FACTORING WEAKLY COMPACT OPERATORS
AND THE INHOMOGENEOUS CAUCHY PROBLEM

DIÓMEDES BÁRCENAS

(Communicated by David R. Larson)

ABSTRACT. By using the technique of factoring weakly compact operators through reflexive Banach spaces we prove that a class of ordinary differential equations with Lipschitz continuous perturbations has a strong solution when the problem is governed by a closed linear operator generating a strongly continuous semigroup of compact operators.

1. Introduction

Consider a Banach space $X$ and the abstract Cauchy problem

$$\begin{cases}
\dot{x}(t) = Ax + f(t, x), \\
x(t_0) = x_0 \in D(A)
\end{cases} \quad (1)$$

where $0 \leq t_0 < T < \infty$ and $A$ generates a strongly continuous semigroup $\{T_t\}_{t \geq 0}$. It is known that the problem (1) does not have to have any solution on $[t_0, T]$ as can be seen by considering a variation of an example given in [4], Chapter X, exercise 5, section X.5, if $X = c_0$, $f(t, x) = y$ where $y_n = \sqrt{|x_n|}$ and $A = 0$.

In [6] it is proved that if $f$ is Lipschitz continuous in both variables, then (1) has always a mild solution; but according to Webb [7], this mild solution does not need to be a strong solution.

The strongness of a mild solution of (1) is obtained by Pazy [6], p. 189, according to the following hypothesis:

If $f : [0, T] \times X \to X$ is Lipschitz continuous in both variables and $X$ is a reflexive Banach space, then a mild solution of (1) is a strong solution.

In this paper we use the factorization scheme announced in the abstract in order to show that the same conclusion holds in non-reflexive Banach spaces when some extra hypotheses are imposed on either the operator $A$ or the perturbation $f$.
2. The results

We start with the following definitions:

**Definition 2.1.** A function \( x : [t_0, T] \rightarrow X \) is called a mild solution of (1) if

\[
x(t) = T_{t-t_0} x_0 + \int_{t_0}^{t} T_{t-s} f(s, x(s)) ds.
\]

A mild solution \( x \) of (1) is called **strong solution** if \( x \) is differentiable almost everywhere with

\[
x'(t) \in L^1_{[t_0, T; X]} \quad \text{and} \quad x'(t) = Ax(t) + f(t, x(t))
\]

for almost every \( t \) in \([t_0, T]\).

**Lemma 2.1.** If \( A \) generates a strongly continuous semigroup of weakly compact operators, then, for each \( t' > 0 \), the problem

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + T_{t'} f(t), \\
x(t_0) &= x_0 \in D(A)
\end{align*}
\]

(2)

has a strong solution on \([t_0, T]\) whenever \( f : [t_0, T] \rightarrow X \) is Lipschitz continuous.

**Proof.** Since \( T_{t'} \) is weakly compact, then by [2] there are a reflexive Banach space \( Z \) and bounded linear operators \( u, v \) such that

\[
T_{t'} = u \circ v, \quad v : X \rightarrow Z, \quad u : Z \rightarrow X.
\]

\( vf : [t_0, T] \rightarrow Z \) is then Lipschitz continuous and by the reflexivity of \( Z \), \( vf \) is differentiable almost everywhere with derivative belonging to \( L^1_{[t_0, T; Z]} \). This implies that \( T_{t'} f = u \circ vf : [t_0, T] \rightarrow X \) is differentiable almost everywhere with derivative belonging to \( L^1_{[t_0, T; X]} \), so by [6], Corollary 4.2.10, the proof is over. \( \square \)

**Theorem 2.1.** Suppose that \( A \) generates a strongly continuous compact semigroup of bounded linear operators and \( f : [t_0, T] \rightarrow X \) is Lipschitz. Then the Cauchy problem

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + f(t), \\
x(t_0) &= x_0 \in D(A)
\end{align*}
\]

(3)

has a strong solution on \([t_0, T]\).

**Proof.** Take a decreasing sequence of positive numbers \( t_n \) going to 0. Then, by Lemma [2.1] for each \( n \in \mathbb{N} \), the Cauchy problem

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + T_{t_n} f(t), \\
x(t_0) &= x_0 \in D(A)
\end{align*}
\]

(4)

has a strong solution \( x_n \) given by

\[
x_n(t) = T_{t-t_0} x_0 + \int_{t_0}^{t} T_{t-s} T_{t_n} f(s) ds = T_{t-t_0} x_0 + T_{t_n} \left( \int_{t_0}^{t} T_{t-s} f(s) ds \right).
\]
Now, we notice that, for \( t \in [t_0, T] \),
\[
(1) \; \lim_{n \to \infty} x_n(t) = T_{t-t_0}x_0 + \int_{t_0}^{t} T_{t-s}f(s)\,ds = x(t).
\]
(2) For each \( n \in \mathbb{N} \), \( x'_n(t) \) exists almost everywhere,
\[
x'_n(t) = AT_{t-t_0}x_0 + T_{t-t_0}T_{t_n}f(t_0) + \int_{t_0}^{t} T_{t-s}(T_{t_n}f)'(s)\,ds
\]
and \( x'_n \in L_{[t_0,T;X]} \).

Since \( f \) is Lipschitz continuous, there is \( K > 0 \) so that
\[
\|f(t) - f(s)\| \leq K\|s - t\| \quad \forall s, t \in [t_0, T].
\]
Considering that \( \{T_t\}_{t \geq 0} \) is a strongly continuous semigroup, we find \( M > 0 \) so that
\[
\|T_t\| = M \quad \forall t \in [t_0, T].
\]
Therefore
\[
\sup_{n,s} \|(T_{t_n}f)'(s)\| \leq KM,
\]
which implies that \( \{(T_{t_n}f)'(s)\}_{n=1}^{\infty} \) is uniformly integrable in \( L^1_{[t_0,T;X]} \).

Since \( \{T_t\}_{t \geq 0} \) is a compact semigroup, by [5] (alternatively [1])
\[
y_n(t) = \int_{t_0}^{t} T_{-s}(T_{t_n}f)'(s)\,ds
\]
has a subsequence relabeled as \( y_n \), converging to \( g \) in the uniform topology of \( C_{[t_0,T;X]} \), so for almost every \( t \in [t_0, T] \),
\[
\lim_{n \to \infty} x'_n = \lim_{n \to \infty} AT_{t-t_0}x_0 + T_{t-t_0}f(t_0) + g(t),
\]
which implies that, for almost every \( t \in [t_0, T] \),
\[
\lim_{n \to \infty} x_n(t) = \int_{t_0}^{t} (g(s) + AT_{t-s}x_0 + T_{t-s}f(t_0))\,ds
\]
and this implies that
\[
x(t) = \int_{t_0}^{t} (g(s) + AT_{t-s}x_0 + T_{t-s}f(t_0))\,ds.
\]
Hence,
\[
x'(t) = AT_{t-t_0}x_0 + T_{t-t_0}f(t_0) + g(t)
\]
almost everywhere. This means that \( x \) is differentiable almost everywhere and
\( x' \in L^1_{[t_0,T;X]} \).

Under additional hypotheses, the strong compactness of \( T_t \) can be removed.

**Theorem 2.2.** Suppose that \( A \) generates a strongly continuous semigroup of weakly compact operators and \( f : [t_0, T] \to X \) is Lipschitz continuous and \( \{t_n\}_{n=1}^{\infty} \) is a sequence as in foregoing theorem. If there is a compact subset \( K \) of \( X \) for which the sequence of derivatives \( (T_{t_n}f')(s) \in K \) for every \( n \) and almost every \( s \), then the Cauchy problem
\[
(5) \quad \begin{cases}
\dot{x}(t) = Ax(t) + f(t), \\
x(t_0) = x_0
\end{cases}
\]
has a strong solution on \([t_0, T]\).
Proof. By Lemma 2.1 and Theorem 6.2 of [1], the sequence \( \{y_n\} \) defined by

\[
y_n(t) = \int_{t_0}^t T_{t-s}(T_{t_n}f)'(s)ds
\]

is relatively compact in \( C[t_0,T;X] \) and the proof follows as in the above theorem. \( \square \)

Combining the techniques used in the proofs of Theorem 2.1 and Theorem 2.2 together with that of [6] in Theorem 1.6 of Chapter 6, we obtain:

**Theorem 2.3.** If \( A \) generates a strongly continuous semigroup of compact operators and \( f : [t_0,T] \times X \to X \) is Lipschitz continuous in both variables, then the Cauchy problem

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + f(t,x), \\
x(t_0) &= x_0 
\end{aligned}
\]

has a strong solution on \( [t_0,T] \).

**Theorem 2.4.** If \( A \) generates a semigroup of weakly compact operators, \( f : [t_0,T] \times X \to X \) is Lipschitz continuous in both variables, \( \{t_n\} \) is a sequence of positive numbers going to zero, and \( K \) a compact subset of \( X \) for which \( (T_{t_n}f)'(s) \in K \) for each \( n \in \mathbb{N} \) and almost every \( s \in [t_0,T] \), then \( \mathbb{A} \) has a strong solution on \( [t_0,T] \).

**Remark.** An important class of differential equations on which our results find applications are the so-called delay equations, which have the particularity of being the semigroup strongly compact for time greater than or equal to the delaying time, say \( t' \) (see [3] for a recent reference). We also notice that the diffusion process also generates compact semigroups ([6], p. 234).

**References**


Departamento de Matemáticas, Universidad de los Andes, Mérida 5101, Venezuela
E-mail address: barcenas@ciens.ula.ve