PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 5, Pages 1313–1317
S 0002-9939(00)05624-0
Article electronically published on February 7, 2000

DECIDABILITY OF THE REPRESENTATION EXTENSION PROPERTY FOR FINITE SEMIGROUPS

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(Communicated by Ronald M. Solomon)

Abstract. We prove that the decision problem of whether or not a finite semigroup has the representation extension property is decidable.

1. The main theorem and preliminaries

It is an immediate consequence of the normal form theorem for amalgamated free products of groups that every amalgam of groups embeds in some group. However, this result fails for semigroup amalgams: an early result of Kimura [6] shows that amalgams of semigroups do not necessarily embed in any semigroup (see also [4], Vol. II, page 138). More recently, Sapir [7] has shown that it is in fact undecidable whether an amalgam of (finite) semigroups embeds in any (finite) semigroup. A semigroup S is called an *amalgamation base* for semigroups if every amalgam of semigroups containing S as a subsemigroup embeds in some semigroup. It is natural to ask if it is decidable whether or not a finite semigroup is an amalgamation base.

According to [3], we say that a semigroup S has the *representation extension property* if for any right S-set X_S and any left S-set ${}_SM$ containing S^1 as a left S-subset, the canonical map: $X \to X \otimes_S M$ $(x \longmapsto x \otimes 1)$ is injective. Hall [5] proved that any semigroup which is an amalgamation base in the class of all semigroups has the representation extension property. In this paper we prove

The Main Theorem. *It is decidable whether or not a finite semigroup has the representation extension property.*

Let S be a semigroup. Let M be a nonempty set with a unitary and associative operation of $S : S^1 \times M \longrightarrow M((s,w) \longmapsto sw)$, where S^1 is the monoid obtained from S by adjoining a new identity 1. Then M is called a *left S-set*. Dually, a *right* S-set is defined. If a left S-set [resp. right S-set] M contains elements $m_1, \cdots, m_n$ such that $M = S^1m_1 \cup \cdots \cup S^1m_n$ [resp. $M = m_1S^1 \cup \cdots \cup m_nS^1$], then we say that $m_1, \cdots, m_n$ are *generators* of M.

A relation ρ on a left [resp. right] S-set M is called an *S-congruence* if $(m, m') \in \rho$ and $s \in S$ implies $(sm, sm') \in \rho$ [resp. $(ms, m's) \in \rho$]. Let M, N be left [resp.

Received by the editors July 1, 1998.
1991 *Mathematics Subject Classification.* Primary 20M10.
Key words and phrases. Finite semigroup, representation extension property, algorithm, decidability.
This work was done while the author was visiting the University of Nebraska, Lincoln. He thanks the Department of Mathematics, UNL for their kind hospitality.

right] S-sets. Then a map $\phi : M \longrightarrow N$ is called an S-map if $\phi(sm) = s\phi(m)$ [resp. $\phi(ms) = \phi(m)s$] for any $m \in M$ and $s \in S$.

Result ([4, Proposition 1.5])**.** Let S be a semigroup and A, B, C, D left [resp. right] S-sets such that C is a left [resp. right] S-subset of A and D is a left [resp. right] S-subset of B. Let α be a bijective S-map: $C \longrightarrow D$. Then there exist a left [resp. right] S-set W and injective S-maps $\beta : A \longrightarrow W$, $\lambda : B \longrightarrow W$ such that $\alpha\lambda = \beta$ on C, $W = A\beta \cup B\lambda$, $A\beta \cap B\lambda = C\beta$.

In this case, we say that the left [resp. right] S-set W is the left [resp. right] S-set *obtained by gluing A and B with* α and write $W = A\#_\alpha B$.

If C, D are generated by x and y respectively and $\alpha(x) = y$, then we write $A \#_{x=y} B$ instead of $A\#_\alpha B$.

Let X be a right S-set. Consider a set Σ of equations $x_i t_i = x_{i+1} s_{i+1}$ $(1 \le i \le n-1)$ on X, where $t_i, s_{i+1} \in S$, $x_i \in X$. Then for any $1 \le i \le n$, we define the S-congruence ρ_i on the right S-set S^1 as follows:

$(s,t) \in \rho_i$ if and only if $x_i s = x_i t$ in X for all $s, t \in S^1$.

Let $\overline{X_i} = S^1/\rho_i$ and $\overline{x_i} = 1\rho_i$. Then we can obtain a right S-set $\overline{X}$ such that

$$\overline{X} = \overline{X}_1 \#_{\overline{x}_1 t_1 = \overline{x}_2 s_2} \overline{X}_2 \cdots \# \overline{X}_{i-1} \#_{\overline{x}_i t_i = \overline{x}_{i+1} s_{i+1}} \overline{X}_{i+1} \# \cdots \#_{\overline{x}_{n-1} t_{n-1} = \overline{x}_n s_n} \overline{X}_n.$$

Then we have the set of equations $\overline{x}_i t_i = \overline{x}_{i+1} s_{i+1}$ $(1 \le i \le n-1)$ in $\overline{X}$. We call $\overline{X}$ the *relatively free* right S-set associated to X with respect to Σ.

2. Decision problem for the representation extension property

Let S be a finite semigroup and let $\mathcal{T}(S^1)$ denote the set of all mappings of S^1. Then we define an operation of S on $\mathcal{T}(S^1)$: tf is defined by $(s)tf = (st)f$ for all $s, t \in S^1$ and $f \in \mathcal{T}(S^1)$. Then $\mathcal{T}(S^1)$ is a left S-set.

Next let $\rho : S^1 \to \mathcal{T}(S^1)$ denote the right regular representation of S^1. Then by ρ we regard the left S-set S^1 is a left S-subset of the left S-set $\mathcal{T}(S^1)$. By [1, Theorem 6], the left S-set $\mathcal{T}(S^1)$ is injective. As a consequence, we have (see Remark after Corollary 2.3 of [8])

Lemma. *Let S be a finite semigroup and $\mathcal{T}(S^1)$ as above. Then S has the representation extension property if and only if for any right S-set X_S, the canonical map: $X \to X \otimes_S \mathcal{T}(S^1)$ $(x \longmapsto x \otimes 1)$ is injective.*

Proof. This follows from Theorem 2.1 of [8] by substituting $\mathcal{T}(S^1)$ for the left S-set W. □

Then we shall prove that for any finite semigroup S, there is a positive integer k dependinig only on the order $|S|$ of S such that if there exist a right S-set X and a scheme of length greater than k which shows $x \otimes 1 = x' \otimes 1$ in $X \otimes_S \mathcal{T}(S^1)$ but $x \ne x'$ in X, then there exist a right S-set Y with fewer generators than k and $y, y' \in Y$ such that $y \otimes 1 = y' \otimes 1$ in $Y \otimes_S \mathcal{T}(S^1)$ but $y \ne y'$ in Y (see [2] for undefined terms).

Proof of the main theorem. Let S be a finite semigroup. By the lemma, suppose that there exist a right S-set X and $x, x' \in X$ such that $x \ne x'$ in X and $x \otimes 1 = x' \otimes 1$ in $X \otimes_S \mathcal{T}(S^1)$. Then by Lemma 1.2 of [2], there exist $x_1, \cdots, x_n \in X$,

$y_2, \cdots, y_n \in \mathcal{T}(S^1)$, $s_1, \cdots, s_n$, $t_1, \ldots, t_n \in S$ such that

$$\begin{array}{rclcrcl} x & = & x_1 s_1, & & s_1 & = & t_1 y_2 \\ x_1 t_1 & = & x_2 s_2, & & s_2 y_2 & = & t_2 y_3 \\ & \vdots & & & & \vdots & \\ x_{n-1} t_{n-1} & = & x_n s_n, & & s_n y_n & = & t_n \\ x_n t_n & = & x'. & & & & \end{array} \tag{1}$$

Then we can assume that $X = \overline{X}$, the relatively free right S-set of the right S-set associated to X with respect to the set of equations (1). Thus we have

$$\begin{aligned} X = X_1 \#_{x_1 t_1 = x_2 s_2} X_2 \# \cdots \# X_i \#_{x_i t_i = x_{i+1} s_{i+1}} X_{i+1} \\ \# \cdots \# X_{n-1} \#_{x_{n-1} t_{n-1} = x_n s_n} X_n. \end{aligned} \tag{2}$$

Next we let $C_r(S)$ be the set of S-congruences on the right S-set S^1. In particular, let ρ_i denote the S-congruence on the right S-set S^1 given by

$$(s,t) \in \rho_i \ \text{ if and only if } \ x_i s = x_i t \ \text{ for all } \ s, t \in S.$$

Let $m = |S \times C_r(S) \times S \times C_r(S) \times \mathcal{T}(S^1)|$ and $k = 2m$. If $n > k$, then there exist $1 \le i_1 < i_2 < i_3 \le n$ such that

$$\begin{aligned} (t_{i_1}, \rho_{i_1}, s_{i_1+1}, \rho_{i_1+1}, y_{i_1+1}) &= (t_{i_2}, \rho_{i_2}, s_{i_2+1}, \rho_{i_2+1}, y_{i_2+1}) \\ &= (t_{i_3}, \rho_{i_3}, s_{i_3+1}, \rho_{i_3+1}, y_{i_3+1}), \end{aligned}$$

where $y_{n+1} = 1$.

By the construction of X, we know that X_i is S-isomorphic to S^1/ρ_i as a right S-set with an S-map mapping x_i on $1\rho_i$. Then, for $(p,q) = (1,2), (1,3), (2,3)$, there are bijective S-maps:

$$x_{i_p} t_{i_p} S \to x_{i_q} t_{i_q} S \ \ (x_{i_p} t_{i_p} \longmapsto x_{i_q} t_{i_q}), \tag{3}$$

$$x_{i_p+1} s_{i_p+1} S \to x_{i_q+1} s_{i_q+1} S \ \ (x_{i_p+1} s_{i_p+1} \longmapsto x_{i_q+1} s_{i_q+1}) \tag{4}$$

and

$$x_{i_p} t_{i_p} S \to x_{i_q+1} s_{i_q+1} S \ \ (x_{i_p} t_{i_p} \longmapsto x_{i_q+1} s_{i_q+1}). \tag{5}$$

By (5), for $(p,q) = (1,2), (1,3), (2,3)$, we obtain three right S-sets

$$\begin{aligned} X_{i_p, i_q} = X_1 \#_{\overline{x}_1 t_1 = x_2 s_2} \overline{X}_2 \# \cdots \# X_{i_{p-1}} \#_{x_{i_p} t_{i_p} = x_{i_q+1} s_{i_q+1}} X_{i_q+1} \\ \# \cdots \#_{x_{n-1} t_{n-1} = x_n s_n} X_n. \end{aligned}$$

Then it follows from (1) and

$$(t_{i_p}, \rho_{i_p}, s_{i_p+1}, \rho_{i_p+1}, y_{i_p+1}) = (t_{i_q}, \rho_{i_q}, s_{i_q+1}, \rho_{i_q+1}, y_{i_q+1})$$

that $x \otimes 1 = x' \otimes 1$ in $X_{i_p, i_q} \otimes_S \mathcal{T}(S^1)$ for all $1 \le p < q \le 3$. On the other hand, there must exist at least one pair (p,q) such that $x \ne x'$ in X_{i_p, i_q}. Suppose, on the contrary, that $x = x'$ holds in X_{i_1, i_2}, X_{i_2, i_3} and X_{i_1, i_3}. Then it follows from the first equation $x = x'$ in X_{i_1, i_2} and the way of construction of X_{i_1, i_1} that

$$x \in \bigcap_{i=1}^{i_1} x_i t_i S, \quad x' \in \bigcap_{j=i_2+1}^{n} x_j s_j S$$

and there exists $u \in S$ such that

$$x = x_{i_1} t_{i_1} u \ \text{ in } \ X_1 \#_{x_1 t_1 = x_2 s_2} X_2 \# \cdots \# X_{i_1}$$

and

$$x' = x_{i_2+1}s_{i_2+1}u \ \text{ in } \ X_{i_2+1}\#_{x_{i_2+1}t_{i_2+1}=x_{i_2+2}s_{i_2+2}}X_{i_2+2}\#\cdots\#X_n.$$

Similarly, it follows from the other equations in X_{i_2,i_2} or X_{i_3,i_3} that there exist $v, w \in S$ such that

$$x = x_{i_2}t_{i_2}v \ \text{ in } \ X_1\#_{x_1t_1=x_2s_2}X_2\#\cdots\#X_{i_2},$$

$$x' = x_{i_3+1}s_{i_3+1}v \ \text{ in } \ X_{i_3+1}\#_{x_{i_3+1}t_{i_3+1}=x_{i_3+2}s_{i_3+2}}X_{i_3+2}\#\cdots\#X_n$$

and

$$x = x_{i_1}t_{i_1}w \ \text{ in } \ X_1\#_{x_1t_1=x_2s_2}X_2\#\cdots\#X_{i_1},$$

$$x' = x_{i_3+1}s_{i_3+1}w \ \text{ in } \ X_{i_3+1}\#_{x_{i_3+1}t_{i_3+1}=x_{i_3+2}s_{i_3+2}}X_{i_3+2}\#\cdots\#X_n.$$

Of course, it can be assumed that

$$X_1\#_{x_1t_1=x_2s_2}X_2\#\cdots\#X_{i_1} \subseteq X_1\#_{x_1t_1=x_2s_2}X_2\#\cdots\#X_{i_2}$$

and

$$\begin{aligned} &X_{i_3+1}\#_{x_{i_3+1}t_{i_3+1}=x_{i_3+2}s_{i_3+2}}X_{i_3+2}\#\cdots\#X_n \\ &\qquad \subseteq X_{i_2+1}\#_{x_{i_2+1}t_{i_2+1}=x_{i_2+2}s_{i_2+2}}X_{i_2+2}\#\cdots\#X_n. \end{aligned}$$

Consequently, we have

$$x = x_{i_1}t_{i_1}u = x_{i_2}t_{i_2}v = x_{i_1}t_{i_1}w \ \text{ in } \ X_1\#_{x_1t_1=x_2s_2}X_2\#\cdots\#X_{i_2}$$

and

$$\begin{aligned} x' = x_{i_2+1}s_{i_2+1}u = x_{i_3+1}s_{i_3+1}v = x_{i_3+1}s_{i_3+1}w \\ \text{in } \ X_{i_2+1}\#_{x_{i_2+1}t_{i_2+1}=x_{i_2+2}s_{i_2+2}}X_{i_2+2}\#\cdots\#X_n. \end{aligned}$$

Since $x_{i_1}t_{i_1}u = x_{i_1}t_{i_1}w$, by (3) we have $x_{i_2}t_{i_2}u = x_{i_2}t_{i_2}w$. Also, by (5) we have $x_{i_2+1}s_{i_2+1}u = x_{i_2+1}s_{i_2+1}w$. Since $x_{i_3+1}s_{i_3+1}v = x_{i_3+1}s_{i_3+1}w$, by (4) we have $x_{i_2+1}s_{i_2+1}v = x_{i_2+1}s_{i_2+1}w$. Therefore $x' = x_{i_2+1}s_{i_2+1}u = x_{i_2+1}s_{i_2+1}v = x_{i_2+1}s_{i_2+1}w$ in $X_{i_2+1}\#_{x_{i_2+1}t_{i_2+1}=x_{i_2+2}s_{i_2+2}}X_{i_2+2}\#\cdots\#X_n$, which implies $x = x_{i_2}t_{i_2}u = x_{i_2}t_{i_2}v = x_{i_2}t_{i_2}w$ in $X_1\#_{x_1t_1=x_2s_2}X_2\#\cdots\#X_{i_2}$, because of (4) and (5). Whence, by (2) $x = x_{i_2}t_{i_2}u = x_{i_2+1}s_{i_2+1}u = x'$ in X. This is a contradiction. Consequently, in order to decide whether or not the lemma above holds for the finite semigroup S, it suffices to check whether or not the canonical map: $X \to X \otimes_S \mathcal{T}(S^1)$ is injective for all right S-sets X with generators of its number equal to or less than k. Note that there exist only finitely many right S-sets with generators of its number equal to or less than k, up to S-isomorphism. Hence the main theorem follows. □

As far as we know, the following problems remain open.

Problem. Is it decidable whether or not a finite semigroup has any one of the following properties?

(1) being an amalgamation base for semigroups.

(2) being an amalgamation base for finite semigroups.

Acknowlegdments

I thank John Meakin for his generous hospitality and stimulating conversation without which this work would not have been realized. I thank Tom Hall for reading the manuscript and correcting it. Finally, I thank the referee for good advice.

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