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ON THE STABILITY OF APPROXIMATELY ADDITIVE MAPPINGS

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Abstract. In this paper we prove a generalization of the stability of approximately additive mappings in the spirit of Hyers, Ulam and Rassias.

1. Introduction

In 1941 Hyers [3] showed that if $\delta > 0$ and $f : E_1 \to E_2$, with E_1 and E_2 Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta, \quad \text{for all} \quad x, y \in E_1,$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \delta,$$

for all $x \in E_1$, and if $f(tx)$ is continuous in t for each fixed x, then T is a linear mapping.

Rassias [6] and Gajda [1] gave some generalizations of the Hyers' result in the following ways : Let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leq \theta, \quad \text{for all } x, y \in E_1.$$

Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\frac{\|T(x) - f(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p}, \quad \text{for all} \quad x \in E_1.$$

However, it was showed that the similar result for the case $p = 1$ does not hold (see [7]). Recently, Găvruta [2] also obtained a further generalization of the Hyers-Rassias theorem : Let G be an abelian group and X a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a mapping such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose $f : G \to X$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

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for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\tilde{\varphi}(x, x) \quad \text{for all} \quad x \in G.$$

In this paper we generalize the results of Hyers, Rassias and Găvruta.

2. Main results

Throughout this paper, let a be a fixed rational number with $a > 1$. If a is not an integer, there exist unique nonnegative integers b, p and q such that $a = b + q/p$, $0 < q/p < 1$ and $(p, q) = 1$. If a is an integer, we let $a = b$. We denote by G a vector space, by X a Banach space, and by $\varphi : G \times G \to [0, \infty)$ a mapping such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} a^{-k}\varphi(a^k x, a^k y) < \infty \tag{1}$$

for all $x, y \in G$. In particular, when $a = 2$, we denote $\tilde{\varphi}(x, y)$ by $\tilde{\varphi}_2(x, y)$. We also assume that $\sum_{i=2}^{n}(\cdot) = 0$ if $n < 2$.

Theorem 2.1. *Let $f : G \to X$ be such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad \text{for all } x, y \in G. \tag{2}$$

Then there exists a unique additive mapping $T : G \to X$ such that

$$\begin{aligned} \|T(x) - f(x)\| \leq{}& a^{-1}\tilde{\varphi}(\frac{q}{p}x, bx) + a^{-1}\frac{q}{p}\sum_{i=2}^{p}\tilde{\varphi}(\frac{1}{p}x, \frac{i-1}{p}x) \\ &+ a^{-1}\sum_{i=2}^{q}\tilde{\varphi}(\frac{1}{p}x, \frac{i-1}{p}x) + a^{-1}\sum_{i=2}^{b}\tilde{\varphi}(x, (i-1)x), \end{aligned} \tag{3}$$

for all $x \in G$.

Proof. We first prove the case that a is not an integer. Putting $y = ix$ in (2), we have

$$\|f((i + 1)x) - f(x) - f(ix)\| \leq \varphi(x, ix), \quad \text{for all } x \in G, i \in N.$$

Thus

$$\begin{aligned} \|f((k + 1)x) - (k + 1)f(x)\| &\leq \sum_{i=1}^{k}\|f((i + 1)x) - f(x) - f(ix)\| \\ &\leq \sum_{i=2}^{k+1}\varphi(x, (i - 1)x) \end{aligned} \tag{4}$$

for all $x \in G,\ k \in N$. From (4) it follows that

$$\|a^{-1}f(bx) - a^{-1}bf(x)\| \leq \sum_{i=2}^{b} a^{-1}\varphi(x, (i - 1)x). \tag{5}$$

Replacing x by $\frac{q}{p}x$ and y by bx, (2) gives

$$\|a^{-1}f(ax) - a^{-1}f(\frac{q}{p}x) - a^{-1}f(bx)\| \leq a^{-1}\varphi(\frac{q}{p}x, bx). \tag{6}$$

Replacing x by $\frac{1}{p}x$ and $k+1$ by p, (4) gives

$$\|f(x) - pf(\frac{1}{p}x)\| \le \sum_{i=2}^{p} \varphi(\frac{1}{p}x, \frac{i-1}{p}x). \tag{7}$$

Replacing x by $\frac{1}{p}x$ and $k+1$ by q, (4) gives

$$\|f(\frac{q}{p}x) - qf(\frac{1}{p}x)\| \le \sum_{i=2}^{q} \varphi(\frac{1}{p}x, \frac{i-1}{p}x). \tag{8}$$

From (7) and (8), we obtain

$$\begin{aligned} a^{-1}\|\frac{q}{p}f(x) - f(\frac{q}{p}x)\| &\le a^{-1}\frac{q}{p}\sum_{i=2}^{p} \varphi(\frac{1}{p}x, \frac{i-1}{p}x) \\ &\quad + a^{-1}\sum_{i=2}^{q} \varphi(\frac{1}{p}x, \frac{i-1}{p}x). \end{aligned} \tag{9}$$

From (5), (6) and (9), we get

$$\begin{aligned} \|a^{-1}f(ax) - f(x)\| &\le a^{-1}\|f(ax) - f(\frac{q}{p}x) - f(bx)\| \\ &\quad + a^{-1}\|\frac{q}{p}(x) - f(\frac{q}{p}x)\| + a^{-1}\|f(bx) - bf(x)\| \\ &\le a^{-1}\Big[\varphi(\frac{q}{p}x, bx) + \frac{q}{p}\sum_{i=2}^{p} \varphi(\frac{1}{p}x, \frac{i-1}{p}x) \\ &\quad + \sum_{i=2}^{q} \varphi(\frac{1}{p}x, \frac{i-1}{p}x) + \sum_{i=2}^{b} \varphi(x, (i-1)x)\Big]. \end{aligned} \tag{10}$$

Replacing x by $a^{k-1}x$, (10) gives

$$\begin{aligned} &\|a^{-1}f(a^k x) - f(a^{k-1}x)\| \\ &\le a^{-1}\Big[\varphi(a^{k-1}\frac{q}{p}x, a^{k-1}bx) + \frac{q}{p}\sum_{i=2}^{p} \varphi(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x) \\ &+ \sum_{i=2}^{q} \varphi(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x) + \sum_{i=2}^{b} \varphi(a^{k-1}x, a^{k-1}(i-1)x)\Big]. \end{aligned} \tag{11}$$

From (11) we obtain

$$\begin{aligned}\|a^{-n}f(a^nx)-f(x)\| &\le \sum_{k=1}^{n} a^{-k+1}\|a^{-1}f(a^kx)-f(a^{k-1}x)\| \\ &\le \sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}\frac{q}{p}x, a^{k-1}bx) \\ &+ \frac{q}{p}\sum_{i=2}^{p}\sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x) \\ &+ \sum_{i=2}^{q}\sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x) \\ &+ \sum_{i=2}^{b}\sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}x, a^{k-1}(i-1)x) \qquad (12)\end{aligned}$$

for all $x \in G$.

We claim that the sequence $\{a^{-n}f(a^nx)\}$ is a Cauchy sequence. Indeed, for $n > m$, we have

$$\begin{aligned}\|a^{-n}f(a^nx)-a^{-m}f(a^mx)\| &\le \sum_{k=m+1}^{n} a^{-k+1}\|a^{-1}f(a^kx)-f(a^{k-1}x)\| \\ &\le \sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1}\frac{q}{p}x, a^{k-1}bx) \\ &+ \frac{q}{p}\sum_{i=2}^{p}\sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x) \\ &+ \sum_{i=2}^{q}\sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x) \\ &+ \sum_{i=2}^{b}\sum_{k=m+1}^{n} a^{-k}\varphi(a^{k-1}x, a^{k-1}(i-1)x) \qquad (13)\end{aligned}$$

for all $x \in G$. Taking the limit in (13) as $m \to \infty$ we obtain

$$\lim_{m\to\infty}\|a^{-n}f(a^nx)-a^{-m}f(a^mx)\| = 0.$$

Since X is a Banach space, the sequence $\{a^{-n}f(a^nx)\}$ converges for every $x \in G$. Denote

$$T(x) = \lim_{n\to\infty}\frac{f(a^nx)}{a^n}.$$

From (2) we have

$$\begin{aligned}&\|a^{-n}f(a^nx+a^ny)-a^{-n}f(a^nx)-a^{-n}f(a^ny)\| \\ &\qquad \le a^{-n}\varphi(a^nx, a^ny) \quad \text{for all } x, y \in G. \qquad (14)\end{aligned}$$

From (1) it follows that

$$\lim_{n\to\infty} a^{-n}\varphi(a^nx, a^ny) = 0.$$

Then (14) implies

$$\|T(x+y)-T(x)-T(y)\| = 0.$$

To prove (3), taking the limit in (12) as $n \to \infty$, we obtain

$$\|T(x)-f(x)\| \le a^{-1}\tilde{\varphi}(\frac{q}{p}x, bx) + a^{-1}\frac{q}{p}\sum_{i=2}^{p}\tilde{\varphi}(\frac{1}{p}x, \frac{i-1}{p}x)$$
$$+ a^{-1}\sum_{i=2}^{q}\tilde{\varphi}(\frac{1}{p}x, \frac{i-1}{p}x) + a^{-1}\sum_{i=2}^{b}\tilde{\varphi}(x,(i-1)x) \quad \text{for all} \quad x \in G.$$

It remains to show that T is uniquely defined. Let $F : G \to X$ be another additive mapping satisfying (3). Then

$$\|T(x)-F(x)\| = \|a^{-n}T(a^n x) - a^{-n}F(a^n x)\|$$
$$\le \|a^{-n}T(a^n x) - a^{-n}f(a^n x)\| + \|a^{-n}f(a^n x) - a^{-n}F(a^n x)\|$$
$$\le 2\Big[a^{-n-1}\tilde{\varphi}(a^n\frac{q}{p}x, a^n bx) + a^{-n-1}\frac{q}{p}\sum_{i=2}^{p}\tilde{\varphi}(a^n\frac{1}{p}x, a^n\frac{i-1}{p}x)$$
$$+ a^{-n-1}\sum_{i=2}^{q}\tilde{\varphi}(a^n\frac{1}{p}x, a^n\frac{i-1}{p}x) + a^{-n-1}\sum_{i=2}^{b}\tilde{\varphi}(a^n x, a^n(i-1)x)\Big]$$
$$= 2a^{-1}\Big[\sum_{j=n}^{\infty}a^{-j}\varphi(a^j\frac{q}{p}x, a^j bx) + \frac{q}{p}\sum_{i=2}^{p}\sum_{j=n}^{\infty}a^{-j}\varphi(a^j\frac{1}{p}x, a^j\frac{i-1}{p}x)$$
$$+ \sum_{i=2}^{q}\sum_{j=n}^{\infty}a^{-j}\varphi(a^j\frac{1}{p}x, a^j\frac{i-1}{p}x) + \sum_{i=2}^{b}\sum_{j=n}^{\infty}a^{-j}\varphi(a^j x, a^j(i-1)x)\Big].$$

Thus

$$\|T(x)-F(x)\| = \|a^{-n}T(a^n x) - a^{-n}F(a^n x)\|$$
$$\le 2a^{-1}\Big[\sum_{j=n}^{\infty}a^{-j}\varphi(a^j\frac{q}{p}x, a^j bx) + \frac{q}{p}\sum_{i=2}^{p}\sum_{j=n}^{\infty}a^{-j}\varphi(a^j\frac{1}{p}x, a^j\frac{i-1}{p}x)$$
$$+ \sum_{i=2}^{q}\sum_{j=n}^{\infty}a^{-j}\varphi(a^j\frac{1}{p}x, a^j\frac{i-1}{p}x) + \sum_{i=2}^{b}\sum_{j=n}^{\infty}a^{-j}\varphi(a^j x, a^j(i-1)x)\Big] \tag{15}$$

for all $x \in G$. Taking the limit (15) as $n \to \infty$ we obtain

$$T(x) = F(x) \quad \text{for all} \quad x \in G.$$

Now we prove the case: $a = b$. From (5) we obtain

$$\|a^{-1}f(ax) - f(x)\| \le \sum_{i=2}^{a} a^{-1}\varphi(x,(i-1)x). \tag{16}$$

Hence we have

$$\|a^{-n}f(a^n x) - f(x)\| \le \sum_{i=2}^{a}\sum_{k=1}^{n} a^{-k}\varphi(a^{k-1}x, a^{k-1}(i-1)x) \tag{12$'$}$$

for all $x \in G$. Denote

$$T(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^n}.$$

Taking the limit in $(12')$ as $n \to \infty$, we obtain

$$\|T(x) - f(x)\| \leq a^{-1} \sum_{i=2}^{a} \tilde{\varphi}(x, (i-1)x) \quad \text{for all } \; x \in G.$$

It is easy to show that T is uniquely defined. □

Lemma 2.2. *Let $T : G \to X$ be an additive mapping and let $x_0 \in G$. If there are an interval (c, d) and $y \in G$ such that $C = \{\|T(ux_0 + y)\| : u \in (c, d)\}$ is bounded, then*

$$T(ux_0) = uT(x_0) \qquad \textit{for all real numbers } u.$$

Proof. Assume that there exists a real number r such that $T(rx_0) \neq rT(x_0)$. Let $m = \|T(rx_0) - rT(x_0)\|$. Let $\{r_n\}$ be a rational number sequence such that

$$\|(r - r_n)T(x_0)\| \leq m/2 \quad \text{and} \quad \lim_{n\to\infty} r_n = r.$$

Choose a rational number sequence $\{r'_n\}$ such that $r'_n(r-r_n) \in (c, d)$ and $\lim_{n\to\infty} r'_n = \infty$. Since

$$\begin{aligned} &\|T(r'_n(r - r_n)x_0 + y) - r'_n rT(x_0) + r'_n r_n T(x_0) - T(y)\| \\ &\quad = \|r'_n T(rx_0) - r'_n rT(x_0)\| \\ &\quad = |r'_n| m, \end{aligned}$$

we have

$$\|T(r'_n(r - r_n)x_0 + y)\| \geq |r'_n|(m/2) - \|T(y)\| \quad \text{for all} \quad n \in N.$$

This contradicts the fact that C is bounded. □

Remarks. In Theorem 2.1, (a) if there exist an interval (c, d) and $\varepsilon > 0$ such that $\{\|f(ux_0)\| : u \in (c, d)\}$ and $\{\tilde{\varphi}(sx_0, tx_0) : d/(p+\varepsilon) \leq s, t \leq (b-1)d\}$ are bounded for a fixed x_0, then $T(rx_0) = rT(x_0)$ for all real numbers r. In fact, choose an interval $(c', d') \subset (c, d) \cap (dp/(p + \varepsilon), d)$. From (3) we obtain $C = \{\|T(ux_0)\| : u \in (c', d')\}$ is bounded.

(b) If G is a normed space and $f(tx)$ is continuous in t for each fixed x and $\tilde{\varphi}$ is bounded on $G \times G$, then T is linear by (a).

Theorem 2.3. *Let G be a normed space and f be as in Theorem 2.1. If f is bounded for some open subset A of G and $\tilde{\varphi}$ is bounded on $G \times G$, then there exists a unique continuous linear mapping $T : G \to X$ such that*

$$\begin{aligned} \|T(x) - f(x)\| &\leq a^{-1}\tilde{\varphi}(\frac{q}{p}x, bx) + a^{-1}\frac{q}{p}\sum_{i=2}^{p} \tilde{\varphi}(\frac{1}{p}x, \frac{i-1}{p}x) \\ &\quad + a^{-1}\sum_{i=2}^{q} \tilde{\varphi}(\frac{1}{p}x, \frac{i-1}{p}x) + a^{-1}\sum_{i=2}^{b} \tilde{\varphi}(x, (i-1)x) \qquad \textit{for all } \; x \in G. \end{aligned}$$

Proof. Let T be a mapping as in Theorem 2.1. From (4) we obtain that T is bounded on A. Let z be an interior point of A. For each fixed $x \in G$ there exists an interval (c, d) such that $\{ux + z : u \in (c, d)\} \subset A$. By the preceding remark, T is linear. Since T is bounded on open set A, T is continuous. □

Corollary 2.4. *Let G and f be as in Theorem 2.3. If f is bounded for some open subset A of G and $\tilde{\varphi}_2$ is bounded on $A \times A$, then there exists a unique continuous linear mapping $T : G \to X$ such that*

$$\|T(x) - f(x)\| \leq 2^{-1}\tilde{\varphi}_2(x, x) \qquad \textit{for all} \;\; x \in G.$$

Proof. By Theorem 2.1, there exists a unique additive mapping $T : G \to X$ such that $\|T(x) - f(x)\| \leq 2^{-1}\tilde{\varphi}_2(x, x)$ for all $x \in A$. We can apply the similar method as in Theorem 2.3. □

Theorem 2.5. *Let $f : E_1 \to E_2$ be a mapping with E_1 and E_2 Banach spaces. If for each fixed $x, y \in E_1$ there exist real numbers $\theta_{xy}, p_{xy}, s_{xy}$ such that $0 \leq p_{xy} < 1$ and*

$$\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\|tx\|^{p_{xy}} + \|ty\|^{p_{xy}}) \qquad \textit{for } t > s_{xy},$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^{p_{xx}}}{2 - 2^{p_{xx}}} \qquad \textit{for} \;\; t > s_{xx}$$

for all $x \in E_1$. In particular, if for a fixed $x_0 \in E_1$ there exist real numbers M_{x_0}, s_{x_0} such that $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then $T(rx_0) = rT(x_0)$ for all real numbers r.

Proof. Let

$$\varphi(tx, ty) = \|f(tx + ty) - f(tx) - f(ty)\|.$$

Then $\tilde{\varphi}_2(x, y) < \infty$ for all $x, y \in E_1$ and $2^{-1}\tilde{\varphi}_2(tx, tx) < 2\theta_{xx}\|tx\|^{p_{xx}}/(2 - 2^{p_{xx}})$ for $t > s_{xx}$ for each $x \in E_1$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^{p_{xx}}}{2 - 2^{p_{xx}}} \qquad \text{for} \;\; t > s_{xx}$$

for all $x, y \in E_1$. If for a fixed $x_0 \in E_1$ there exist real numbers M_{x_0}, s_{x_0} with $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then

$$\|T(tx_0)\| \leq \frac{2\theta_{x_0x_0}\|tx_0\|^{p_{x_0x_0}}}{2 - 2^{p_{x_0x_0}}} + M_{x_0}t \;\; \text{for} \;\; t > \max(s_{x_0x_0}, s_{x_0}).$$

Therefore $\{\|T(ux_0)\| : u \in (\max(s_{x_0x_0}, s_{x_0}), 2\max(s_{x_0x_0}, s_{x_0}))\}$ is bounded. Apply Lemma 2.2. □

The following theorem is a generalization of Theorem 1 in [4].

Theorem 2.6. *Let a function $\psi : R^+ \to R^+$ satisfy*

(i) $\psi(ts) \leq \psi(t)\psi(s)$ *for all $t, s \in R^+$ and*
(ii) $\lim_{t\to\infty} \psi(t)/t = 0$

and let $f : E_1 \to E_2$ be a mapping with E_1 and E_2 Banach spaces. If for each fixed $x, y \in E_1$, there exists a real number θ_{xy} such that

$$\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\psi(\|tx\|) + \psi(\|ty\|)) \;\; \textit{for all} \;\; t \in R^+,$$

then there exist a unique additive mapping $T : E_1 \to E_2$ and a rational number $a > 1$ such that

$$\begin{aligned} \|f(tx) - T(tx)\| \le a^{-1}(1 - \frac{\psi(a)}{a})\Big[&\theta_{(q/p)x,bx}(\psi(\|\frac{q}{p}tx\|) + \psi(\|btx\|)) \\ &+ \frac{q}{p}\sum_{i=2}^{p}\theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|\frac{i-1}{p}tx\|)) \\ &+ \sum_{i=2}^{q}\theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|\frac{i-1}{p}tx\|)) \\ &+ \sum_{i=2}^{b}\theta_{x,(i-1)x}(\psi(\|tx\|) + \psi(\|(i-1)tx\|))\Big]. \end{aligned} \tag{17}$$

In particular, if for each fixed $x \in E_1$ there exist positive real numbers c_x, d_x such that $A_x = \{\|f(ux)\| : u \in (c_x, d_x)\}$ is bounded, then T is linear.

Proof. From (ii), there exists a rational number a such that $\psi(a) < a$. Let $\varphi(x, y) = \|f(tx + ty) - f(tx) - f(ty)\|$. From (i) we get

$$\begin{aligned} \tilde{\varphi}(tx, ty) &= \sum_{n=1}^{\infty} a^{-n}\varphi(a^n tx, a^n ty) \\ &\le \sum_{n=1}^{\infty} a^{-n}\theta_{xy}(\psi(\|a^n tx\|) + \psi(\|a^n ty\|)) \\ &\le \sum_{n=1}^{\infty}(\psi(a)/a)^n\theta_{xy}(\psi(\|tx\|) + \psi(\|ty\|)) \\ &= \frac{\theta_{xy}(\psi(\|tx\|) + \psi(\|ty\|))}{1 - \psi(a)/a} < \infty \end{aligned}$$

for all $x, y \in E_1$ and $t \in R^+$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\begin{aligned} \|f(tx) - T(tx)\| \le a^{-1}(1 - \frac{\psi(a)}{a})\Big[&\theta_{(q/p)x,bx}(\psi(\|\frac{q}{p}tx\|) + \psi(\|btx\|)) \\ &+ \frac{q}{p}\sum_{i=2}^{p}\theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|\frac{i-1}{p}tx\|)) \\ &+ \sum_{i=2}^{q}\theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|\frac{i-1}{p}tx\|)) \\ &+ \sum_{i=2}^{b}\theta_{x,(i-1)x}(\psi(\|tx\|) + \psi(\|(i-1)tx\|))\Big] \end{aligned} \tag{17}$$

for $x \in E_1$ and $t \in R^+$. Since $\lim_{t\to\infty} \psi(t)/t = 0$, there exists a positive number M such that $\psi(t)/t < 1$ for all $t > M$. Choose N such that $c_x N > M$. From (17) we have

$$\begin{aligned}\|f(tx)-T(tx)\| \le a^{-1}\psi(tN)(1-\frac{\psi(a)}{a})\Big[&\theta_{(q/p)x,bx}(\psi(\|\frac{q}{pN}x\|)+\psi(\|\frac{b}{N}x\|))\\ &+\frac{q}{p}\sum_{i=2}^{p}\theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{pN}x\|)+\psi(\|\frac{i-1}{pN}x\|))\\ &+\sum_{i=2}^{q}\theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{pN}x\|)+\psi(\|\frac{i-1}{pN})x\|))\\ &+\sum_{i=2}^{b}\theta_{x,(i-1)x}(\psi(\|\frac{1}{N}x\|)+\psi(\|\frac{i-1}{N})x\|))\Big].\end{aligned} \tag{18}$$

Since $\psi(Nt) < Nt$ for all $t \in (c_x, d_x)$, the right-hand side of the inequality of (18) is bounded for $t \in (c_x, d_x)$. From $A_x = \{\|f(ux)\| : u \in (c_x, d_x)\}$ is bounded, $C_x = \{\|T(ux)\| : u \in (c_x, d_x)\}$ is bounded. Applying Lemma 2.2, T is linear. □

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