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REPRESENTATIONS OF SKEW POLYNOMIAL ALGEBRAS

SØREN JØNDRUP

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Abstract. C. De Concini and C. Procesi have proved that in many cases the degree of a skew polynomial algebra is the same as the degree of the corresponding quasi polynomial algebra. We prove a slightly more general result. In fact we show that in case the skew polynomial algebra is a P.I. algebra, then its degree is the degree of the quasi polynomial algebra.

Our argument is then applied to determine the degree of some algebras given by generators and relations.

1. Introduction

Many of the quantized algebras are iterated skew polynomial algebras. An important invariant for these algebras is the degree. It was proved by C. De Concini and C. Procesi that for many such algebras the degree can be found if one can find the rank of a certain matrix [1, 7.1 Proposition]. But for an algebra of the above type considered by De Concini and Procesi, it can be quite complicated to find this rank (cf. [3] and [4]).

One of the main goals of this paper is to give an alternative method for calculation of the degree of certain algebras being iterated skew polynomial algebras. The method also indicates a way of constructing representations of maximal degree, i.e. representation of degree equal to the degree of the algebra.

Our argument for the above results also shows that if a prime skew polynomial algebra has finite degree, then the degree is equal to the degree of the associated quasi polynomial algebra; this result is a generalization of [1, 6.4 Theorem] and also gives one more argument for the fact that Weyl algebras are not P.I. algebras.

All algebras in this paper are associative algebras over a field and all the algebras have an identity element.

2. Generalities

We recall some definitions and some more or less well-known facts about the degree of an algebra; [6] may serve as a general reference for results on P.I. algebras.

A skew derivation on a k-algebra R is a pair (α, δ), where α is a k-automorphism of R and δ an α-derivation.

In this situation one can form the skew polynomial algebra $R[\Theta; \alpha, \delta]$ (cf. [2]).

The associated quasi polynomial algebra is the algebra $R[\Theta; \alpha]$.

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The degree of a prime P.I. algebra, A, is the least integer h, such that A satisfies all identities of $(h \times h)$-matrices over a commutative ring and this degree equals the p.i. degree of A [6, 13.6.7 Corollary], and in case A is an affine k-algebra over an algebraically closed field k, the degree is the greatest integer r such that there exists a maximal ideal M of A with $A/M \cong \mathrm{Mat}_r(k)$ [5].

Thus for a prime affine algebra A over an algebraically closed field k and a finite set of regular elements $a_1, \cdots, a_n$ of A, we have that

$$\operatorname{degree} A = \operatorname{degree} A[a_1^{-1}, \cdots, a_n^{-1}] .$$

Thus if we take a simple representation of $A[a_1^{-1}, \cdots, a_n^{-1}]$ of maximal degree, then the representation induces a representation ρ of A such that $\rho(a_i)$ is an invertible matrix and by the Cayley–Hamilton Theorem its inverse lies in $\rho(A)$; hence ρ is a simple representation of A.

It now follows that for any finite set of regular elements of A, there is a representation of A of maximal degree taking each element of the finite set to an invertible matrix.

We recall three useful lemmas from [2].

Lemma 2.1. *Let $A = R[\Theta; \alpha, \delta]$ and let X be a right Ore set in R such that $\alpha(X) = X$. Then (α, δ) extends to $R[X^{-1}]$, X is a right Ore set in A and $A[X^{-1}]$ is isomorphic to $(R[X^{-1}])[\Theta; \alpha, \delta]$.*

Lemma 2.2. *Let (α, δ) be a skew derivation on a ring R.*

If there exists a central element c in R such that $c - \alpha(c)$ is invertible in R, then δ is inner and $R[\Theta; \alpha, \delta]$ is isomorphic to $R[\Theta'; \alpha]$, where $\Theta' = (c - \alpha(c))^{-1}\delta(c)$.

Lemma 2.3. *Let (α, δ) be a skew derivation on a ring R.*

If α is an inner automorphism, say $\alpha(r) = u^{-1}ru$ for all $r \in R$, then $u\delta$ is an ordinary derivation and $R[\Theta'; u\delta] \simeq R[\Theta; \alpha, \delta]$, where $\Theta' = u\theta$.

3. Main theorem

In this section we formulate and prove the main result of this paper.

R denotes a prime algebra over a field k of characteristic 0 and (α, δ) is a skew derivation on R.

If $0 \neq c \in Z(R)$ (the center of R), then

$$\{\alpha^{i_1}(c)\alpha^{i_2}(c) \cdots \alpha^{i_k}(c) \mid 0 \leq i_j\ ,\ k \in \mathbb{N} \cup \{0\}\}$$

is a multiplicatively closed α invariant set of central elements of R which also is an Ore set. We denote this set by X_c. X_c is the least multiplicative closed α invariant subset of R containing c.

We can, by Lemma 2.1, extend (α, δ) to $R[X_c^{-1}]$.

In this section we also denote this extension of (α, δ) by (α, δ).

Theorem 3.1. *Let R and (α, δ) be as above.*

If $\operatorname{degree} R[\Theta; \alpha, \delta]$ *is finite, then* $\operatorname{degree} R[\Theta; \alpha, \delta] = \operatorname{degree} R[\Theta; \alpha]$.

Proof. Since all algebras are prime and P.I. we get from the results in Section 1 that the degree coincides with the p.i. degree.

If a homogeneous multilinear polynomial f vanishes on $R[\Theta; \alpha, \delta]$, then we claim that f vanishes on $R[\Theta; \alpha]$ as well. This follows if we evaluate f on elements of the form $x_i = r_i\Theta^{k_i}$, $1 \le i \le l$; then $f(x_1, \cdots, x_l)$ in $R[\Theta;\alpha]$ is the term of highest degree in $R[\Theta;\alpha,\delta]$. Thus

$$\operatorname{degree} R[\Theta;\alpha] \le \operatorname{degree} R[\Theta;\alpha,\delta]\ ,$$

and degree $R[\Theta;\alpha]$ is finite.

It is a little less obvious to obtain the other inequality. There are two different cases to consider.

Case 1. There exists an element $c \in Z(R)$ such that $\alpha(c) - c = d \ne 0$. By the lemmas in Section 2 we get

$$R[X_d^{-1}][\Theta;\alpha,\delta] \simeq R[X_d^{-1}][\Theta;\alpha]$$

and since the p.i. degree of a prime ring is the same as the p.i. degree of the quotient ring, the theorem is proved in Case 1.

Notice that in Case 1 we have only used that R is a P.I. algebra.

Case 2. α is the identity on $Z(R)$.

We will show that this will imply that $\operatorname{degree} R[\Theta;\alpha,\delta] = \operatorname{degree} R$ and the theorem is then proved.

Let X denote $Z \setminus \{0\}$. Clearly X is invariant under α and by Lemma 2.1 $R[X^{-1}][\Theta;\alpha.\delta]$ is isomorphic to $(R[\Theta;\alpha,\delta])X^{-1}$.

$R[X^{-1}]$ is the quotient ring of R and is a simple artinian ring with center F, the quotient field of Z.

The automorphism induced by α on $R[X^{-1}]$ is an F-automorphism and therefore it is inner by the Noether–Skolem Theorem; hence by Lemma 2.3

$$R[X^{-1}][\Theta;\alpha,\delta] \simeq R[X^{-1}][\Theta,\delta'],$$

where δ' is an ordinary derivation.

$R[X^{-1}][\Theta,\delta']$ is a P.I. algebra and $R[X^{-1}]$, being simple, cannot have a proper δ' stable ideal. Thus $R[X^{-1}][\Theta,\delta']$ is simple unless δ' is inner [6, 1.8.4 Theorem]. In the latter case $R[X^{-1}][\Theta,\delta'] \cong R[X^{-1}][\Theta]$ is an ordinary polynomial ring which clearly has the same degree as $R[X^{-1}]$, which has the same p.i. degree as R.

Thus we must show that $R[X^{-1}][\Theta;\delta']$ cannot be a simple P.I. algebra.

To ease notation: We have a simple algebra Q with center F and a derivation δ on Q such that $Q[X;\delta]$ is a simple P.I. algebra, and we have to obtain a contradiction.

By Kaplansky's Theorem for P.I. algebras $Q[x,\delta]$ is a finite module over its center.

The center, Z', of $Q[x,\delta]$ is a field.

Suppose $c = \sum_0^m q_i x^i \in Z'$. Since $xc - cx = 0$ we get that $\delta(q_i) = 0$ for all i and, moreover, since $cq = qc$ for all q, $q_m \in F$.

But c cannot be inverted in $Q[x,\delta]$ unless $m = 0$; thus

$$\{c \in F \mid \delta(c) = 0\} = Z'\ .$$

Clearly $Q[x,\delta]$ is not a finite Z'-module. □

One might note that the assumptions in the result by De Concini and Procesi [1, p. 58] imply that $R[\Theta;\alpha,\delta]$ is a finite module over its center, and hence is P.I.

Suppose we have an iterated skew polynomial algebra

$$A_n = k[\Theta_1;\alpha_1,\delta_1]\cdots[\Theta_n,\alpha_n,\delta_n]$$

such that $\alpha_i(\Theta_j) = k_{ij}\Theta_j$ for some $k_{ij} \in k$ and $i > j$ $(\Theta_i\Theta_j = k_{ij}\Theta_j\Theta_i + \delta_i(\Theta_j))$.

If A_n is a P.I. algebra, then our argument above shows that in case α_n is the identity on $Z(A_{n-1})$, $\operatorname{degree} A_n = \operatorname{degree} A_{n-1}$.

The other case: α_n is not the identity on $Z(A_{n-1})$; then

$$(\dagger)\qquad \begin{aligned}\operatorname{degree} A_n &= \operatorname{degree} Q(A_{n-1})[\Theta_n, \alpha_n]\\ &= \operatorname{degree} k[\Theta_n]([\Theta_1, \alpha_1', \delta_1'] \cdots [\Theta_{n-1}; \alpha_{n-1}', \delta_{n-1}']) ,\end{aligned}$$

where (α_i', δ_i') is the skew derivation defined on $k[\Theta_n] \cdots [\Theta_{n-1}; \alpha_{n-1}', \delta_{n-1}']$ by

$$\begin{aligned}&\alpha_i'(\Theta_n) = k_{n_i}^{-1}\Theta_n,\ \delta_i'(\Theta_n) = 0 ,\\ &\alpha_i'(\Theta_j) = \alpha_i(\Theta_j),\ \delta_i'(\Theta_j) = \delta_i(\Theta_j), 1 \le j < i .\end{aligned}$$

The last "$=$ "in ($\dagger$) comes from the fact that the assumptions imply that the 2 rings have isomorphic quotient rings.

If A_n is as above and $k_H[\Theta_1, \dots, \Theta_n]$ is the algebra of regular functions on the quantum hyperplane associated to the sequence of parameters $H = (k_{ij})_{1 \le j < i \le n}$, then one gets by induction on ℓ, the greatest integer such that $\delta_{n-\ell} \neq 0$, that

$$\operatorname{degree} A_n = \operatorname{degree} k_H[\Theta_1, \dots, \Theta_n] .$$

The author wishes to thank the referee for pointing out the above remark; cf. also [1, page 59, Theorem].

The set-up here is more general than the one in [1, 6.4].

To find the degree of an algebra $R[\Theta; \alpha, \delta]$, it follows from our arguments that in case α is the identity on $Z(R)$, there exist a regular element r in R, an element s in R and a non-zero central element t such that $r\Theta + st^{-1}$ is central in $Q(R)[\Theta; \alpha, \delta]$. (See Section 4 for details.) On the other hand, if there exist such elements r, s and t, then clearly $\operatorname{degree} R[\Theta; \alpha, \delta] = \operatorname{degree} R[\Theta] = \operatorname{degree} R$.

This happens, for instance, for quantum (2×2)-matrices:

Let us say the algebra of quantum (2×2)-matrices $M_2(q)$ is generated by a, b, c, d with relations

$$\begin{gathered}ad - da = (q - 1/q)bc, \qquad ab = qba, \qquad ac = qca,\\ bc = cb, \qquad bd = qdb, \qquad cd = qdc\end{gathered}$$

and q an m'th root of unity.

Then $ad - qbc$ is central; hence

$$\operatorname{degree} M_2(q) = \operatorname{degree} k[a, b, c],$$

but bc^{m-1} is central in $k[a, b, c]$, so

$$\operatorname{degree} k[a, b, c] = \operatorname{degree} k[a, c],$$

which is easily seen to be equal to m.

In a forthcoming paper with H. P. Jakobsen the same simple argument is applied to find the degree of $M_n(q)$ and the degree for some classes of related algebras.

Moreover, in case α has order m and there exists an element $c \in Z(R)$ such that $|\{\alpha^i(c) \mid i \in \mathbb{N}_0\}| = m$, then $\operatorname{degree} R[\Theta, \alpha] = m \operatorname{degree} R$; [1] and [5].

4. When is $R[\Theta; \alpha, \delta]$ a P.I. algebra ?

In view of the results of Section 3 it seems to be relevant to ask when $R[\Theta; \alpha, \delta]$ is a P.I. algebra for a prime P.I. algebra R. The following result is probably well known, but we have not been able to find a reference.

We found in Section 3 that in case α is not the identity on $Z(R)$, then $Q(R)[\Theta;\alpha]$ and $Q(R)[\Theta;\alpha,\delta]$ are isomorphic. Moreover, $R[\Theta;\alpha]$ is a P.I. algebra exactly when $Q(R)[\Theta;\alpha]$ is.

Since $Q(R)$ is a finite module over its center F, we get that $Q(R)[\Theta;\alpha]$ is P.I. if and only if $F[\Theta;\alpha]$ is [6, 13. 4.9 Corollary].

$F[\Theta;\alpha]$ is a prime noetherian hereditary algebra and hence a P.I. algebra precisely when it is a finite module over its center. This is easily seen to happen only when α has finite order.

The other case where α is the identity on $Z(R)$: if $R[\Theta;\alpha,\delta]$ is a P.I. algebra, then R is of course a P.I. algebra and α is inner on $Q(R)$. This means that

$$Q(R)[\Theta;\alpha,\delta] \simeq Q(R)[\Theta';u\delta]\ ,$$

where $\Theta' = u\Theta$ and u is a unit in $Q(R)$, i.e. $u = rc^{-1}$, $0 \neq c \in Z(R)$. We have that

$$\alpha(x) = cr^{-1}xrc^{-1} = r^{-1}xr;$$

thus we can take $u = r$ to define α. If $Q(R)[\Theta', u\delta]$ is a P.I. algebra, then $u\delta$ is an inner derivation on $Q(R)$, $u\delta = \delta_b$.

Thus $Q(R)[\Theta', u\delta] \simeq Q(R)[\Theta' - b]$, where $\Theta' - b$ is a commuting indeterminant.

We have shown that in case $R[Q;\alpha,\delta]$ is a P.I. algebra, then $r\Theta - st^{-1}$ is a central element in $R[\Theta;\alpha,\delta]$, for regular element r, central element t and s in R.

On the other hand, if we can find such elements, then clearly $R[\Theta;\alpha,\delta]$ is a P.I. algebra with same degree as R.

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Mathematics Institute, Universitetsparken 5, DK 2100 Copenhagen Ø, Denmark
E-mail address: jondrup@math.ku.dk