PARTITIONS WITH PARTS IN A FINITE SET

MELVYN B. NATHANSON

(Communicated by David E. Rohrlich)

Abstract. Let \( A \) be a nonempty finite set of relatively prime positive integers, and let \( p_A(n) \) denote the number of partitions of \( n \) with parts in \( A \). An elementary arithmetic argument is used to prove the asymptotic formula

\[
p_A(n) = \left( \frac{1}{\prod_{a \in A} a} \right) \frac{n^{k-1}}{(k-1)!} + O\left(n^{k-2}\right).
\]

Let \( A \) be a nonempty set of positive integers. A partition of a positive integer \( n \) with parts in \( A \) is a representation of \( n \) as a sum of not necessarily distinct elements of \( A \). Two partitions are considered the same if they differ only in the order of their summands. The partition function of the set \( A \), denoted \( p_A(n) \), counts the number of partitions of \( n \) with parts in \( A \).

If \( A \) is a finite set of positive integers with no common factor greater than 1, then every sufficiently large integer can be written as a sum of elements of \( A \) (see Nathanson \[3\] and Han, Kirfel, and Nathanson \[2\]), and so \( p_A(n) \geq 1 \) for all \( n \geq n_0 \).

In the special case that \( A \) is the set of the first \( k \) integers, it is known that

\[
p_A(n) \sim \frac{n^{k-1}}{k!(k-1)!}.
\]

Erdős and Lehner \[1\] proved that this asymptotic formula holds uniformly for \( k = o(n^{1/3}) \). If \( A \) is an arbitrary finite set of relatively prime positive integers, then

\[
p_A(n) \sim \left( \frac{1}{\prod_{a \in A} a} \right) \frac{n^{k-1}}{(k-1)!}.
\]

The usual proof of this result (Netto \[4\], Pólya–Szegö \[5\] Problem 27) is based on the partial fraction decomposition of the generating function for \( p_A(n) \). The purpose of this note is to give a simple, purely arithmetic proof of \( (1) \).

We define \( p_A(0) = 1 \).

Theorem 1. Let \( A = \{a_1, \ldots, a_k\} \) be a set of \( k \) relatively prime positive integers, that is,

\[
gcd(A) = (a_1, \ldots, a_k) = 1.
\]
Let $p_A(n)$ denote the number of partitions of $n$ into parts belonging to $A$. Then

$$p_A(n) = \left( \frac{1}{\prod_{a \in A} a} \right) n^{k-1} \frac{n^{k-2}}{(k-1)!} + O(n^{k-2}).$$

Proof. Let $k = |A|$. The proof is by induction on $k$. If $k = 1$, then $A = \{1\}$ and

$$p_A(n) = 1,$$

since every positive integer has a unique partition into a sum of 1’s.

Let $k \geq 2$, and assume that the theorem holds for $k - 1$. Let 

$$d = (a_1, \ldots, a_{k-1}).$$

Then

$$(d, a_k) = 1.$$

For $i = 1, \ldots, k - 1$, we set

$$a_i' = \frac{a_i}{d}.$$

Then

$$A' = \{a_1', \ldots, a_{k-1}'\}$$

is a set of $k - 1$ relatively prime positive integers, that is,

$$\gcd(A') = 1.$$

Since the induction assumption holds for $A'$, we have

$$p_{A'}(n) = \left( \frac{1}{\prod_{i=1}^{k-1} a_i'} \right) n^{k-2} \frac{n^{k-3}}{(k-2)!} + O(n^{k-3})$$

for all nonnegative integers $n$.

Let $n \geq (d - 1)a_k$. Since $(d, a_k) = 1$, there exists a unique integer $u$ such that

$$0 \leq u \leq d - 1$$

and

$$n \equiv ua_k \pmod{d}.$$

Then

$$m = \frac{n - ua_k}{d}$$

is a nonnegative integer, and

$$m = O(n).$$

If $v$ is any nonnegative integer such that

$$n \equiv va_k \pmod{d},$$

then

$$va_k \equiv ua_k \pmod{d}.$$
and so \( v \equiv u \pmod{d} \), that is, \( v = u + \ell d \) for some nonnegative integer \( \ell \). If
\[
 n - va_k = n - (u + \ell d)a_k \geq 0,
\]
then
\[
0 \leq \ell \leq \left\lfloor \frac{n}{da_k} - \frac{u}{d} \right\rfloor = \left\lfloor \frac{m}{a_k} \right\rfloor = r.
\]
We note that
\[
r = O(n).
\]
Let \( \pi \) be a partition of \( n \) into parts belonging to \( A \). If \( \pi \) contains exactly \( v \) parts equal to \( a_k \), then \( n - va_k \geq 0 \) and \( n - va_k \equiv 0 \pmod{d} \), since \( n - va_k \) is a sum of elements in \( \{a_1, \ldots, a_{k-1}\} \), and each of the elements in this set is divisible by \( d \). Therefore, \( v = u + \ell d \), where \( 0 \leq \ell \leq r \). Consequently, we can divide the partitions of \( n \) with parts in \( A \) into \( r + 1 \) classes, where, for each \( \ell = 0, 1, \ldots, r \), a partition belongs to class \( \ell \) if it contains exactly \( u + \ell d \) parts equal to \( a_k \). The number of partitions of \( n \) with exactly \( u + \ell d \) parts equal to \( a_k \) is exactly the number of partitions of \( n - \ell d a_k \) into parts belonging to the set \( \{a_1, \ldots, a_{k-1}\} \), or, equivalently, the number of partitions of
\[
\frac{n - (u + \ell d) a_k}{d}
\]
into parts belonging to \( A' \), which is exactly
\[
p_{A'} \left( \frac{n - (u + \ell d) a_k}{d} \right) = p_{A'} \left( m - \ell a_k \right).
\]
Therefore,
\[
p_{A}(n) = \sum_{\ell=0}^{r} p_{A'}(m - \ell a_k)
\]
\[
= \left( \frac{1}{\prod_{i=1}^{k-1} a_i^i} \right) \sum_{\ell=0}^{r} \left( \frac{(m - \ell a_k)^{k-2}}{(k-2)!} + O(m^{k-3}) \right)
\]
\[
= \left( \frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \sum_{\ell=0}^{r} \left( \frac{(m - \ell a_k)^{k-2}}{(k-2)!} + O(n^{k-2}) \right).
\]
To evaluate the inner sum, we note that
\[
\sum_{\ell=0}^{r} \ell^j = \frac{r^{j+1}}{(j+1)} + O(r^j)
\]
and
\[
\sum_{j=0}^{k-2} (-1)^j \binom{k-1}{j+1} = -\sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} = 1.
\]
Then
\[
\sum_{\ell=0}^{r} \frac{(m - \ell a_k)^{k-2}}{(k-2)!} = \frac{1}{(k-2)!} \sum_{\ell=0}^{r} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-\ell a_k)^j \sum_{\ell=0}^{r} \ell^j
\]
\[
= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \sum_{\ell=0}^{r} \ell^j
\]
\[
= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \left( \frac{r^{j+1}}{(j+1)} + O(r^j) \right)
\]
\[
= \frac{1}{(k-2)!} \sum_{j=0}^{k-2} \binom{k-2}{j} m^{k-2-j} (-a_k)^j \left( \frac{m^{j+1}}{a_k^{j+1}(j+1)} + O(m^j) \right)
\]
\[
= \frac{m^{k-1}}{a_k} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(k-2)! (j+1)} + O(m^{k-2})
\]
\[
= \frac{m^{k-1}}{a_k} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(k-2-j)! (j+1)!} + O(m^{k-2})
\]
\[
= \frac{m^{k-1}}{a_k} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^j}{(k-1-(j+1))! (j+1)!} + O(m^{k-2})
\]
\[
= \frac{m^{k-1}}{a_k (k-1)!} \sum_{j=0}^{k-2} (-1)^j \binom{k-1}{j+1} + O(m^{k-2})
\]
\[
= \frac{m^{k-1}}{a_k (k-1)!} + O(m^{k-2}).
\]

Therefore,
\[
p_A(n) = \left( \frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \sum_{\ell=0}^{r} \frac{(m - \ell a_k)^{k-2}}{(k-2)!} + O(n^{k-2})
\]
\[
= \left( \frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \left( \frac{m^{k-1}}{a_k (k-1)!} + O(n^{k-2}) \right) + O(n^{k-2})
\]
\[
= \left( \frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \left( \frac{1}{a_k (k-1)!} \right) \left( \frac{n}{d} - \frac{a_k}{d} \right)^{k-1} + O(n^{k-2})
\]
\[
= \left( \frac{d^{k-1}}{\prod_{i=1}^{k-1} a_i} \right) \left( \frac{1}{a_k (k-1)!} \right) \left( \frac{n}{d} \right)^{k-1} + O(n^{k-2})
\]
\[
= \left( \frac{1}{\prod_{i=1}^{k} a_i} \right) \frac{n^{k-1}}{(k-1)!} + O(n^{k-2}).
\]

This completes the proof.
References


Department of Mathematics, Lehman College (CUNY), Bronx, New York 10468

E-mail address: nathansn@alpha.lehman.cuny.edu

Current address: School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

E-mail address: nathansn@ias.edu