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FACTORING WEAKLY COMPACT OPERATORS AND THE INHOMOGENEOUS CAUCHY PROBLEM

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ABSTRACT. By using the technique of factoring weakly compact operators through reflexive Banach spaces we prove that a class of ordinary differential equations with Lipschitz continuous perturbations has a strong solution when the problem is governed by a closed linear operator generating a strongly continuous semigroup of compact operators.

1. INTRODUCTION

Consider a Banach space X and the abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax + f(t,x), \\ x(t_0) = x_0 \in D(A) \end{cases} \tag{1}$$

where $0 \leq t_0 < T < \infty$ and A generates a strongly continuous semigroup $\{T_t\}_{t\geq 0}$. It is known that the problem (1) does not have to have any solution on $[t_0, T]$ as can be seen by considering a variation of an example given in [4], Chapter X, exercise 5, section X.5, if $X = c_0$, $f(t,x) = y$ where $y_n = \sqrt{|x_n|}$ and $A = 0$.

In [6] it is proved that if f is Lipschitz continuous in both variables, then (1) has always a mild solution; but according to Webb [7], this mild solution does not need to be a strong solution.

The strongness of a mild solution of (1) is obtained by Pazy [6], p. 189, according to the following hypothesis:

If $f : [0,T] \times X \to X$ is Lipschitz continuous in both variables and X is a reflexive Banach space, then a mild solution of (1) is a strong solution.

In this paper we use the factorization scheme announced in the abstract in order to show that the same conclusion holds in non-reflexive Banach spaces when some extra hypotheses are imposed on either the operator A or the perturbation f.

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2. The results

We start with the following definitions:

Definition 2.1. A function $x : [t_0, T] \to X$ is called a mild solution of (1) if

$$x(t) = T_{t-t_0}x_0 + \int_{t_0}^{t} T_{t-s}f(s, x(s))ds.$$

A mild solution x of (1) is called **strong solution** if x is differentiable almost everywhere with

$$x' \in L^1_{[t_0,T;X]} \qquad \text{and} \qquad x'(t) = Ax(t) + f(t, x(t))$$

for almost every t in $[t_0, T]$.

Lemma 2.1. *If A generates a strongly continuous semigroup of weakly compact operators, then, for each $t' > 0$, the problem*

$$\begin{cases} \dot{x}(t) = Ax(t) + T_{t'}f(t), \\ x(t_0) = x_0 \in D(A) \end{cases} \tag{2}$$

has a strong solution on $[t_0, T]$ whenever $f : [t_0, T] \to X$ is Lipschitz continuous.

Proof. Since T_t' is weakly compact, then by [2] there are a reflexive Banach space Z and bounded linear operators u, v such that

$$T_t' = u \circ v, \qquad v\colon\ X \to Z,\ u : Z \to X.$$

$vf : [t_0, T] \to Z$ is then Lipschitz continuous and by the reflexivity of Z, vf is differentiable almost everywhere with derivative belonging to $L^1_{[t_0,T;Z]}$. This implies that $T_{t'}f = u \circ vf : [t_0, T] \to X$ is differentiable almost everywhere with derivative belonging to $L^1_{[t_0,T;X]}$, so by [6], Corollary 4.2.10, the proof is over. $\square$

Theorem 2.1. *Suppose that A generates a strongly continuous compact semigroup of bounded linear operators and $f : [t_0, T] \to X$ is Lipschitz. Then the Cauchy problem*

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t), \\ x(t_0) = x_0 \in D(A) \end{cases} \tag{3}$$

has a strong solution on $[t_0, T]$.

Proof. Take a decreasing sequence of positive numbers t_n going to 0. Then, by Lemma 2.1, for each $n \in N$, the Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t) + T_{t_n}f(t), \\ x(t_0) = x_0 \in D(A) \end{cases} \tag{4}$$

has a strong solution x_n given by

$$\begin{aligned} x_n(t) &= T_{t-t_0}x_0 + \int_{t_0}^{t} T_{t-s}T_{t_n}f(s)ds \\ &= T_{t-t_0}x_0 + T_{t_n}\left(\int_{t_0}^{t} T_{t-s}f(s)ds\right). \end{aligned}$$

Now, we notice that, for $t \in [t_0, T]$,

(1) $\lim_{n\to\infty} x_n(t) = T_{t-t_0}x_0 + \int_{t_0}^t T_{t-s}f(s)ds = x(t)$.

(2) For each $n \in \mathbb{N}$, $x_n'(t)$ exists almost everywhere,

$$x_n'(t) = AT_{t-t_0}x_0 + T_{t-t_0}T_{t_n}f(t_0) + \int_{t_0}^t T_{t-s}(T_{t_n}f)'(s)ds$$

and $x_n' \in L_{[t_0,T;X]}$.

Since f is Lipschitz continuous, there is $K > 0$ so that

$$\|f(t) - f(s)\| \le K\|s - t\| \quad \forall s, t \in [t_0, T].$$

Considering that $\{T_t\}_{t\ge 0}$ is a strongly continuous semigroup, we find $M > 0$ so that $\|T_t\| \le M \quad \forall t \in [t_0, T]$. Therefore

$$\sup_{n,s} \|(T_{t_n}f)'(s)\| \le KM,$$

which implies that $\{(T_{t_n}f)'(s)\}_{n=1}^\infty$ is uniformly integrable in $L^1_{[t_0,T;X]}$.

Since $\{T_t\}_{t\ge 0}$ is a compact semigroup, by [5] (alternatively [1])

$$y_n(\cdot) = \int_{t_0}^{\cdot} T_{\cdot-s}(T_{t_n}f)'(s)ds$$

has a subsequence relabeled as y_n, converging to g in the uniform topology of $C_{[t_o,T;X]}$, so for almost every $t \in [t_0, T]$,

$$\lim_{n\to\infty} x_n' = \lim_{n\to\infty} AT_{t-t_0}x_0 + T_{t-t_0}f(t_0) + g(t),$$

which implies that, for almost every $t \in [t_0, T]$,

$$\lim_{n\to\infty} x_n(t) = \int_{t_0}^t (g(s) + AT_{t-t_o}x_0 + T_{t-t_0}f(t_0))ds$$

and this implies that

$$x(t) = \int_{t_0}^t (g(s) + AT_{t-t_0}x_0 + T_{t-t_0}f(t_0))ds.$$

Hence,

$$x'(t) = AT_{t-t_0}x_0 + T_{t-t_0}f(t_0) + g(t)$$

almost everywhere. This means that x is differentiable almost everywhere and $x' \in L^1_{[t_0,T;X]}$ □

Under additional hypotheses, the strong compactness of T_t can be removed.

Theorem 2.2. *Suppose that A generates a strongly continuous semigroup of weakly compact operators and $f : [t_0, T] \to X$ is Lipschitz continuous and $\{t_n\}_{n=1}^\infty$ is a sequence as in foregoing theorem. If there is a compact subset K of X for which the sequence of derivatives $(T_{t_n}f')(s) \in K$ for every n and almost every s, then the Cauchy problem*

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t), \\ x(t_0) = x_0 \end{cases} \tag{5}$$

has a strong solution on $[t_0, T]$.

Proof. By Lemma 2.1 and Theorem 6.2 of [1], the sequence $\{y_n\}$ defined by

$$y_n(t) = \int_{t_0}^{t} T_{t-s}(T_{t_n} f)'(s)ds$$

is relatively compact in $C_{[t_0,T;X]}$ and the proof follows as in the above theorem. □

Combining the techniques used in the proofs of Theorem 2.1 and Theorem 2.2 together with that of [6] in Theorem 1.6 of Chapter 6, we obtain:

Theorem 2.3. *If A generates a strongly continuous semigroup of compact operators and $f : [t_o, T] \times X \to X$ is Lipschitz continuous in both variables, then the Cauchy problem*

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t,x), \\ x(t_0) = x_0 \in D(A) \end{cases} \tag{6}$$

has a strong solution on $[t_o, T]$.

Theorem 2.4. *If A generates a semigroup of weakly compact operators, $f : [t_0, T] \times X \to X$ is Lipschitz continuous in both variables, $\{t_n\}$ is a sequence of positive numbers going to zero, and K a compact subset of X for which $(T_{t_n} f)'(s) \in K$ for each $n \in N$ and almost every $s \in [t_0, T]$, then (6) has a strong solution on $[t_0, T]$.*

Remark. An important class of differential equations on which our results find applications are the so-called *delay equations*, which have the particularity of being the semigroup strongly compact for time greater than or equal to the delaying time, say t' (see [3] for a recent reference). We also notice that the diffusion process also generates compact semigroups ([6], p. 234).

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