{2}\mathbb{N} \), \( Q_{\nu}(s) \) has at most a simple pole. Furthermore, for any \( b \in \mathbb{R}, \delta, \rho_o > 0 \), there exists \( K = K(b, \delta, \rho_o) \) such that \( |Q_{\nu}(r)| \leq K \) for any \( r \geq \rho_o, \nu \in S_{b, \delta} \).
(b) Where defined, \( \phi_{\nu} = e(\nu)Q_{\nu} + e(\nu)Q_{-\nu} \).
(c) As \( r \to 0 \), \( Q_{\nu}(r) \sim d(\nu)r^{-p-q+1}|\log r|^{p+q+1} \), for some meromorphic function \( d(\nu) \) on \( \mathbb{C} \), holomorphic if \( 2\nu \notin -\mathbb{N} \).
(d) \( \lim_{r \to 0^+} J(r) \frac{Q_{\nu}(r)}{r^2} = -2\nu c(\nu) \).
(e) If \( f \in C^\infty_c(S) \) and \( 2\nu \notin -\mathbb{N} \), then

\[
\int_{S} Q_{\nu}(x^{-1}y)(L - \lambda(\nu))f(y)dy = -2\nu c(\nu)f(x).
\]

**Proof.** We look for a solution of (1) of the form \( q_{\nu}(r) = \sum_{j=0}^\infty a_j(\nu)e^{-(\nu+Q/2+j)r} \).

Substituting in (1) and using \( \coth(r) = \frac{1+e^{-2r}}{1-e^{-2r}} \), we get that

\[
\sum_{j \geq 0}(Q + j)(2\nu + Q + j)a_j(\nu)e^{-jr} + p \sum_{j \geq 1}(v + Q/2 + j + 1)a_{j+1}(\nu)e^{-jr}
+ \sum_{j \geq 2}(j + 2)(2\nu + j + 2)a_{j+2}(\nu)e^{-jr} = 0.
\]

Thus, the coefficients \( a_j(\nu) \) must satisfy the recurrence relations

\[
(8) \quad a_1(\nu) = a_0(\nu)f_{-1}(\nu), \quad a_{j+2}(\nu) = a_{j+1}(\nu)f_j(\nu) + a_j(\nu)g_j(\nu),
\]

where \( f_j(\nu) = \frac{Q+j+(Q+j)}{(Q+j)(2\nu+j+2)} \) and \( g_j(\nu) = \frac{(Q+j)(2\nu+j+2)}{(j+2)(2\nu+j+2)} \), for \( j \geq 0 \).

We thus set \( q_{\nu}(r) = e^{-(\nu+Q/2)r}\sum_{j=0}^\infty a_j(\nu)e^{-jr} \), where \( a_0 = 1 \), and if \( 2\nu \notin -\mathbb{N} \), then the \( a_j(\nu) \) are given by (8).

If \( b \in \mathbb{R}, \delta > 0 \) and \( \nu \in S_{b, \delta} \), we have

\[
|f_j(\nu)| \leq \frac{p}{2j + 4} \left( 1 + \frac{Q+j}{2\nu+j+2} \right) \leq \frac{p}{2j + 4} \left( 1 + \frac{Q+j}{j+2-2k} \right),
\]

\[
|g_j(\nu)| \leq \frac{Q+j}{2j+2} \left( 1 + \frac{|Q-2|}{2\nu+j+2} \right) \leq \frac{Q+j}{j+2} \left( 1 + \frac{|Q-2|}{j+2-2k} \right)
\]

for \( j + 2 > 2|k| \), where \( k \) is the first integer such that \( k \leq b \). These estimates clearly imply that given \( \varepsilon > 0 \) there exist \( j_0 \) and \( M = M(\varepsilon) \) such that \( |f_j(\nu)| \leq \varepsilon, |g_j(\nu)| \leq 1 + \varepsilon \), if \( j \geq j_0, |f_j(\nu)| \leq M, |g_j(\nu)| \leq M \), if \( j < j_0 \), uniformly for \( \nu \in S_{b, \delta} \). Using these estimates we see that if \( \nu \in S_{b, \delta} \), if \( M' = M'(\varepsilon) = j_0 M^{j_0} \), then

\[
(9) \quad |a_j(\nu)| \leq \left\{ \begin{array}{ll} j M_j & j \leq j_0, \\
M'(1+2\varepsilon)^{j-j_0+1} & j > j_0. \end{array} \right.
\]

Now, by (9) \( |q_{\nu}(r)| \leq e^{-(\Re \nu + \frac{Q}{2})r}M' \left( j_0 + \sum_{j \geq 0} (1+2\varepsilon)^{l+1} e^{-(l+j_0)r} \right) \); hence the series defining \( q_{\nu} \) converges absolutely and uniformly for \( \nu \in S_{b, \delta} \) and \( r > r_o \), for
Theorem 3.1. Otherwise, it has simple poles lying at a finite rank operator, for each value of $k > \Re T$.

From now on we shall write $Q_\nu(r) = q_\nu(r)$, for simplicity. By the asymptotic behavior as $r \to +\infty$, it follows that if $2\nu \notin \Z$, $Q_\nu(r), Q_{-\nu}(r)$ form a fundamental system of solutions of (4). Writing $\phi_\nu$ in terms of $Q_\nu$ and $Q_{-\nu}$, the functional equation in (b) follows as in the symmetric case (see [6], p. 671).

We now prove (c). Equation (4) has a regular singular point at $r = 0$ and the corresponding indicial equation is $s(s - 1) + (p + q)s = 0$, with roots $s = 0$, $s = 1 - p - q$. The solution $\phi_\nu(r)$ is associated to the root $s = 0$ and is continuous at $r = 0$. If $2\nu \notin \N$, and if $p + q > 1$, $Q_\nu$ is a second linearly independent solution; hence limit$_{r \to 0^+} Q_\nu(r)^{p + q - 1} := d(\nu)$ exists and the meromorphy of $Q_\nu$ implies that of $d(\nu)$. Similarly, if $p + q = 1$, $Q_\nu(r) \sim d(\nu) \log r$ as $r \to 0^+$. Thus (c) follows.

The proof of (d) is similar to that of [3], Lemma 1.3, and will be omitted.

To see (e) we may assume that $x = e$. We have, for any $f \in C^\infty_c(S)$,

$$
\int_S Q_\nu(y)(L - \lambda(\nu)I)f(y) \, d\mu(y) = \int_0^\infty \int_0^\infty \tilde{Q}_\nu(r\sigma)(L - \lambda(\nu)I)\tilde{f}(r\sigma)J(r)\, dr\, d\sigma
$$

$$
= \int_0^\infty Q_\nu(r)J(r)(L - \lambda(\nu))\pi f(r) \, dr.
$$

Now we observe that for a radial function $h$ on $S$, $J(r)^{1/2}Lh(r) = \frac{d^2}{dr^2}J^{1/2}(r)h(r) + J(r)^{1/2}\eta(r)h(r)$, where $\eta = \frac{J''(r)^{2/3} - 2J'(r)^{1/3}}{J(r)^{2/3}}$. Hence we see that the above equals

$$
\int_0^\infty \frac{d^2}{dr^2} \left( J(r)^{1/2} \pi f(r) \right) J(r)^{1/2}Q_\nu(r) - J(r)^{1/2} \pi f(r) \frac{d^2}{dr^2} \left( J(r)^{1/2}Q_\nu(r) \right) \, dr
$$

$$
= \int_0^\infty \frac{d}{dr} \left[ J(r)^{1/2} \pi f(r) \right] J(r)^{1/2}Q_\nu(r) - J(r)^{1/2} \pi f(r) \frac{d}{dr} \left( J(r)^{1/2}Q_\nu(r) \right) \, dr
$$

$$
= -2\nu c(\nu) f(e)
$$

using (c) and (d). This gives (e); hence the theorem follows.

3. The residues of the resolvent

Let $\tilde{R}(\lambda(\nu))$ denote the kernel operator with kernel $K_\nu(x, y) = -\frac{Q_\nu(x^{-1}y)}{2\nu c(\nu)}$. If $\Re \nu > 0$, then $\tilde{R}(\lambda(\nu)) = R(\lambda(\nu))$.

Theorem 3.1. If $p, q$ are both even, then $\tilde{R}(\lambda(\nu))$ is everywhere holomorphic. Otherwise, it has simple poles lying at $\nu_k = -Q/2 - k$ with $k \in \N \cup \{0\}$. If $\nu = \nu_k$, set $T_{\nu_k}(f) := \Res_{\nu = \nu_k} \tilde{R}(\lambda(\nu))(f)$. Then $T_{\nu_k}(f) = (2\pi \nu_k)^{-1} p(\nu_k) f * \phi_\nu$, and $T_{\nu_k}$ is a finite rank operator, for each value of $k$.

Proof. The possible poles of $K_\nu(x, y)$ lie at $-\frac{1}{2} \N$ or at the zeros of $c(\nu)$. By using formula (3) one sees that $c(\nu)$ has no zeros in $\C$, if $p$ and $q$ are both even. Otherwise, $q$ is odd and $c(\nu)$ has simple zeros at $\nu_k = -Q/2 - k$, for any $k \in \N \cup \{0\}$, and possibly simple poles at $\nu \in -\frac{1}{2} \N$. 

Since \( \nu = 0 \) is a simple pole of \( c(\nu) \), and \( Q_\nu \) is holomorphic at 0, \( \frac{Q_0}{2\nu c(\nu)} \) is holomorphic at \( \nu = 0 \).

On the other hand \( c(-\nu) \) and \( Q_{-\nu} \) are holomorphic and nonvanishing on \( \mathbb{R}^{-0} \), \( \phi_\nu \) is everywhere holomorphic and \( \phi_\nu(1) = 1 \). So Theorem 2.2 (b) implies that a pole of \( Q_\nu \) must be compensated by a pole of \( c(\nu) \) and a zero of \( c(\nu) \) cannot be a zero of \( Q_\nu \).

Therefore, \( \frac{Q_\nu}{2\nu c(\nu)} \) has a pole at \( \nu \) if and only if \( \nu \) is a zero of \( c(\nu) \), that is, \( \nu = \nu_k = -Q/2-k, k \in \mathbb{N} \cup \{0\} \). On the other hand, \( \frac{Q_{-\nu}}{2\nu c(-\nu)} \) is analytic at \( \nu = \nu_k \).

Thus, if \( f \in C_\infty^\infty(S) \) and using that \( \frac{Q_\nu}{2\nu c(\nu)} = \frac{Q_{-\nu}}{2\nu c(-\nu)} = \frac{\mu(\nu)\phi_\nu}{2\nu} \), we have

\[
T_{\nu_k}(f) = \text{Res}_{\nu=\nu_k} \tilde{R}(\lambda(\nu))(f) = \frac{p(\nu_k)}{2\pi \nu_k} f \ast \tilde{\phi}_{\nu_k}.
\]

From (5) and the expression for \( \cosh(\frac{z}{2}) \) in the Preliminaries, we have that

\[
\phi_\nu(X, Z, a) = \sum_{i\geq 0} \frac{(Q/2 - \nu)_i (Q/2 + \nu)_i}{i! (n/2)_i} \left[ \frac{(a + \frac{1}{2}[X]^2 + |Z|^2)^i}{4a} \right],
\]

where \((u)_i = \prod_{i=0}^{i-1} u + l \) for \( u \in \mathbb{C} \). Hence we see that the coefficients in the expansion (11) are zero for \( i \geq k+1 \), for the special values \( \nu_k = -Q/2-k \). Fix \( \{V_i\} \) and \( \{W_j\} \), orthonormal bases of \( \mathfrak{v} \) and \( \mathfrak{z} \) respectively, and write \( X = \sum_{i=1}^{p} x_i V_i \) and \( Z = \sum_{j=1}^{q} z_j W_j \). If \( I = (i_1, \ldots, i_p) \), \( J = (j_1, \ldots, j_q) \), set \( X^I = \prod x_{i_j}^{i_j} \), \( Z^J = \prod z_{j_i}^{j_i} \), \(|I| = \sum_{j} i_j \) and similarly for \(|J|\). Let \( \mathcal{F}_k \) be the linear span of the functions \( a^i X^I Z^J : i \in \mathbb{Z}, \) \(|i| \leq k, |I|, |J| \leq 2k \). Clearly \( \phi_{\nu_k} \in \mathcal{F}_k \). If \( t = (Y, U, b) \) with \( Y = \sum_{i=1}^{p} y_i V_i \in \mathfrak{v} \), \( U = \sum_{i=1}^{q} u_i W_i \in \mathfrak{z} \), \( b \in A \) and \( s = (X, Z, a) \in S \), then

\[
t^{-1} s = \left( b^{\frac{-1}{2}}(X - Y), b^{-1}(Z - U + \frac{1}{2}[X, Y]), b^{-1}a \right)
\]

\[
= \left( b^{\frac{-1}{2}} \sum_{i} (x_i - y_i)V_i, b^{-1} \sum (z_j - u_j)W_j + \frac{1}{2} \sum_{i,j} x_i y_j a_{i,j} W_i b^{-1}a \right),
\]

where \([X, Y] = \sum_{i,j} x_i y_j a_{i,j} W_i \). Hence, by (10) \( \phi_{\nu_k}(t^{-1} s) \) is a linear combination of functions of the form \( a^{i_1} b^{i_2} X^{2I_1} Y^{2I_2} Z^{J_1} U^{J_2} \) with \( j_i \in \mathbb{Z}, |j_i| \leq k, i = 1, 2 \), and \(|I_i|, |J_i| \leq 2k \) for \( i = 1, 2 \).

Therefore, if \( f \in C_\infty^\infty(S) \), it follows that \( f \ast \tilde{\phi}_{\nu_k}(t) = \int_s f(s) \phi_{\nu_k}(t^{-1} s) ds \) is a linear combination of expressions of the form

\[
t \mapsto b^{i_2} Y^{2I_2} U^{-J_2} \int_{J_1} \int_{A} f(X, Z, a) a^{i_1} X^{2I_1} Z^{J_1} a^{\frac{-Q}{1-a}} da dX dZ.
\]

Therefore, \( f \ast \tilde{\phi}_{\nu_k} \) belongs to \( \mathcal{F}_k \), a finite dimensional space, as asserted.
4. The symmetric case

In the case when $S$ is of symmetric type one can get more precise information on the operators $T_{\nu_k}$ by using representation theory.

The group of isometries $G$ of $S$ is a noncompact semisimple Lie group of real rank one. Let $\mathfrak{g}$, $\mathfrak{t}$, $N$, and $A$ be as in Section 4 let $M$ be the centralizer of $A$ in $K$, let $P = MAN$ and let $\mathfrak{p}$ be the Lie algebra of $P$. Extend $\mathfrak{a}$ in the usual way to a Cartan subalgebra $\mathfrak{h}_c = \mathfrak{a}_c + h_\mathfrak{c}^-$ of $\mathfrak{g}$, where $\mathfrak{h}_-^\mathfrak{c}$ is a maximal abelian subalgebra of $\mathfrak{m}$, and introduce compatible orderings in the dual spaces of $\mathfrak{a}$ and $\mathfrak{a} + \sqrt{-1}\mathfrak{h}_-^\mathfrak{c}$. Let $\Sigma^+ (\Delta^+)$ denote the corresponding set of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$ (respectively $(\mathfrak{g}_c, \mathfrak{h}_c)$). Since $\mathfrak{g}$ has real rank one, there is only one real root $\check{\alpha} \in \Delta^+$. It satisfies $\check{\alpha}_{|\mathfrak{h}^-} = 0$ and $\check{\alpha}_{|\mathfrak{a}} = \alpha$.

For $\nu \in \mathbb{C}$, let $(\pi_\nu, H^\nu)$ be the spherical principal series representation of $G$ (see [7], Section 3.6). The zonal spherical function $\phi_\nu$ is given by $\phi_\nu (g) = \langle \pi_\nu (g) 1_\nu, 1_\nu \rangle$, where $1_\nu \in H^\nu$ is such that $1_\nu (nak) = \alpha^{(\nu + p) \alpha}, n \in N, a \in A, k \in K$, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $H^\nu$.

**Theorem 4.1.** Let $S = G/K$ be a noncompact symmetric space of real rank one and let $\nu_k = -\rho - k$ with $k \in \mathbb{N} \cup \{0\}$. Then $\text{Im} (T_{\nu_k})$ is an irreducible $\mathfrak{g}_c$-module of highest weight $k\check{\alpha}$.

**Proof.** By a result of Helgason (see [4], Ch. V, Theorem 4.1), the $K$-spherical finite dimensional representations of $G$ can be characterized as the representations of $\mathfrak{g}_c$ of highest weight $\Lambda \in \mathfrak{h}_c^*$ such that: $\Lambda_{|\mathfrak{h}^-} = 0$ and $\langle \Lambda, \lambda \rangle / \langle \lambda, \lambda \rangle \in \mathbb{Z}_{\geq 0}$, for any $\lambda \in \Sigma^+$. Since in our case $\Sigma^+ = \{ \alpha, \alpha/2 \}$ or $\{ \alpha \}$, this is equivalent to $\Lambda_{|\mathfrak{a}} = k\check{\alpha}$, with $k \in \mathbb{Z}_{\geq 0}$, and $\check{\alpha}$ the real root. We shall denote by $V_{k\check{\alpha}}$ the $\mathfrak{g}_c$-module with highest weight $k\check{\alpha}$.

Our claim is that $1_{\nu_k}$ generates a finite dimensional $(\mathfrak{g}, K)$-submodule $V_{\nu_k}$ of $H^\nu$, isomorphic to $V_{k\check{\alpha}}$.

In the notation of Lemma 3.8.2 in [7], we have that

$$\text{Hom}_{\mathfrak{g}, K} (V_{k\check{\alpha}}, H^\nu) \cong \text{Hom}_{\mathfrak{p}, M} (V_{k\check{\alpha}} / nV_{k\check{\alpha}}, C_{\nu_k}),$$

where $C_{\nu_k}$ denotes the $\text{MAN}$-module $C$, with $M$ acting trivially and $\alpha \in A$ acting by multiplication by $\alpha^{(\nu + p) \alpha}$. To prove our claim it will thus be sufficient to show that there exists a nontrivial $(\mathfrak{p}, M)$-morphism $f : V_{k\check{\alpha}} / nV_{k\check{\alpha}} \to C_{\nu_k}$. We denote by $\Lambda_\alpha$ the lowest weight of $V_{k\alpha}$ and by $\nu_0$ the corresponding lowest weight vector. Then $\Lambda_\alpha = s_\alpha \Lambda$, $s_\alpha$ the long element of the Weyl group of $(\mathfrak{g}_c, \mathfrak{h}_c)$. Since $\Lambda' = -s_\alpha \Lambda$ is the highest weight of the dual representation of $V_{k\check{\alpha}}$, which is also $K$-spherical, $\Lambda'$ satisfies Helgason’s conditions. This implies that $s_\alpha \Lambda_{|\mathfrak{h}^-} = 0$ and $s_\alpha \Lambda_{|\alpha} = -\kappa \alpha$. Arguing as in the proof of Theorem 4.1, Ch. V in [4], one shows that $\pi_{\Lambda} (M) \nu_0 = \nu_0$.

Since $s_\alpha \Lambda_{|\mathfrak{h}^-} = 0$, it follows that $V = C_{\nu_k} \oplus (n + \mathfrak{m}) V$. Now we can define a $(\mathfrak{p}, M)$-morphism $f : V / nV \to C_{\nu}$ such that $f : [v_0] \mapsto 1$, where $[v_0]$ is the class of $v_0$ and $f = 0$ on $nV$. Hence, by (12), there is a nonzero $G$-map of $V_{k\check{\alpha}}$ onto a subspace $V_{\nu_k}$ of $H^\nu$, which must contain $1_{\nu_k}$.

Now we prove the statement in the theorem. If $f \in C_c^\infty (G/K)$, and $x \in G$, we have by (3.1)

$$T_{\nu_k} (f) = p_k f * \tilde{\phi}_{\nu_k} (x) = p_k (\pi (x^{-1}) \pi (f)) 1_{\nu_k},$$

where $p_k = \frac{p(\nu_k)}{\pi_{\nu_k}} \neq 0$, for all $k$ (see the formula of $p(\nu)$ in Section 2).
By irreducibility, as \( f \) varies, \( \pi(f) \) \( V_{\nu} \) fills \( V_{\nu} \simeq V_{\kappa} \). Hence, the image of \( T_{\nu} \) coincides with the image of the \( G \)-morphism \( T_{\kappa} : V_{\nu} \to C^\infty(G/K) \) given by \( T_{\kappa}(v)(x) = \langle \pi_{\nu}(x^{-1})v, 1_{\nu} \rangle \), for \( v \in V_{\nu} \). This proves the theorem.

**Remark 4.2.** We will now use the Weyl dimension formula to calculate the dimension of the \( g \)-module \( V_{\kappa} \) in each case. The real roots \( \alpha \) can be read from the Satake diagram of \( g \). They are listed, for each rank one group, in [5] (for instance).

We shall thus use the notation in [5].

(i) \( g = \mathfrak{so}(n, 1) \) (\( n \) even). In this case, the real root is \( \alpha_1 = \epsilon_1 \), the first fundamental weight. The corresponding \( g \)-module \( V_{\kappa} \) is isomorphic to the representation of \( G \) on \( \mathcal{H}_k \), the space of homogeneous harmonic polynomials of degree \( k \) in \( n + 1 \) variables, which has dimension \( \frac{(k-n-2)!(2k+n-1)}{k(n-1)!} \). This can easily be computed by the Weyl dimension formula.

(ii) \( g = \mathfrak{su}(n, 1) \). Here, the real root is \( \alpha_2 = \epsilon_1 - \epsilon_{n+1} \), and the positive roots are \( \epsilon_i - \epsilon_j \), \( i < j \) and \( 2\rho = \sum_{j=1}^{n+1} (n - 2j + 2)\epsilon_j \).

\[
\dim(V_{\kappa}) = \prod_{1 \leq i < j \leq n+1} \frac{\langle k(\epsilon_1 - \epsilon_{n+1}) + \rho, \epsilon_i - \epsilon_j \rangle}{\langle \rho, \epsilon_i - \epsilon_j \rangle} = \prod_{2 \leq j \leq n} \frac{k + j - 1}{j - 1} \prod_{2 \leq i \leq n} \frac{k + n + 1 - i}{n + 1 - i} \frac{2k + n}{n} = \left( \frac{k+n-1}{k} \right)^2 \frac{2k+n}{n}.
\]

(iii) \( g = \mathfrak{sp}(n, 1) \). In this case, \( \alpha_2 = \epsilon_1 + \epsilon_2 \), and the positive roots are \( \epsilon_i \pm \epsilon_j \), \( 1 \leq i < j \leq n + 1 \), and \( 2\epsilon_i \), \( 1 \leq i \leq n + 1 \). Also, \( \rho = \sum_{j=1}^{n+1} (2n - j)\epsilon_j \).

\[
\dim(V_{\kappa}) = \left( \prod_{i=1}^{n+1} \prod_{j=3}^{n+1} \frac{2n+4-j-i+k}{2n+4-j-i} \frac{j-i+k}{j-i} \right) \frac{2n+1+2k}{2n+1} \frac{n+k}{n} \frac{n+k+1}{n+1} = \left( \frac{2n+k+1}{k} \right)^2 \frac{2n+k}{(2n+1)(2n)} \frac{2n+2k+1}{k+1}.
\]

(iv) \( g = \mathfrak{f}_4 \). The real root is \( \alpha_4 = \lambda_4 \) (\( \epsilon_1 \)), the fourth fundamental weight. The positive roots are \( \epsilon_i, \epsilon_i \pm \epsilon_j, 1 \leq i < j \leq 4 \), \( \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \) and \( 2\rho = 11\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + \epsilon_4 \).

Using the Weyl dimension formula we obtain in this case

\[
\dim(V_{\kappa}) = \frac{2k+11}{11} \prod_{j=1}^{10} \frac{k+j}{j} \prod_{j=4}^{7} \frac{k+j}{j} \frac{k+j}{j}.
\]

### 4.1 The real hyperbolic \( n \)-space.

If \( S \approx H^n \), one can make the results in Theorem 2.2 more precise. In this case one can solve the recurrence in [8], obtaining an explicit series expression for \( Q_\nu \). Indeed, since \( p = 0 \), by [8] we see that \( a_{2j+1} = 0 \) for \( j \geq 0 \); hence \( a_{2j} = a_{2j-2} \frac{(j-1+\rho)(j-1+\rho+1)}{j(j+1)(\nu+j)} \), \( j \geq 0 \). Thus, if we set \( c_j := a_{2j} \), for \( j \geq 0 \), and if \( c_0 := 1 \), we obtain for \( j \geq 1 \)

\[
c_j(\nu) = \frac{(\rho)_j (\nu + \rho)_j}{j! (\nu + 1)_j}.
\]
Furthermore, \( c(\nu) = \frac{2^{2n-1}\Gamma(n/2)}{\pi^{1/2}} \frac{\Gamma(\nu)}{\Gamma(\nu + \rho)} \) hence, using (13) and the duplication formula for the Gamma function, we obtain

\[
(14) \quad Q_{\nu}(r) = \frac{2^{-2n+3}}{\nu c(\nu)} \frac{e^{-(\nu + \rho)r}}{(n-2)!} \sum_{j=0}^{\infty} \frac{\Gamma(\nu + j)}{j! \Gamma(\nu + j + 1)} e^{-2jr},
\]

Now, if \( S_{b, \delta} \) is as in Theorem 2.2 one sees, by using Stirling’s estimates, that there exists a constant \( K = K(b, \delta) \) such that the coefficients in (14) are bounded by \( K \rho^{-1} \nu + j \rho^{-1} \), uniformly for \( \nu \) in \( S_{b, \delta} \). This gives an alternative proof of the convergence, as stated in Theorem 2.2 (a).

Regarding the poles, we see that if \( n \) is odd, since \( \rho = \frac{n-1}{2} \in \mathbb{N} \), the coefficients in (14) are polynomial functions in \( \nu \); hence \( \tilde{R}(\lambda(\nu)) \) is everywhere holomorphic in this case.

If \( n \) is even, (14) implies that the kernel is meromorphic with poles at \( \nu_k = -\rho - k, \ k \in \mathbb{N} \cup \{0\} \). Since \( \Gamma(\nu + \rho + j) \) is holomorphic at \( \nu = \nu_k \) for \( j > k \), we get

\[
\text{Res}_{\nu=\nu_k} \frac{Q_{\nu}(r)}{\nu c(\nu)} = \frac{2^{-2n+3}}{(n-2)!} \sum_{j=0}^{k} \frac{\Gamma(\nu + j)(-1)^{k-j}}{j!(k-j)! \Gamma(-\rho + j + 1)} e^{-2jr},
\]

since \( \frac{\Gamma(\nu+j)(-1)^k}{\Gamma(-\rho - k + 1)} = \frac{\Gamma(\rho + k - j)}{\Gamma(-\rho - j + 1)} \), for \( 0 \leq j \leq k \).

**Remark 4.3.** We note that in [3], Section 2, Guillopé-Zworski consider the resolvent kernel for the real hyperbolic \( n \)-space, giving the location of the poles and showing that the residues define operators of finite rank.

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