ON THE STABILITY
OF APPROXIMATELY ADDITIVE MAPPINGS

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Abstract. In this paper we prove a generalization of the stability of approximately additive mappings in the spirit of Hyers, Ulam and Rassias.

1. Introduction

In 1941 Hyers [3] showed that if $δ > 0$ and $f : E_1 → E_2$, with $E_1$ and $E_2$ Banach spaces, such that

$$\|f(x + y) - f(x) - f(y)\| \leq δ, \text{ for all } x, y \in E_1,$$

then there exists a unique additive mapping $T : E_1 → E_2$ such that

$$\|f(x) - T(x)\| \leq δ,$$

for all $x \in E_1$, and if $f(tx)$ is continuous in $t$ for each fixed $x$, then $T$ is a linear mapping.

Rassias [6] and Gajda [1] gave some generalizations of the Hyers’ result in the following ways: Let $f : E_1 → E_2$ be a mapping such that $f(tx)$ is continuous in $t$ for each fixed $x$. Assume that there exist $θ ≥ 0$ and $p ≠ 1$ such that

$$\frac{\|f(x + y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} ≤ θ, \text{ for all } x, y \in E_1.$$

Then there exists a unique linear mapping $T : E_1 → E_2$ such that

$$\frac{\|T(x) - f(x)\|}{\|x\|^p} ≤ \frac{2θ}{2 - 2^p}, \text{ for all } x \in E_1.$$

However, it was showed that the similar result for the case $p = 1$ does not hold (see [7]). Recently, Gavruta [2] also obtained a further generalization of the Hyers-Rassias theorem: Let $G$ be an abelian group and $X$ a Banach space. Denote by $φ : G × G → [0, ∞)$ a mapping such that

$$φ(x, y) = \sum_{k=0}^{∞} 2^{-k}φ(2^k x, 2^k y) < ∞$$

for all $x, y \in G$. Suppose $f : G → X$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| ≤ φ(x, y)$$

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for all \(x, y \in G\). Then there exists a unique additive mapping \(T : G \to X\) such that

\[
\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x) \quad \text{for all} \quad x \in G.
\]

In this paper we generalize the results of Hyers, Rassias and Găvruta.

2. Main results

Throughout this paper, let \(a\) be a fixed rational number with \(a > 1\). If \(a\) is not an integer, there exist unique nonnegative integers \(b, p\) and \(q\) such that \(a = b + q/p\), \(0 < q/p < 1\) and \((p, q) = 1\). If \(a\) is an integer, we let \(a = b\). We denote by \(G\) a vector space, by \(X\) a Banach space, and by \(\varphi : G \times G \to [0, \infty)\) a mapping such that

\[
\varphi(x, y) = \sum_{k=0}^{\infty} a^{-k} \varphi(a^k x, a^k y) < \infty
\]

for all \(x, y \in G\). In particular, when \(a = 2\), we denote \(\varphi(x, y)\) by \(\varphi_2(x, y)\). We also assume that \(\sum_{n=2}^{\infty} x^n = 0\) if \(n < 2\).

**Theorem 2.1.** Let \(f : G \to X\) be such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad \text{for all} \quad x, y \in G.
\]

Then there exists a unique additive mapping \(T : G \to X\) such that

\[
\|T(x) - f(x)\| \leq a^{-1} \varphi\left(\frac{2}{p} x, bx\right) + a^{-1} \sum_{i=1}^{p} \varphi\left(\frac{1}{p} x, \frac{i-1}{p} x\right)
\]

\[
+ a^{-1} \sum_{i=2}^{q} \varphi\left(\frac{1}{p} x, \frac{i-1}{p} x\right) + a^{-1} \sum_{i=2}^{b} \varphi(x, (i - 1)x),
\]

for all \(x \in G\).

**Proof.** We first prove the case that \(a\) is not an integer. Putting \(y = ix\) in (2), we have

\[
\|f((i + 1)x) - f(x) - f(ix)\| \leq \varphi(x, ix), \quad \text{for all} \quad x \in G, i \in N.
\]

Thus

\[
\|f((k + 1)x) - (k + 1)f(x)\| \leq \sum_{i=1}^{k} \|f((i + 1)x) - f(x) - f(ix)\|
\]

\[
\leq \sum_{i=2}^{k+1} \varphi(x, (i - 1)x)
\]

for all \(x \in G, k \in N\). From (4) it follows that

\[
\|a^{-1} f(bx) - a^{-1} bf(x)\| \leq \sum_{i=2}^{b} a^{-1} \varphi(x, (i - 1)x).
\]

Replacing \(x\) by \(\frac{2}{p} x\) and \(y\) by \(bx\), (2) gives

\[
\|a^{-1} f(ax) - a^{-1} f\left(\frac{q}{p} x\right) - a^{-1} f(bx)\| \leq a^{-1} \varphi\left(\frac{q}{p} x, bx\right).
\]
Replacing $x$ by $\frac{1}{x}$ and $k+1$ by $p$, (4) gives

\begin{equation}
\|f(x) - pf\left(\frac{1}{p}x\right)\| \leq \sum_{i=2}^{p} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right).
\end{equation}

Replacing $x$ by $\frac{1}{p}x$ and $k+1$ by $q$, (4) gives

\begin{equation}
\|f\left(\frac{q}{p}x\right) - qf\left(\frac{1}{p}x\right)\| \leq \sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right).
\end{equation}

From (7) and (8), we obtain

\begin{equation}
a^{-1}\left\|\frac{q}{p}f(x) - f\left(\frac{q}{p}x\right)\right\| \leq a^{-1}\sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right)
\end{equation}

\begin{equation}
+ a^{-1}\sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right).
\end{equation}

From (5), (6) and (9), we get

\begin{equation}
\|a^{-1}f(ax) - f(x)\| \leq \|a^{-1}f(ax) - f\left(\frac{q}{p}x\right) - f(bx)\|
\end{equation}

\begin{equation}
+ a^{-1}\|\frac{q}{p}f(x) - f\left(\frac{q}{p}x\right)\| + a^{-1}\|f(bx) - bf(x)\|
\end{equation}

\begin{equation}
\leq a^{-1}\left[\varphi\left(\frac{q}{p}x, bx\right) + \sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right)
\end{equation}

\begin{equation}
+ \sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right) + \sum_{i=2}^{b} \varphi\left(x, (i-1)x\right)\right].
\end{equation}

Replacing $x$ by $a^{-1}x$, (10) gives

\begin{equation}
\|a^{-1}f(a^k x) - f(a^k x)\|
\end{equation}

\begin{equation}
\leq a^{-1}\left[\varphi(a^{-1}\frac{q}{p}x, a^{-1}bx) + \sum_{i=2}^{q} \varphi(a^{-1}\frac{1}{p}x, a^{-1}\frac{i-1}{p}x)
\end{equation}

\begin{equation}
+ \sum_{i=2}^{q} \varphi(a^{-1}\frac{1}{p}x, a^{-1}\frac{i-1}{p}x) + \sum_{i=2}^{b} \varphi(a^{-1}x, a^{-1}(i-1)x)\right].
\end{equation}
From (11) we obtain

\[ \|a^{-n}f(a^n x) - f(x)\| \leq \sum_{k=1}^{n} a^{-k+1}\|a^{-1}f(a^k x) - f(a^{k-1} x)\| \]

\[ \leq \sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}\frac{q}{p} x, a^{k-1} I x) \]

\[ + \frac{q}{p} \sum_{i=2}^{p} \sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p} x, a^{k-1} I - \frac{1}{p} x) \]

\[ + \sum_{i=2}^{q} \sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p} x, a^{k-1} I - \frac{1}{p} x) \]

\[ + \sum_{i=2}^{b} \sum_{k=1}^{n} a^{-k}\varphi(a^{k-1} x, a^{k-1} (i - 1) x) \]

(12)

for all \( x \in G \).

We claim that the sequence \( \{a^{-n}f(a^n x)\} \) is a Cauchy sequence. Indeed, for \( n > m \), we have

\[ \|a^{-n}f(a^n x) - a^{-m}f(a^m x)\| \leq \sum_{k=m+1}^{n} a^{-k+1}\|a^{-1}f(a^k x) - f(a^{k-1} x)\| \]

\[ \leq \sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1}\frac{q}{p} x, a^{k-1} I x) \]

\[ + \frac{q}{p} \sum_{i=2}^{p} \sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p} x, a^{k-1} I - \frac{1}{p} x) \]

\[ + \sum_{i=2}^{q} \sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p} x, a^{k-1} I - \frac{1}{p} x) \]

\[ + \sum_{i=2}^{b} \sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1} x, a^{k-1} (i - 1) x) \]

(13)

for all \( x \in G \). Taking the limit in (13) as \( m \to \infty \) we obtain

\[ \lim_{m \to \infty} \|a^{-n}f(a^n x) - a^{-m}f(a^m x)\| = 0. \]

Since \( X \) is a Banach space, the sequence \( \{a^{-n}f(a^n x)\} \) converges for every \( x \in G \). Denote

\[ T(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^n}. \]

From (2) we have

\[ \|a^{-n}f(a^n x + a^n y) - a^{-n}f(a^n x) - a^{-n}f(a^n y)\| \]

\[ \leq a^{-n}\varphi(a^n x, a^n y) \quad \text{for all} \; x, y \in G. \]

(14)

From (1) it follows that

\[ \lim_{n \to \infty} a^{-n}\varphi(a^n x, a^n y) = 0. \]
Then (14) implies
\[ ||T(x + y) - T(x) - T(y)|| = 0.\]
To prove (3), taking the limit in (12) as \( n \to \infty \), we obtain
\[
\begin{align*}
||T(x) - f(x)|| &\leq a^{-1} \tilde{\varphi}\left(\frac{q}{p} x, bx\right) + \frac{q}{p} \sum_{i=2}^{p} \tilde{\varphi}\left(\frac{1}{p} x, \frac{i-1}{p} x\right) \\
&\quad + a^{-1} \sum_{i=2}^{q} \tilde{\varphi}\left(\frac{1}{p} x, \frac{i-1}{p} x\right) + a^{-1} \sum_{i=2}^{b} \tilde{\varphi}(x, (i-1)x) \quad \text{for all } x \in G.
\end{align*}
\]
It remains to show that \( T \) is uniquely defined. Let \( F : G \to X \) be another additive mapping satisfying (3). Then
\[
||T(x) - F(x)|| = ||a^{-n}T(a^n x) - a^{-n}F(a^n x)||
\leq ||a^{-n}T(a^n x) - a^{-n}f(a^n x)|| + ||a^{-n}f(a^n x) - a^{-n}F(a^n x)||
\leq 2 \left[ a^{-n-1} \tilde{\varphi}(a^n \frac{q}{p} x, a^n bx) + a^{-n} \tilde{\varphi}(a^n \frac{1}{p} x, a^n \frac{i-1}{p} x) \right]
\]
\[
+ a^{-n-1} \sum_{i=2}^{q} \tilde{\varphi}(a^n \frac{1}{p} x, a^n \frac{i-1}{p} x) + a^{-n-1} \sum_{i=2}^{b} \tilde{\varphi}(a^n x, a^n (i-1)x) \]
\[
= 2a^{-1} \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} \frac{q}{p} x, a^{j} bx) + \frac{q}{p} \sum_{i=2}^{p} \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} \frac{1}{p} x, a^{j} \frac{i-1}{p} x) \\
+ \sum_{i=2}^{\infty} \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} \frac{1}{p} x, a^{j} \frac{i-1}{p} x) + \sum_{i=2}^{b} \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} x, a^{j} (i-1)x)
\]
Thus
\[
||T(x) - F(x)|| = ||a^{-n}T(a^n x) - a^{-n}F(a^n x)||
\leq 2a^{-1} \left[ \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} \frac{q}{p} x, a^{j} bx) + \frac{q}{p} \sum_{i=2}^{p} \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} \frac{1}{p} x, a^{j} \frac{i-1}{p} x) \\
+ \sum_{i=2}^{\infty} \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} \frac{1}{p} x, a^{j} \frac{i-1}{p} x) + \sum_{i=2}^{b} \sum_{j=n}^{\infty} a^{-j} \varphi(a^{j} x, a^{j} (i-1)x) \right]
\]
for all \( x \in G \). Taking the limit (15) as \( n \to \infty \) we obtain
\[
T(x) = F(x) \quad \text{for all } x \in G.
\]
Now we prove the case: \( a = b \). From (5) we obtain
\[
||a^{-1}f(ax) - f(x)|| \leq \sum_{i=2}^{a} a^{-1} \varphi(x, (i-1)x).
\]
Hence we have
\[
||a^{-n}f(a^n x) - f(x)|| \leq \sum_{i=2}^{a} \sum_{k=1}^{n} a^{-k} \varphi(a^{k-1} x, a^{k-1} (i-1)x)
\]
for all \( x \in G \). Denote

\[ T(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^n}. \]

Taking the limit in (12') as \( n \to \infty \), we obtain

\[ \|T(x) - f(x)\| \leq a^{-1} \sum_{i=2}^{n} \phi(x, (i-1)x) \quad \text{for all } x \in G. \]

It is easy to show that \( T \) is uniquely defined.

**Lemma 2.2.** Let \( T : G \to X \) be an additive mapping and let \( x_0 \in G \). If there are an interval \((c, d)\) and \( y \in G \) such that \( C = \{\|T(ux_0 + y)\| : u \in (c, d)\} \) is bounded, then

\[ T(ux_0) = uT(x_0) \quad \text{for all real numbers } u. \]

**Proof.** Assume that there exists a real number \( r \) such that \( T(rx_0) \neq rT(x_0) \). Let \( m = \|T(rx_0) - rT(x_0)\| \). Let \( \{r_n\} \) be a rational number sequence such that

\[ \|(r - r_n)T(x_0)\| \leq m/2 \quad \text{and} \quad \lim_{n \to \infty} r_n = r. \]

Choose a rational number sequence \( \{r'_n\} \) such that \( r'_n(r - r_n) \in (c, d) \) and \( \lim_{n \to \infty} r'_n = \infty \). Since

\[ \|T(r'_n(r - r_n)x_0 + y) - r'_n rT(x_0) + r'_n r_n T(x_0) - T(y)\| \\
=\| r'_n T(rx_0) - r'_n rT(x_0)\| \\
= |r'_n| m, \]

we have

\[ \|T(r'_n(r - r_n)x_0 + y)\| \geq |r'_n|(m/2) - \|T(y)\| \quad \text{for all } n \in N. \]

This contradicts the fact that \( C \) is bounded.

**Remarks.** In Theorem 2.1, (a) if there exist an interval \((c, d)\) and \( \varepsilon > 0 \) such that \( \{\|f(ux_0)\| : u \in (c, d)\} \) and \( \{\phi(sx_0, tx_0) : d/(p+\varepsilon) \leq s, t \leq (b-1)d\} \) are bounded for a fixed \( x_0 \), then \( T(rx_0) = rT(x_0) \) for all real numbers \( r \). In fact, choose an interval \((c', d') \subset (c, d) \cap (dp/(p + \varepsilon), d)\). From (3) we obtain \( C = \{\|T(ux_0)\| : u \in (c', d')\} \) is bounded.

(b) If \( G \) is a normed space and \( f(tx) \) is continuous in \( t \) for each fixed \( x \) and \( \phi \) is bounded on \( G \times G \), then \( T \) is linear by (a).

**Theorem 2.3.** Let \( G \) be a normed space and \( f \) be as in Theorem 2.1. If \( f \) is bounded for some open subset \( A \) of \( G \) and \( \phi \) is bounded on \( G \times G \), then there exists a unique continuous linear mapping \( T : G \to X \) such that

\[ \|T(x) - f(x)\| \leq a^{-1} \phi(\frac{a}{p} x, bx) + a^{-1} \sum_{i=2}^{p} \phi(\frac{1}{p} x, \frac{i-1}{p} x) \\
+ a^{-1} \sum_{i=2}^{q} \phi(\frac{1}{p} x, \frac{i-1}{p} x) + a^{-1} \sum_{i=2}^{b} \phi(x, (i-1)x) \quad \text{for all } x \in G. \]

**Proof.** Let \( T \) be a mapping as in Theorem 2.1. From (4) we obtain that \( T \) is bounded on \( A \). Let \( z \) be an interior point of \( A \). For each fixed \( x \in G \) there exists an interval \((c, d)\) such that \( \{ux + z : u \in (c, d)\} \subset A \). By the preceding remark, \( T \) is linear. Since \( T \) is bounded on open set \( A \), \( T \) is continuous.
Corollary 2.4. Let $G$ and $f$ be as in Theorem 2.3. If $f$ is bounded for some open subset $A$ of $G$ and $\varphi_2$ is bounded on $A \times A$, then there exists a unique continuous linear mapping $T : G \to X$ such that

$$
\|T(x) - f(x)\| \leq 2^{-1}\varphi_2(x,x) \quad \text{for all } x \in G.
$$

Proof. By Theorem 2.1, there exists a unique additive mapping $T : G \to X$ such that $\|T(x) - f(x)\| \leq 2^{-1}\varphi_2(x,x)$ for all $x \in A$. We can apply the similar method as in Theorem 2.3. \qed

Theorem 2.5. Let $f : E_1 \to E_2$ be a mapping with $E_1$ and $E_2$ Banach spaces. If for each fixed $x, y \in E_1$ there exist real numbers $\theta_{xy}, p_{xy}, s_{xy}$ such that $0 \leq p_{xy} < 1$ and

$$
\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\|tx\|^p_{xy} + \|ty\|^p_{xy}) \quad \text{for } t > s_{xy},
$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$
\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^p_{xx}}{2 - 2^p_{xx}} \quad \text{for } t > s_{xx}
$$

for all $x \in E_1$. In particular, if for a fixed $x_0 \in E_1$ there exist real numbers $M_{x_0}, s_{x_0}$ such that $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then $T(rx_0) = rT(x_0)$ for all real numbers $r$.

Proof. Let

$$
\varphi(tx, ty) = \|f(tx + ty) - f(tx) - f(ty)\|.
$$

Then $\varphi_2(x, y) < \infty$ for all $x, y \in E_1$ and $2^{-1}\varphi_2(tx, tx) < 2\theta_{xx}\|tx\|^p_{xx}/(2 - 2^p_{xx})$ for $t > s_{xx}$ for each $x \in E_1$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$
\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^p_{xx}}{2 - 2^p_{xx}} \quad \text{for } t > s_{xx}
$$

for all $x, y \in E_1$. If for a fixed $x_0 \in E_1$ there exist real numbers $M_{x_0}, s_{x_0}$ with $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then

$$
\|T(tx_0)\| \leq \frac{2\theta_{xx_0}\|tx_0\|^p_{xx_0x_0}}{2 - 2^p_{xx_0x_0}} + M_{x_0}t \quad \text{for } t > \max(s_{x_0x_0}, s_{x_0}).
$$

Therefore $\{\|T(ux_0)\| : u \in (\max(s_{x_0x_0}, s_{x_0}), 2\max(s_{x_0x_0}, s_{x_0}))\}$ is bounded. Apply Lemma 2.2. \qed

The following theorem is a generalization of Theorem 1 in [4].

Theorem 2.6. Let a function $\psi : R^+ \to R^+$ satisfy

(i) $\psi(ts) \leq \psi(t)\psi(s)$ for all $t, s \in R^+$ and

(ii) $\lim_{t \to \infty} \psi(t)/t = 0$

and let $f : E_1 \to E_2$ be a mapping with $E_1$ and $E_2$ Banach spaces. If for each fixed $x, y \in E_1$, there exists a real number $\theta_{xy}$ such that

$$
\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\|tx\| + \psi(\|ty\|)) \quad \text{for all } t \in R^+,
$$

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then there exist a unique additive mapping $T : E_1 \to E_2$ and a rational number $a > 1$ such that 

$$
\|f(tx) - T(tx)\| \leq a^{-1}(1 - \frac{\psi(a)}{a}) \left[ \theta_{(q/p)x,bx}(\psi(\|\frac{q}{p}tx\|) + \psi(\|btx\|)) 
+ \frac{q}{p} \sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx\|)) 
+ \sum_{i=2}^{q} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx\|)) 
+ \sum_{k=2}^{b} \theta_{x,(i-1)x}(\psi(\|tx\|) + \psi(\|(i - 1)tx\|)))
\right].
$$

(17)

In particular, if for each fixed $x \in E_1$ there exist positive real numbers $c_x, d_x$ such that $A_x = \{\|f(u)x\| : u \in (c_x,d_x)\}$ is bounded, then $T$ is linear.

Proof. From (ii), there exists a rational number $a$ such that $\psi(a) < a$. Let $\varphi(x,y) = \|f(tx + ty) - f(tx) - f(ty)\|$. From (i) we get

$$
\varphi(tx,ty) = \sum_{n=1}^{\infty} a^{-n} \varphi(a^n tx, a^n ty) 
\leq \sum_{n=1}^{\infty} a^{n-1} \theta_{x^p}(\psi(\|a^n tx\|) + \psi(\|a^n ty\|)) 
\leq \sum_{n=1}^{\infty} (\psi(a)/a)^n \theta_{x^p}(\psi(\|tx\|) + \psi(\|ty\|)) 
= \theta_{x^p}(\psi(\|tx\|) + \psi(\|ty\|)) \frac{1 - \psi(a)/a}{1 - \psi(a)/a} < \infty
$$

for all $x,y \in E_1$ and $t \in R^+$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that 

$$
\|f(tx) - T(tx)\| \leq a^{-1}(1 - \frac{\psi(a)}{a}) \left[ \theta_{(q/p)x,bx}(\psi(\|\frac{q}{p}tx\|) + \psi(\|btx\|)) 
+ \frac{q}{p} \sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx\|)) 
+ \sum_{i=2}^{q} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx\|)) 
+ \sum_{k=2}^{b} \theta_{x,(i-1)x}(\psi(\|tx\|) + \psi(\|(i - 1)tx\|)))
\right].
$$

(17)

for $x \in E_1$ and $t \in R^+$. Since $\lim_{t \to \infty} \psi(t)/t = 0$, there exists a positive number $M$ such that $\psi(t)/t < 1$ for all $t > M$. Choose $N$ such that $c_x N > M$. From (17)
we have
\[\|f(tx) - T(tx)\| \leq a^{-1}\psi(tN)(1 - \frac{\psi(a)}{a})\left[\theta_{(q/p)x,bx}(\psi\left(\|\frac{q}{pN}x\|\right) + \psi\left(\frac{b}{N}\|x\|\right))
+ \frac{q}{p}\sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi\left(\|\frac{1}{pN}x\|\right) + \psi\left(\frac{i-1}{pN}\|x\|\right))\right]
+ \frac{g}{p}\sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi\left(\|\frac{1}{pN}x\|\right) + \psi\left(\frac{i-1}{pN}\|x\|\right))\right]
\]
(18)
+ \frac{b}{p}\sum_{i=2}^{p} \theta_{x,(i-1)x}(\psi\left(\|\frac{1}{N}x\|\right) + \psi\left(\frac{i-1}{N}\|x\|\right))\].

Since \(\psi(Nt) < Nt\) for all \(t \in (c_x, d_x)\), the right-hand side of the inequality of (18) is bounded for \(t \in (c_x, d_x)\). From \(A_x = \{\|f(ux)\| : u \in (c_x, d_x)\}\) is bounded, \(C_x = \{\|T(ux)\| : u \in (c_x, d_x)\}\) is bounded. Applying Lemma 2.2, \(T\) is linear. □

REFERENCES


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