ON THE STABILITY OF APPROXIMATELY ADDITIVE MAPPINGS

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Abstract. In this paper we prove a generalization of the stability of approximately additive mappings in the spirit of Hyers, Ulam and Rassias.

1. Introduction

In 1941 Hyers [3] showed that if \( \delta > 0 \) and \( f : E_1 \to E_2 \), with \( E_1 \) and \( E_2 \) Banach spaces, such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \delta, \quad \text{for all } x, y \in E_1,
\]
then there exists a unique additive mapping \( T : E_1 \to E_2 \) such that
\[
\| f(x) - T(x) \| \leq \delta,
\]
for all \( x \in E_1 \), and if \( f(tx) \) is continuous in \( t \) for each fixed \( x \), then \( T \) is a linear mapping.

Rassias [4] and Gajda [1] gave some generalizations of the Hyers’ result in the following ways: Let \( f : E_1 \to E_2 \) be a mapping such that \( f(tx) \) is continuous in \( t \) for each fixed \( x \). Assume that there exist \( \theta \geq 0 \) and \( p \neq 1 \) such that
\[
\frac{\| f(x + y) - f(x) - f(y) \|}{\| x \|^p + \| y \|^p} \leq \theta, \quad \text{for all } x, y \in E_1.
\]
Then there exists a unique linear mapping \( T : E_1 \to E_2 \) such that
\[
\frac{\| T(x) - f(x) \|}{\| x \|^p} \leq \frac{2\theta}{2 - 2^p}, \quad \text{for all } x \in E_1.
\]

However, it was showed that the similar result for the case \( p = 1 \) does not hold (see [7]). Recently, Gavruta [2] also obtained a further generalization of the Hyers-Rassias theorem: Let \( G \) be an abelian group and \( X \) a Banach space. Denote by \( \varphi : G \times G \to [0, \infty) \) a mapping such that
\[
\hat{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty
\]
for all \( x, y \in G \). Suppose \( f : G \to X \) is a mapping satisfying
\[
\| f(x + y) - f(x) - f(y) \| \leq \varphi(x, y)
\]
for all \(x, y \in G\). Then there exists a unique additive mapping \(T : G \to X\) such that
\[
\|f(x) - T(x)\| \leq \frac{1}{2}\bar{\varphi}(x, x) \quad \text{for all } x \in G.
\]

In this paper we generalize the results of Hyers, Rassias and Găvruta.

2. Main results

Throughout this paper, let \(a\) be a fixed rational number with \(a > 1\). If \(a\) is not an integer, there exist unique nonnegative integers \(b, p\) and \(q\) such that \(a = b + q/p\), \(0 < q/p < 1\) and \((p, q) = 1\). If \(a\) is an integer, we let \(a = b\). We denote by \(G\) a vector space, by \(X\) a Banach space, and by \(\varphi : G \times G \to [0, \infty)\) a mapping such that
\[
\bar{\varphi}(x, y) = \sum_{k=0}^{\infty} a^{-k}\varphi(a^kx, a^ky) < \infty
\]
for all \(x, y \in G\). In particular, when \(a = 2\), we denote \(\bar{\varphi}(x, y)\) by \(\bar{\varphi}_2(x, y)\). We also assume that \(\sum_{i=2}^{\infty} i\varphi(1, 1) = 0\) if \(n < 2\).

**Theorem 2.1.** Let \(f : G \to X\) be such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad \text{for all } x, y \in G.
\]
Then there exists a unique additive mapping \(T : G \to X\) such that
\[
\|T(x) - f(x)\| \leq a^{-1}\bar{\varphi}(\frac{q}{p}x, bx) + a^{-1}\sum_{i=2}^{p}\varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right)
\]
\[
+ a^{-1}\sum_{i=2}^{q}\varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right) + a^{-1}\sum_{i=2}^{b}\varphi(x, (i-1)x),
\]
for all \(x \in G\).

**Proof.** We first prove the case that \(a\) is not an integer. Putting \(y = ix\) in (2), we have
\[
\|f((i+1)x) - f(x) - f(ix)\| \leq \varphi(x, ix), \quad \text{for all } x \in G, i \in N.
\]
Thus
\[
\|f((k+1)x) - (k+1)f(x)\| \leq \sum_{i=1}^{k}\|f((i+1)x) - f(x) - f(ix)\|
\]
\[
\leq \sum_{i=2}^{k+1}\varphi(x, (i-1)x)
\]
for all \(x \in G, k \in N\). From (4) it follows that
\[
\|a^{-1}f(bx) - a^{-1}bf(x)\| \leq \sum_{i=2}^{b}a^{-1}\varphi(x, (i-1)x).
\]
Replacing \(x\) by \(\frac{q}{p}x\) and \(y\) by \(bx\), (2) gives
\[
\|a^{-1}f(ax) - a^{-1}f\left(\frac{q}{p}x\right) - a^{-1}f(bx)\| \leq a^{-1}\varphi\left(\frac{q}{p}x, bx\right).
\]
Replacing $x$ by $\frac{1}{p}x$ and $k+1$ by $p$, (4) gives

\begin{equation}
\|f(x) - pf(\frac{1}{p}x)\| \leq \sum_{i=2}^{p} \phi(\frac{1}{p} x, \frac{i-1}{p} x).
\end{equation}

Replacing $x$ by $\frac{1}{p}x$ and $k+1$ by $q$, (4) gives

\begin{equation}
\|f(\frac{q}{p}x) - qf(\frac{1}{p}x)\| \leq \sum_{i=2}^{q} \phi(\frac{1}{p} x, \frac{i-1}{p} x).
\end{equation}

From (7) and (8), we obtain

\begin{equation}
\begin{aligned}
a^{-1} q \phi(\frac{q}{p}x) - qf(\frac{1}{p}x) &\leq a^{-1} q \sum_{i=2}^{p} \phi(\frac{1}{p} x, \frac{i-1}{p} x) \\ &+ a^{-1} \sum_{i=2}^{q} \phi(\frac{1}{p} x, \frac{i-1}{p} x).
\end{aligned}
\end{equation}

From (5), (6) and (9), we get

\begin{equation}
\|a^{-1} f(ax) - f(x)\| \leq a^{-1} \|f(ax) - f(\frac{q}{p}x) - f(bx)\| \\
+ a^{-1} \|\frac{q}{p}(x) - f(\frac{q}{p}x)\| + a^{-1} \|f(b) - bf(x)\| \\
\leq a^{-1} \left[ \phi(\frac{q}{p}x, bx) + \sum_{i=2}^{p} \phi(\frac{1}{p} x, \frac{i-1}{p} x) \\
+ \sum_{i=2}^{q} \phi(\frac{1}{p} x, \frac{i-1}{p} x) + \sum_{i=2}^{b} \phi(x, (i-1)x) \right].
\end{equation}

Replacing $x$ by $a^{k-1}x$, (10) gives

\begin{equation}
\begin{aligned}
\|a^{-1} f(a^k x) - f(a^{k-1} x)\| \\
\leq a^{-1} \left[ \phi(a^{k-1} \frac{q}{p}x, a^{k-1}bx) + \sum_{i=2}^{p} \phi(a^{k-1} \frac{1}{p} x, a^{k-1} \frac{i-1}{p} x) \\
+ \sum_{i=2}^{q} \phi(a^{k-1} \frac{1}{p} x, a^{k-1} \frac{i-1}{p} x) + \sum_{i=2}^{b} \phi(a^{k-1} x, a^{k-1} (i-1)x) \right].
\end{aligned}
\end{equation}
From (11) we obtain
\[
\|a^{-n}f(a^n x) - f(x)\| \leq \sum_{k=1}^{n} a^{-k+1} |a^{-1}f(a^k x) - f(a^{k-1} x)|
\]
\[
\leq \sum_{k=1}^{n} a^{-k} \varphi(a^{k-1} x, a^{k-1} x)
\]
\[
+ \frac{q}{p} \sum_{i=2}^{p} \sum_{k=1}^{n} a^{-k} \varphi(a^{k-1} \frac{1}{p} x, a^{k-1} \frac{i-1}{p} x)
\]
\[
+ \sum_{i=2}^{q} \sum_{k=1}^{n} a^{-k} \varphi(a^{k-1} \frac{1}{p} x, a^{k-1} \frac{i-1}{p} x)
\]
\[
+ \sum_{i=2}^{b} \sum_{k=1}^{n} a^{-k} \varphi(a^{k-1} x, a^{k-1} (i-1) x)
\]
(12)

for all \(x \in G\).

We claim that the sequence \(\{a^{-n}f(a^n x)\}\) is a Cauchy sequence. Indeed, for \(n > m\), we have
\[
\|a^{-n}f(a^n x) - a^{-m}f(a^m x)\| \leq \sum_{k=m+1}^{n} a^{-k+1} |a^{-1}f(a^k x) - f(a^{k-1} x)|
\]
\[
\leq \sum_{k=m+1}^{n} a^{-k} \varphi(a^{k-1} x, a^{k-1} x)
\]
\[
+ \frac{q}{p} \sum_{i=2}^{p} \sum_{k=m+1}^{n} a^{-k} \varphi(a^{k-1} \frac{1}{p} x, a^{k-1} \frac{i-1}{p} x)
\]
\[
+ \sum_{i=2}^{q} \sum_{k=m+1}^{n} a^{-k} \varphi(a^{k-1} \frac{1}{p} x, a^{k-1} \frac{i-1}{p} x)
\]
\[
+ \sum_{i=2}^{b} \sum_{k=m+1}^{n} a^{-k} \varphi(a^{k-1} x, a^{k-1} (i-1) x)
\]
(13)

for all \(x \in G\). Taking the limit in (13) as \(m \to \infty\) we obtain
\[
\lim_{m \to \infty} \|a^{-n}f(a^n x) - a^{-m}f(a^m x)\| = 0.
\]

Since \(X\) is a Banach space, the sequence \(\{a^{-n}f(a^n x)\}\) converges for every \(x \in G\).

Denote
\[
T(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^n}.
\]

From (2) we have
\[
\|a^{-n}f(a^n x + a^n y) - a^{-n}f(a^n x) - a^{-n}f(a^n y)\|
\]
\[
\leq a^{-n} \varphi(a^n x, a^n y) \quad \text{for all } x, y \in G.
\]
(14)

From (1) it follows that
\[
\lim_{n \to \infty} a^{-n} \varphi(a^n x, a^n y) = 0.
\]
Then (14) implies
\[ ||T(x + y) - T(x) - T(y)|| = 0. \]
To prove (3), taking the limit in (12) as \( n \to \infty \), we obtain
\[
\| T(x) - f(x) \| \leq a^{-1} \frac{q}{p} x + a^{-1} \frac{q}{p} \sum_{i=2}^{p} \psi \left( \frac{1}{p}, i - 1, x \right) \\
+ a^{-1} \sum_{i=2}^{q} \psi \left( \frac{1}{p}, i - 1, x \right) + a^{-1} \sum_{i=2}^{b} \psi(x, (i-1)x) \quad \text{for all } x \in G.
\]
It remains to show that \( T \) is uniquely defined. Let \( F : G \to X \) be another additive mapping satisfying (3). Then
\[
\| T(x) - F(x) \| = \| a^{-n}T(a^n x) - a^{-n}F(a^n x) \| \\
\leq \| a^{-n}T(a^n x) - a^{-n}F(a^n x) \| + \| a^{-n}f(a^n x) - a^{-n}F(a^n x) \| \\
\leq 2 \left[ a^{-n} \frac{q}{p} x + a^{-n} \sum_{i=2}^{p} \psi \left( \frac{1}{p}, b, x \right) \\
+ a^{-n} \sum_{i=2}^{q} \psi \left( \frac{1}{p}, a^n i - 1, x \right) + a^{-n} \sum_{i=2}^{b} \psi(x, (i-1)x) \right] \\
= 2a^{-1} \left[ \sum_{j=n}^{\infty} a^{-j} \psi(a^{-j} x, a^{j} bx) + \sum_{i=2}^{p} \sum_{j=n}^{\infty} \psi \left( \frac{1}{p}, a^{j} x, a^{j} i - 1, x \right) \\
+ \sum_{i=2}^{q} \sum_{j=n}^{\infty} \psi \left( \frac{1}{p}, a^{j} x, a^{j} i - 1, x \right) + \sum_{i=2}^{b} \sum_{j=n}^{\infty} \psi(x, (i-1)x) \right].
\]
Thus
\[
\| T(x) - F(x) \| = \| a^{-n}T(a^n x) - a^{-n}F(a^n x) \| \\
\leq 2a^{-1} \left[ \sum_{j=n}^{\infty} a^{-j} \psi(a^{-j} x, a^{j} bx) + \sum_{i=2}^{p} \sum_{j=n}^{\infty} \psi \left( \frac{1}{p}, a^{j} x, a^{j} i - 1, x \right) \\
+ \sum_{i=2}^{q} \sum_{j=n}^{\infty} \psi \left( \frac{1}{p}, a^{j} x, a^{j} i - 1, x \right) + \sum_{i=2}^{b} \sum_{j=n}^{\infty} \psi(x, (i-1)x) \right].
\]
for all \( x \in G \). Taking the limit (15) as \( n \to \infty \) we obtain
\[ T(x) = F(x) \quad \text{for all } x \in G. \]

Now we prove the case: \( a = b \). From (5) we obtain
\[
\| a^{-1}f(ax) - f(x) \| \leq \sum_{i=2}^{a} a^{-i} \psi(x, (i-1)x).
\]
Hence we have
\[
\| a^{-n}f(a^n x) - f(x) \| \leq \sum_{i=2}^{a} \sum_{k=1}^{n} a^{-k} \psi(a^{k-1} x, a^{k-1} (i-1)x)
\]
Theorem 2.3. Let $A$ be a bounded set for some open subset $G$. Then there exists a unique continuous linear mapping $G$ bounded on $A$. Therefore, we have

$$T(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^n}.$$  

Taking the limit in (12) as $n \to \infty$, we obtain

$$\|T(x) - f(x)\| \leq a^{-1} \sum_{i=2}^{a} \hat{\varphi}(x, (i-1)x) \quad \text{for all } x \in G.$$  

It is easy to show that $T$ is uniquely defined.$\blacksquare$

**Lemma 2.2.** Let $T : G \to X$ be an additive mapping and let $x_0 \in G$. If there are an interval $(c, d)$ and $y \in G$ such that $C = \{\|T(ux_0 + y)\| : u \in (c, d)\}$ is bounded, then

$$T(u x_0) = uT(x_0) \quad \text{for all real numbers } u.$$  

**Proof.** Assume that there exists a real number $r$ such that $T(r x_0) \neq rT(x_0)$. Let $m = \|T(r x_0) - rT(x_0)\|$. Let $\{r_n\}$ be a rational number sequence such that

$$\|(r - r_n)T(x_0)\| \leq m/2 \quad \text{and} \quad \lim_{n \to \infty} r_n = r.$$  

Choose a rational number sequence $\{r_n\}$ such that $r_n(r - r_n) \in (c, d)$ and $\lim_{n \to \infty} r_n = \infty$. Since

$$\|T(r_n(r - r_n)x_0 + y) - r_nT(x_0) + r_nT(x_0) - T(y)\|$$
$$= \|r_nT(x_0) - r_nT(x_0)\|$$
$$= \|r_n(m),$$

we have

$$\|T(r_n(r - r_n)x_0 + y)\| \geq |r_n|(m/2) - \|T(y)\| \quad \text{for all } n \in N.$$  

This contradicts the fact that $C$ is bounded.$\blacksquare$

**Remarks.** In Theorem 2.1, (a) if there exist an interval $(c, d)$ and $\varepsilon > 0$ such that $\{\|f(ux_0)\| : u \in (c, d)\}$ and $\{\hat{\varphi}(sx_0, tx_0) : d/(p+\varepsilon) \leq s, t \leq (b-1)d\}$ are bounded for some fixed $x_0$, then $T(rx_0) = rT(x_0)$ for all real numbers $r$. In fact, choose an interval $(c', d') \subset (c, d) \cap (d/p + \varepsilon, d)$. From (3) we obtain $C = \{\|T(ux_0)\| : u \in (c', d')\}$ is bounded.

(b) If $G$ is a normed space and $f(tx)$ is continuous in $t$ for each fixed $x$ and $\hat{\varphi}$ is bounded on $G \times G$, then $T$ is linear by (a).

**Theorem 2.3.** Let $G$ be a normed space and $f$ be as in Theorem 2.1. If $f$ is bounded for some open subset $A$ of $G$ and $\hat{\varphi}$ is bounded on $G \times G$, then there exists a unique continuous linear mapping $T : G \to X$ such that

$$\|T(x) - f(x)\| \leq a^{-1}\hat{\varphi}(x, bx) + a^{-1}\sum_{i=2}^{a} \hat{\varphi}(x, \frac{i-1}{p} x)$$
$$+ a^{-1}\sum_{i=2}^{a} \hat{\varphi}(x, \frac{i-1}{p} x) + a^{-1}\sum_{i=2}^{b} \hat{\varphi}(x, (i-1)x) \quad \text{for all } x \in G.$$  

**Proof.** Let $T$ be a mapping as in Theorem 2.1. From (4) we obtain that $T$ is bounded on $A$. Let $z$ be an interior point of $A$. For each fixed $x \in G$ there exists an interval $(c, d)$ such that $\{uz + z : u \in (c, d)\} \subset A$. By the preceding remark, $T$ is linear. Since $T$ is bounded on open set $A$, $T$ is continuous.$\blacksquare$
Corollary 2.4. Let $G$ and $f$ be as in Theorem 2.3. If $f$ is bounded for some open subset $A$ of $G$ and $\tilde{\varphi}_2$ is bounded on $A \times A$, then there exists a unique continuous linear mapping $T : G \to X$ such that
\[
\|T(x) - f(x)\| \leq 2^{-1} \tilde{\varphi}_2(x, x) \quad \text{for all } x \in G.
\]

Proof. By Theorem 2.1, there exists a unique additive mapping $T : G \to X$ such that $\|T(x) - f(x)\| \leq 2^{-1} \tilde{\varphi}_2(x, x)$ for all $x \in A$. We can apply the similar method as in Theorem 2.3. \hfill \Box

Theorem 2.5. Let $f : E_1 \to E_2$ be a mapping with $E_1$ and $E_2$ Banach spaces. If for each fixed $x, y \in E_1$ there exist real numbers $\theta_{xy}, p_{xy}, s_{xy}$ such that $0 \leq p_{xy} < 1$ and
\[
\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\|tx\|^{p_{xy}} + \|ty\|^{p_{xy}}) \quad \text{for } t > s_{xy},
\]
then there exists a unique additive mapping $T : E_1 \to E_2$ such that
\[
\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^{p_{xx}}}{2 - 2p_{xx}} \quad \text{for } t > s_{xx}
\]
for all $x \in E_1$. In particular, if for a fixed $x_0 \in E_1$ there exist real numbers $M_{x_0}, s_{x_0}$ such that $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then $T(rx_0) = rT(x_0)$ for all real numbers $r$.

Proof. Let
\[
\varphi(tx, ty) = \|f(tx + ty) - f(tx) - f(ty)\|.
\]
Then $\varphi_2(x, y) < \infty$ for all $x, y \in E_1$ and $2^{-1} \varphi_2(tx, tx) < 2\theta_{xx}\|tx\|^{p_{xx}}/(2 - 2p_{xx})$ for $t > s_{xx}$ for each $x \in E_1$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that
\[
\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^{p_{xx}}}{2 - 2p_{xx}} \quad \text{for } t > s_{xx}
\]
for all $x, y \in E_1$. If for a fixed $x_0 \in E_1$ there exist real numbers $M_{x_0}, s_{x_0}$ with $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then
\[
\|T(tx_0)\| \leq \frac{2\theta_{x_0x_0}\|tx_0\|^{p_{x_0x_0}}}{2 - 2p_{x_0x_0}} + M_{x_0}t \quad \text{for } t > \max(s_{x_0x_0}, s_{x_0}).
\]
Therefore $\{\|T(tx_0)\| : u \in (\max(s_{x_0x_0}, s_{x_0}), 2\max(s_{x_0x_0}, s_{x_0}))\}$ is bounded. Apply Lemma 2.2. \hfill \Box

The following theorem is a generalization of Theorem 1 in [4].

Theorem 2.6. Let a function $\psi : R^+ \to R^+$ satisfy
\begin{enumerate}[(i)]
\item $\psi(ts) \leq \psi(t)\psi(s)$ for all $t, s \in R^+$ and
\item $\lim_{t \to \infty} \psi(t)/t = 0$
\end{enumerate}
and let $f : E_1 \to E_2$ be a mapping with $E_1$ and $E_2$ Banach spaces. If for each fixed $x, y \in E_1$, there exists a real number $\theta_{xy}$ such that
\[
\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\psi(\|tx\|) + \psi(\|ty\|)) \quad \text{for all } t \in R^+,
\]
then there exist a unique additive mapping $T : E_1 \to E_2$ and a rational number $a > 1$ such that

$$
\|f(tx) - T(tx)\| \leq a^{-1}(1 - \frac{\psi(a)}{a}) \left[ \theta_{(q/p)x,bx}(\psi(\|q/p tx\|) + \psi(\|bt x\|))
\right.
\]

$$

\[+
\frac{q}{p} \sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi(\|1/p tx\|) + \psi(\|i-1/p tx\|))
\]

$$
\left.+
\sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi(\|1/p tx\|) + \psi(\|i-1/p tx\|))
\right]

(17)

In particular, if for each fixed $x \in E_1$ there exist positive real numbers $c_x, d_x$ such that $A_x = \{\|f(ux)\| : u \in (c_x, d_x)\}$ is bounded, then $T$ is linear.

**Proof.** From (ii), there exists a rational number $a$ such that $\psi(a) < a$. Let $\varphi(x, y) = \|f(tx + ty) - f(tx) - f(ty)\|$. From (i) we get

$$
\varphi(tx, ty) = \sum_{n=1}^{\infty} a^{-n} \varphi(a^n tx, a^n ty)
$$

\[\leq \sum_{n=1}^{\infty} a^{-n} \theta_{x^p}(\psi(\|a^n tx\|) + \psi(\|a^n ty\|))
\]

\[\leq \sum_{n=1}^{\infty} (\psi(a)/a)^n \theta_{x^p}(\psi(\|tx\|) + \psi(\|ty\|))
\]

\[= \frac{\theta_{x^p}(\psi(\|tx\|) + \psi(\|ty\|))}{1 - \psi(a)/a} < \infty
\]

for all $x, y \in E_1$ and $t \in R^+$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$
\|f(tx) - T(tx)\| \leq a^{-1}(1 - \frac{\psi(a)}{a}) \left[ \theta_{(q/p)x,bx}(\psi(\|q/p tx\|) + \psi(\|bt x\|))
\right.
\]

$$

\[+
\frac{q}{p} \sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi(\|1/p tx\|) + \psi(\|i-1/p tx\|))
\]

$$
\left.+
\sum_{i=2}^{p} \theta_{(1/p)x,(i-1)x/p}(\psi(\|1/p tx\|) + \psi(\|i-1/p tx\|))
\right]

(17)

for $x \in E_1$ and $t \in R^+$. Since $\lim_{t \to \infty} \psi(t)/t = 0$, there exists a positive number $M$ such that $\psi(t)/t < 1$ for all $t > M$. Choose $N$ such that $c_x N > M$. From (17)
we have
\[
\|f(tx) - T(tx)\| \leq a^{-1} \psi(tN)(1 - \frac{\psi(a)}{a}) \left[ \theta_{(q/p)x,bx}(\psi(\frac{q}{pN}x)) + \psi(\frac{b}{N}x) \right] \\
+ \frac{q}{p} \sum_{i=2}^{p} \theta_{(1/p)x,(i-1)/p}(\psi(\frac{1}{pN}x)) + \psi(\frac{i-1}{pN}x) \\
+ \sum_{i=2}^{q} \theta_{(1/p)x,(i-1)/p}(\psi(\frac{1}{pN}x)) + \psi(\frac{i-1}{pN}x) \\
+ \sum_{i=2}^{b} \theta_{x,(i-1)/x}(\psi(\frac{1}{N}x)) + \psi(\frac{i-1}{N}x)) \right].
\]
(18)

Since \( \psi(Nt) < Nt \) for all \( t \in (c_x,d_x) \), the right-hand side of the inequality of (18) is bounded for \( t \in (c_x,d_x) \). From \( A_x = \{ \|f(ux)\| : u \in (c_x,d_x) \} \) is bounded, \( C_x = \{ \|T(ux)\| : u \in (c_x,d_x) \} \) is bounded. Applying Lemma 2.2, \( T \) is linear. □

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