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STABILITY OF n-VORTICES IN THE GINZBURG-LANDAU EQUATION

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ABSTRACT. We consider the class of n-vortex solutions to the time-independent Ginzburg-Landau equation on $\mathbf{R}^2$. We prove an inequality governing the solutions of a particular boundary value problem. This inequality is crucial for an elementary proof by Ovchinnikov and Sigal that such n-vortices are unstable in the case $|n| \geq 2$.

We consider here the time-independent Ginzburg-Landau equation

$$\Delta\varphi + (1 - |\varphi|^2)\varphi = 0, \tag{1}$$

where $\varphi : \mathbf{R}^2 \to \mathbf{C}$ satisfies the boundary condition $|\varphi(r, \theta)| \to 1$ as $r \to \infty$. The simplest family of solutions of (1) are the n-vortices [1] where n is an arbitrary integer. They are defined by $\varphi_n(r, \theta) = e^{in\theta} f_n(r)$, where $f_n : [0, \infty) \to \mathbf{R}$ satisfies

$$f_n''(r) + \frac{1}{r} f_n'(r) + \left(1 - \frac{n^2}{r^2}\right) f_n(r) - f_n^3(r) = 0, \quad f_n(0) = 0, \quad f_n(\infty) = 1. \tag{2}$$

The existence and uniqueness of a solution for (2) are shown in [2, 4].

Of particular interest is the question of the stability of n-vortices under small perturbations, originally studied by Hagan [3]. Later, it was shown in [7] using variational methods that n-vortices are unstable for $|n| \geq 2$. An alternative, elementary proof was proposed by Ovchinnikov and Sigal [5] and makes use of an inequality whose validity was only checked experimentally. In this note, we present a simple proof of this inequality, as follows.

Proposition. *Let f_n be the solution of (2) with $|n| \geq 2$. Then $f_n \leq f_n^{\max}$, where*

$$f_n^{\max}(r) = \frac{r^2}{r^2 + \frac{n^2}{2}}, \qquad r \in [0, \infty). \tag{3}$$

For the proof, we recall the following comparison result for ODE's [6].

Lemma. *Let (a, b) be an interval in $\mathbf{R}$, let $\Omega = \mathbf{R}^2 \times (a, b)$, and let $H \in C^1(\Omega, \mathbf{R})$. Suppose $f \in C^2(a, b)$ satisfies $f'' + H(f, f', r) = 0$. If $H_f \leq 0$ on Ω and if there exist functions $M, m \in C^2(a, b)$ satisfying $M''(r) + H(M(r), M'(r), r) \leq 0$ and $m''(r) + H(m(r), m'(r), r) \geq 0$, as well as the boundary conditions $m(a) \leq f(a) \leq M(a)$ and $m(b) \leq f(b) \leq M(b)$, then for all $r \in (a, b)$ we have $m(r) \leq f(r) \leq M(r)$.* □

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Proof of the Proposition. We apply the lemma to our problem, taking $a = 0$, $b = \infty$, $M = f_n^{\max}$, $m = f_n$, and

$$H(f_n, f_n', r) = \frac{1}{r} f_n' + \left(1 - \frac{n^2}{r^2}\right) f_n - f_n^3. \tag{4}$$

We easily check that all the conditions of the lemma are satisfied, except possibly $H_{f_n} \leq 0$. Now,

$$H_{f_n}(f_n, f_n', r) = \left(1 - \frac{n^2}{r^2}\right) - 3f_n^2, \tag{5}$$

and so $H_{f_n} \leq 0$ as long as f_n lies above the curve g_n given by

$$g_n(r) = \begin{cases} 0, & 0 \leq r \leq n; \\ \sqrt{\frac{1}{3}\left(1 - \frac{n^2}{r^2}\right)}, & r \geq n. \end{cases} \tag{6}$$

Let $U = \{r \in (0, \infty),\ f_n(r) > g_n(r)\}$. If $U = (0, \infty)$, then the theorem follows immediately. Otherwise, since U is open it consists of countably many disjoint open intervals. If (c, d) is any such interval, then applying the lemma to (c, d) we have $f_n \leq f_n^{\max}$ on all of (c, d). Hence the inequality holds on U. But it is easily checked that $g_n(r) \leq f_n^{\max}(r)$ for all $r \in (0, \infty)$, and so $f_n \leq f_n^{\max}$ for all $r \in [0, \infty)$. $\square$

With this inequality in place, the proof of instability of n-vortices for $|n| \geq 2$ proceeds along the lines given in [5]. Begin by introducing the renormalized energy functional $\mathcal{E}$, defined by

$$\mathcal{E}(\varphi) = \frac{1}{2} \int_{\mathbf{R}^2} \left(|\nabla \varphi(x)|^2 - \frac{\deg(\varphi)^2}{|x|^2} \chi(x) + \frac{1}{2}(|\varphi(x)|^2 - 1)^2 \right) d^2x, \tag{7}$$

where χ is a smooth weight function inserted to make the integral finite, and $\deg(\varphi)$ is the degree of the function φ (in particular, $\deg(\varphi_n) = n$). We will work with the space X defined by $X = \{\varphi \in H^1(\mathbf{R}^2),\ \mathcal{E}(\varphi) < \infty\}$. The tangent space $T_\varphi X$ to X at the point φ will be identified with the Sobolev space $H^1(\mathbf{R}^2)$.

A function φ is a solution of (1) if and only if it is a critical point of the energy functional (7). Our goal is to determine whether a given solution is an extremum or a saddle point of this functional. This can be done by considering the usual first and second variations [8] $d\mathcal{E}_\varphi : T_\varphi X \to \mathbf{R}$ and $d^2\mathcal{E}_\varphi : T_\varphi X \times T_\varphi X \to \mathbf{R}$. To prove the instability of a solution φ it suffices to find a $u \in T_\varphi X$ such that $d^2\mathcal{E}_\varphi(u, u) < 0$.

Now, we claim that $u(r, \theta) = e^{-r^2/4}$ satisfies this requirement as long as $|n| \geq 2$. Using the fact that $f_n \leq f_n^{\max}$, we have

$$\begin{aligned} d^2\mathcal{E}_{\varphi_n}(u, u) &= \operatorname{Re} \int_{\mathbf{R}^2} \bar{u} \left(-\Delta u + (2|\varphi_n|^2 - 1)u + \varphi_n^2 \bar{u} \right) \\ &= \operatorname{Re} \int_0^{2\pi} \int_0^\infty e^{-r^2/4} \left(-\Delta \left(e^{-r^2/4} \right) + (2f_n^2 - 1)e^{-r^2/4} + f_n^2 e^{2in\theta} e^{-r^2/4} \right) r\, dr\, d\theta \\ &\leq -\frac{\pi}{2} \int_0^\infty \frac{r^3 \left[(2r^2 + n^2 - 8)^2 + 16n^2 - 64 \right] e^{-r^2/2}}{(2r^2 + n^2)^2} dr. \end{aligned}$$

For $|n| \geq 2$ this expression is clearly negative, and the result follows immediately.

References

[1] F. Bethuel, H. Brézis and F. Hélein, Ginzburg-Landau Vortices, Birkhauser, 1994. MR **95c:**58044

[2] X. Chen, C.M. Elliott and T. Qi, Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation, Proc. Roy. Soc. Edinburgh Sect. A. 124 (1994), no. 6, 1075-1088. MR **95j:**35208

[3] P. Hagan, Spiral waves in reaction-diffusion equations, SIAM J. of Appl. Math. 42 (1982), no. 4, 762-786. MR **84c:**92069

[4] R.M. Hervé and M. Hervé, Etude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg-Landau, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (1994), no. 4, 427-440. MR **95g:**35194

[5] Y.M. Ovchinnikov and I.M. Sigal, Ginzburg-Landau equation I. Static vortices, Partial Differential Equations and their Applications, 199-220, CRM Publications, 1997. MR **98k:**35179

[6] M.H. Protter and H.F. Weinberger, Maximum Principles in Differential Equations, Springer, 1984. MR **86f:**35034

[7] I. Shafrir, Remarks on solutions of $-\Delta u = (1 - |u|^2)u$ in $\mathbf{R}^2$, C.R. Acad. Sci. Paris Sér I Math. 318 (1994), no. 4, 327-331. MR **95c:**35091

[8] M. Struwe, Variational Methods, Springer, 1990. MR **92b:**49002

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