ON THE STABILITY OF APPROXIMATELY ADDITIVE MAPPINGS

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ABSTRACT. In this paper we prove a generalization of the stability of approximately additive mappings in the spirit of Hyers, Ulam and Rassias.

1. Introduction

In 1941 Hyers [3] showed that if $\delta > 0$ and $f : E_1 \rightarrow E_2$, with $E_1$ and $E_2$ Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta,$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \delta,$$

for all $x \in E_1$, and if $f(tx)$ is continuous in $t$ for each fixed $x$, then $T$ is a linear mapping.

Rassias [6] and Gajda [1] gave some generalizations of the Hyers’ result in the following ways: Let $f : E_1 \rightarrow E_2$ be a mapping such that $f(tx)$ is continuous in $t$ for each fixed $x$. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leq \theta,$$

for all $x, y \in E_1$. Then there exists a unique linear mapping $T : E_1 \rightarrow E_2$ such that

$$\frac{\|T(x) - f(x)\|}{\|x\|^p} \leq \frac{2\theta}{2 - 2^p},$$

for all $x \in E_1$.

However, it was showed that the similar result for the case $p = 1$ does not hold (see [7]). Recently, Gavruta [2] also obtained a further generalization of the Hyers-Rassias theorem: Let $G$ be an abelian group and $X$ a Banach space. Denote by $\varphi : G \times G \rightarrow [0, \infty)$ a mapping such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose $f : G \rightarrow X$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y).$$
for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that
\[
\|f(x) - T(x)\| \leq \frac{1}{2} \varphi(x, x) \quad \text{for all } x \in G.
\]

In this paper we generalize the results of Hyers, Rassias and Găvruta.

2. Main results

Throughout this paper, let $a$ be a fixed rational number with $a > 1$. If $a$ is not an integer, there exist unique nonnegative integers $b, p$ and $q$ such that $a = b + q/p, 0 < q/p < 1$ and $(p, q) = 1$. If $a$ is an integer, we let $a = b$. We denote by $G$ a vector space, by $X$ a Banach space, and by $\varphi : G \times G \to [0, \infty)$ a mapping such that
\[
\varphi(x, y) = \sum_{k=0}^{\infty} a^{-k} \varphi(a^k x, a^k y) < \infty
\]
for all $x, y \in G$. In particular, when $a = 2$, we denote $\varphi(x, y)$ by $\varphi_2(x, y)$. We also assume that $\sum_{i=2}^{n} (\cdot) = 0$ if $n < 2$.

**Theorem 2.1.** Let $f : G \to X$ be such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad \text{for all } x, y \in G.
\]
Then there exists a unique additive mapping $T : G \to X$ such that
\[
\|T(x) - f(x)\| \leq a^{-1} \varphi(\frac{q}{p}, bx) + a^{-1} \sum_{p}^{q} \varphi(\frac{i}{p}, \frac{i-1}{p} x)
\]
\[
+ a^{-1} \sum_{i=2}^{b} \varphi(\frac{i}{p}, \frac{i-1}{p} x) + a^{-1} \sum_{i=2}^{b} \varphi(x, (i-1)x),
\]
for all $x \in G$.

**Proof.** We first prove the case that $a$ is not an integer. Putting $y = ix$ in (2), we have
\[
\|f((i+1)x) - f(x) - f(ix)\| \leq \varphi(x, ix), \quad \text{for all } x \in G, i \in N.
\]
Thus
\[
\|f((k+1)x) - (k+1)f(x)\| \leq \sum_{i=1}^{k} \|f((i+1)x) - f(x) - f(ix)\|
\]
\[
\leq \sum_{i=2}^{k+1} \varphi(x, (i-1)x)
\]
for all $x \in G, k \in N$. From (4) it follows that
\[
\|a^{-1} f(bx) - a^{-1} bf(x)\| \leq \sum_{i=2}^{b} a^{-1} \varphi(x, (i-1)x).
\]
Replacing $x$ by $\frac{q}{p} x$ and $y$ by $bx$, (2) gives
\[
\|a^{-1} f(ax) - a^{-1} f(\frac{q}{p} x) - a^{-1} f(bx)\| \leq a^{-1} \varphi(\frac{q}{p}, bx).
\]
Replacing $x$ by $\frac{1}{p}x$ and $k+1$ by $p$, (4) gives

\begin{equation}
\|f(x) - pf\left(\frac{1}{p}x\right)\| \leq \sum_{i=2}^{p} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right).
\end{equation}

Replacing $x$ by $\frac{1}{p}x$ and $k+1$ by $q$, (4) gives

\begin{equation}
\|f\left(\frac{q}{p}x\right) - qf\left(\frac{1}{p}x\right)\| \leq \sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right).
\end{equation}

From (7) and (8), we obtain

\begin{align}
a^{-1} \left| \frac{q}{p}f(x) - f\left(\frac{q}{p}x\right) \right| & \leq a^{-1} \sum_{i=2}^{p} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right) \\
& \quad + \sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right).
\end{align}

From (5), (6) and (9), we get

\begin{align}
\|a^{-1}f(ax) - f(x)\| & \leq a^{-1}\|f(ax) - f\left(\frac{q}{p}x\right)\| \\
& \quad + a^{-1}\|\frac{q}{p}f(x) - f\left(\frac{q}{p}x\right)\| + a^{-1}\|f(bx) - bf(x)\| \\
& \leq a^{-1} \left[ \varphi\left(\frac{q}{p}x, bx\right) + \frac{q}{p} \sum_{i=2}^{p} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right) \\
& \quad + \sum_{i=2}^{q} \varphi\left(\frac{1}{p}x, \frac{i-1}{p}x\right) + \sum_{i=2}^{b} \varphi(x, (i-1)x) \right].
\end{align}

Replacing $x$ by $a^{k-1}x$, (10) gives

\begin{align}
\|a^{-1}f(a^k x) - f(a^{k-1}x)\| & \leq a^{-1} \left[ \varphi(a^{k-1}\frac{q}{p}x, a^{k-1}bx) + \frac{q}{p} \sum_{i=2}^{p} \varphi(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x) \\
& \quad + \sum_{i=2}^{q} \varphi\left(a^{k-1}\frac{1}{p}x, a^{k-1}\frac{i-1}{p}x\right) + \sum_{i=2}^{b} \varphi(a^{k-1}x, a^{k-1}(i-1)x) \right].
\end{align}
From (11) we obtain
\[
\|a^{-n}f(a^n x) - f(x)\| \leq \sum_{k=1}^{n} a^{-k} \|a^{-1}f(a^k x) - f(a^{k-1} x)\|
\]
\[
\leq \sum_{k=1}^{n} a^{-k} \varphi(a^{-1} \frac{q}{p} x, a^{-1} b x)
\]
\[
+ \frac{q}{p} \sum_{i=2}^{p} \sum_{k=1}^{n} a^{-k} \varphi(a^{-1} \frac{1}{p} x, a^{-1} \frac{i-1}{p} x)
\]
\[
+ \sum_{i=2}^{q} \sum_{k=1}^{n} a^{-k} \varphi(a^{-1} \frac{1}{p} x, a^{-1} \frac{i-1}{p} x)
\]
\[
+ \frac{b}{p} \sum_{i=2}^{n} \sum_{k=1}^{n} a^{-k} \varphi(a^{-1} x, a^{-1} (i-1) x)
\]
(12)
for all \(x \in G\).

We claim that the sequence \(\{a^{-n}f(a^n x)\}\) is a Cauchy sequence. Indeed, for \(n > m\), we have
\[
\|a^{-n}f(a^n x) - a^{-m}f(a^m x)\| \leq \sum_{k=m+1}^{n} a^{-k} \|a^{-1}f(a^k x) - f(a^{k-1} x)\|
\]
\[
\leq \sum_{k=m+1}^{n} a^{-k} \varphi(a^{-1} \frac{q}{p} x, a^{-1} b x)
\]
\[
+ \frac{q}{p} \sum_{i=2}^{p} \sum_{k=m+1}^{n} a^{-k} \varphi(a^{-1} \frac{1}{p} x, a^{-1} \frac{i-1}{p} x)
\]
\[
+ \sum_{i=2}^{q} \sum_{k=m+1}^{n} a^{-k} \varphi(a^{-1} \frac{1}{p} x, a^{-1} \frac{i-1}{p} x)
\]
\[
+ \frac{b}{p} \sum_{i=2}^{n} \sum_{k=m+1}^{n} a^{-k} \varphi(a^{-1} x, a^{-1} (i-1) x)
\]
(13)
for all \(x \in G\). Taking the limit in (13) as \(m \to \infty\) we obtain
\[
\lim_{m \to \infty} \|a^{-n}f(a^n x) - a^{-m}f(a^m x)\| = 0.
\]
Since \(X\) is a Banach space, the sequence \(\{a^{-n}f(a^n x)\}\) converges for every \(x \in G\).

Denote
\[
T(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^n}.
\]
From (2) we have
\[
\|a^{-n}f(a^n x + a^n y) - a^{-n}f(a^n x) - a^{-n}f(a^n y)\|
\]
\[
\leq a^{-n} \varphi(a^n x, a^n y) \quad \text{for all } x, y \in G.
\]
(14)
From (1) it follows that
\[
\lim_{n \to \infty} a^{-n} \varphi(a^n x, a^n y) = 0.
\]
Then (14) implies

$$||T(x + y) - T(x) - T(y)|| = 0.$$ 

To prove (3), taking the limit in (12) as $n \to \infty$, we obtain

$$||T(x) - f(x)|| \leq a^{-1} \tilde{\varphi}(\frac{q}{p}x, bx) + a^{-1} \frac{q}{p} \sum_{i=2}^{p} \tilde{\varphi}(\frac{1}{p}, \frac{i - 1}{p} x)$$

$$+ a^{-1} \frac{q}{p} \sum_{i=2}^{p} \tilde{\varphi}(\frac{1}{p}, \frac{i - 1}{p} x) + a^{-1} \sum_{i=2}^{b} \tilde{\varphi}(x, (i - 1)x) \quad \text{for all } x \in G.$$ 

It remains to show that $T$ is uniquely defined. Let $F : G \to X$ be another additive mapping satisfying (3). Then

$$||T(x) - F(x)|| = ||a^{-n}T(a^n x) - a^{-n}F(a^n x)||$$

$$\leq ||a^{-n}T(a^n x) - a^{-n}f(a^n x)|| + ||a^{-n}f(a^n x) - a^{-n}F(a^n x)||$$

$$\leq 2 \left[ a^{-n-1} \tilde{\varphi}(a^n \frac{q}{p}x, a^n bx) + a^{-n-1} \frac{q}{p} \sum_{i=2}^{p} \tilde{\varphi}(a^n \frac{1}{p}, a^n \frac{i - 1}{p} x)$$

$$+ a^{-n-1} \sum_{i=2}^{p} \tilde{\varphi}(a^n \frac{1}{p}, a^n \frac{i - 1}{p} x) + a^{-n-1} \sum_{i=2}^{b} \tilde{\varphi}(a^n x, a^n (i - 1)x) \right]$$

$$= 2a^{-1} \left[ \sum_{j=n}^{\infty} a^{-j} \varphi(a^j \frac{q}{p}x, a^j bx) + \frac{q}{p} \sum_{i=2}^{p} \sum_{j=n}^{\infty} a^{-j} \varphi(a^j \frac{1}{p}, a^j \frac{i - 1}{p} x)$$

$$+ \sum_{i=2}^{p} \sum_{j=n}^{\infty} a^{-j} \varphi(a^j \frac{1}{p}, a^j \frac{i - 1}{p} x) + \sum_{i=2}^{b} \sum_{j=n}^{\infty} a^{-j} \varphi(a^j x, a^j (i - 1)x) \right].$$

Thus

$$||T(x) - F(x)|| = ||a^{-n}T(a^n x) - a^{-n}F(a^n x)||$$

$$\leq 2a^{-1} \left[ \sum_{j=n}^{\infty} a^{-j} \varphi(a^j \frac{q}{p}x, a^j bx) + \frac{q}{p} \sum_{i=2}^{p} \sum_{j=n}^{\infty} a^{-j} \varphi(a^j \frac{1}{p}, a^j \frac{i - 1}{p} x)$$

$$+ \sum_{i=2}^{p} \sum_{j=n}^{\infty} a^{-j} \varphi(a^j \frac{1}{p}, a^j \frac{i - 1}{p} x) + \sum_{i=2}^{b} \sum_{j=n}^{\infty} a^{-j} \varphi(a^j x, a^j (i - 1)x) \right]$$

for all $x \in G$. Taking the limit (15) as $n \to \infty$ we obtain

$$T(x) = F(x) \quad \text{for all } x \in G.$$ 

Now we prove the case: $a = b$. From (5) we obtain

$$||a^{-1}f(ax) - f(x)|| \leq \sum_{i=2}^{a} a^{-1} \varphi(x, (i - 1)x).$$

Hence we have

$$||a^{-n}f(a^n x) - f(x)|| \leq \sum_{i=2}^{a} \sum_{k=1}^{n} a^{-k} \varphi(a^{k-1} x, a^{k-1} (i - 1)x)$$
for all \( x \in G \). Denote 

\[
T(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^n}.
\]

Taking the limit in (12') as \( n \to \infty \), we obtain 

\[
\|T(x) - f(x)\| \leq a^{-1} \sum_{i=2}^{a} \tilde{\varphi}(x, (i-1)x) \quad \text{for all } x \in G.
\]

It is easy to show that \( T \) is uniquely defined. \( \square \)

**Lemma 2.2.** Let \( T : G \to X \) be an additive mapping and let \( x_0 \in G \). If there are an interval \((c, d)\) and \( y \in G \) such that 

\[
C = \{ \|T(ux_0 + y)\| : u \in (c, d) \}
\]

is bounded, then 

\[
T(ux_0) = uT(x_0) \quad \text{for all real numbers } u.
\]

**Proof.** Assume that there exists a real number \( r \) such that \( T(rx_0) \neq rT(x_0) \). Let \( m = \|T(rx_0) - rT(x_0)\| \). Let \( \{r_n\} \) be a rational number sequence such that 

\[
\|(r - r_n)T(x_0)\| \leq m/2 \quad \text{and} \quad \lim_{n \to \infty} r_n = r.
\]

Choose a rational number sequence \( \{r'_n\} \) such that \( r'_n(r - r_n) \in (c, d) \) and \( \lim_{n \to \infty} r'_n = \infty \). Since 

\[
\|T(r'_n(r - r_n)x_0 + y) - r'_n rT(x_0) + r'_n r_n T(x_0) - T(y)\|
\]

\[
= \|r'_n T(rx_0) - r'_n rT(x_0)\|
\]

\[
= |r'_n|m,
\]

we have 

\[
\|T(r'_n(r - r_n)x_0 + y)\| \geq |r'_n|(m/2) - \|T(y)\| \quad \text{for all } n \in N.
\]

This contradicts the fact that \( C \) is bounded. \( \square \)

**Remarks.** In Theorem 2.1, (a) if there exist an interval \((c, d)\) and \( \varepsilon > 0 \) such that 

\[
\{ \|f(ux_0)\| : u \in (c, d) \} \quad \text{and} \quad \{ \|f(sx_0, tx_0)\| : d/(p+\varepsilon) \leq s, t \leq (b-1)d \}
\]

are bounded for a fixed \( x_0 \), then \( T(rx_0) = rT(x_0) \) for all real numbers \( r \). In fact, choose an interval \((c', d') \subseteq (c, d) \cap (dp/(p+\varepsilon), d) \). From (3) we obtain \( C = \{ \|T(ux_0)\| : u \in (c', d') \} \) is bounded.

(b) If \( G \) is a normed space and \( f(tx) \) is continuous in \( t \) for each fixed \( x \) and \( \tilde{\varphi} \) is bounded on \( G \times G \), then \( T \) is linear by (a).

**Theorem 2.3.** Let \( G \) be a normed space and \( f \) be as in Theorem 2.1. If \( f \) is bounded for some open subset \( A \) of \( G \) and \( \tilde{\varphi} \) is bounded on \( G \times G \), then there exists a unique continuous linear mapping \( T : G \to X \) such that 

\[
\|T(x) - f(x)\| \leq a^{-1} \tilde{\varphi}(\frac{q}{p}, bx) + a^{-1} \sum_{i=2}^{p} \tilde{\varphi}(\frac{1}{p}, \frac{i-1}{p}, x) + a^{-1} \sum_{i=2}^{b} \tilde{\varphi}(x, (i-1)x) \quad \text{for all } x \in G.
\]

**Proof.** Let \( T \) be a mapping as in Theorem 2.1. From (4) we obtain that \( T \) is bounded on \( A \). Let \( z \) be an interior point of \( A \). For each fixed \( x \in G \) there exists an interval \((c, d)\) such that \( \{ux + z : u \in (c, d)\} \subseteq A \). By the preceding remark, \( T \) is linear. Since \( T \) is bounded on open set \( A \), \( T \) is continuous. \( \square \)
Corollary 2.4. Let $G$ and $f$ be as in Theorem 2.3. If $f$ is bounded for some open subset $A$ of $G$ and $\tilde{\varphi}_2$ is bounded on $A \times A$, then there exists a unique continuous linear mapping $T : G \to X$ such that

$$
\|T(x) - f(x)\| \leq 2^{-1} \tilde{\varphi}_2(x,x) \quad \text{for all } x \in G.
$$

Proof. By Theorem 2.1, there exists a unique additive mapping $T : G \to X$ such that $\|T(x) - f(x)\| \leq 2^{-1} \tilde{\varphi}_2(x,x)$ for all $x \in A$. We can apply the similar method as in Theorem 2.3. \hfill \Box

Theorem 2.5. Let $f : E_1 \to E_2$ be a mapping with $E_1$ and $E_2$ Banach spaces. If for each fixed $x, y \in E_1$ there exist real numbers $\theta_{xy}, p_{xy}, s_{xy}$ such that $0 \leq p_{xy} < 1$ and

$$
\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\|tx\|^{p_{xy}} + \|ty\|^{p_{xy}}) \quad \text{for } t > s_{xy},
$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$
\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^{p_{xx}}}{2 - 2p_{xx}} \quad \text{for } t > s_{xx}
$$

for all $x \in E_1$. In particular, if for a fixed $x_0 \in E_1$ there exist real numbers $M_{x_0}, s_{x_0}$ such that $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then $T(rx_0) = rT(x_0)$ for all real numbers $r$.

Proof. Let

$$
\varphi(tx, ty) = \|f(tx + ty) - f(tx) - f(ty)\|.
$$

Then $\varphi_2(x, y) < \infty$ for all $x, y \in E_1$ and $2^{-1} \tilde{\varphi}_2(tx, tx) < 2\theta_{xx}\|tx\|^{p_{xx}}/(2 - 2p_{xx})$ for $t > s_{xx}$ for each $x \in E_1$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$
\|T(tx) - f(tx)\| \leq \frac{2\theta_{xx}\|tx\|^{p_{xx}}}{2 - 2p_{xx}} \quad \text{for } t > s_{xx}
$$

for all $x, y \in E_1$. If for a fixed $x_0 \in E_1$ there exist real numbers $M_{x_0}, s_{x_0}$ with $\|f(tx_0)\|/t < M_{x_0}$ for $t > s_{x_0}$, then

$$
\|T(tx_0)\| \leq \frac{2\theta_{x_0x_0}\|tx_0\|^{p_{x_0x_0}}}{2 - 2p_{x_0x_0}} + M_{x_0}t \quad \text{for } t > \max(s_{x_0x_0}, s_{x_0}).
$$

Therefore $\{\|T(ux_0)\| : u \in (\max(s_{x_0x_0}, s_{x_0}), 2\max(s_{x_0x_0}, s_{x_0}))\}$ is bounded. Apply Lemma 2.2. \hfill \Box

The following theorem is a generalization of Theorem 1 in [4].

Theorem 2.6. Let a function $\psi : R^+ \to R^+$ satisfy

(i) $\psi(ts) \leq \psi(t)\psi(s)$ for all $t, s \in R^+$ and

(ii) $\lim_{t \to \infty} \psi(t)/t = 0$

and let $f : E_1 \to E_2$ be a mapping with $E_1$ and $E_2$ Banach spaces. If for each fixed $x, y \in E_1$, there exists a real number $\theta_{xy}$ such that

$$
\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}(\psi(\|tx\|) + \psi(\|ty\|)) \quad \text{for all } t \in R^+,
$$

then there exists a unique continuous mapping $T : G \to X$ such that $\|T(x) - f(x)\| \leq \theta_{xy}^{1/2}(\psi(\|x\|) + \psi(\|y\|))$ for all $x, y \in G$. \hfill \Box
then there exist a unique additive mapping $T : E_1 \to E_2$ and a rational number $a > 1$ such that

$$\|f(tx) - T(tx)\| \leq a^{-1}(1 - \frac{\psi(a)}{a}) \left[ \theta_{(q/p)x, bx}(\psi(\|\frac{q}{p}tx\|) + \psi(\|btx\|)) \ight. \\
+ \frac{q}{p} \sum_{i=2}^{\infty} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx)) \\
+ \frac{q}{p} \sum_{i=2}^{\infty} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx)) \\
\left. + \frac{b}{\psi(a)/a} \sum_{i=2}^{\infty} \theta_{x,(i-1)x}(\psi(\|tx\|) + \psi(\|tx\|)) \right].$$

(17)

In particular, if for each fixed $x \in E_1$ there exist positive real numbers $c_x, d_x$ such that $A_x = \{\|f(u)x\| : u \in (c_x, d_x)\}$ is bounded, then $T$ is linear.

**Proof.** From (ii), there exists a rational number $a$ such that $\psi(a) < a$. Let $\varphi(x, y) = \|f(tx + ty) - f(tx) - f(ty)\|$. From (i) we get

$$\varphi(tx, ty) = \sum_{n=1}^{\infty} a^{-n} \varphi(a^n tx, a^n ty) \\
\leq \sum_{n=1}^{\infty} a^{-n} \theta_{x,y}(\psi(\|a^n tx\|) + \psi(\|a^n ty\|)) \\
\leq \sum_{n=1}^{\infty} (\psi(a)/a)^n \theta_{x,y}(\psi(\|tx\|) + \psi(\|ty\|)) \\
= \frac{\theta_{x,y}(\psi(\|tx\|) + \psi(\|ty\|))}{1 - \psi(a)/a} < \infty$$

for all $x, y \in E_1$ and $t \in R^+$. By Theorem 2.1 there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(tx) - T(tx)\| \leq a^{-1}(1 - \frac{\psi(a)}{a}) \left[ \theta_{(q/p)x, bx}(\psi(\|\frac{q}{p}tx\|) + \psi(\|btx\|)) \ight. \\
+ \frac{q}{p} \sum_{i=2}^{\infty} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx)) \\
+ \frac{q}{p} \sum_{i=2}^{\infty} \theta_{(1/p)x,(i-1)x/p}(\psi(\|\frac{1}{p}tx\|) + \psi(\|i - 1\|tx)) \\
\left. + \frac{b}{\psi(a)/a} \sum_{i=2}^{\infty} \theta_{x,(i-1)x}(\psi(\|tx\|) + \psi(\|tx\|)) \right].$$

(17)

for $x \in E_1$ and $t \in R^+$. Since $\lim_{t \to \infty} \psi(t)/t = 0$, there exists a positive number $M$ such that $\psi(t)/t < 1$ for all $t > M$. Choose $N$ such that $c_x N > M$. From (17)
we have

\[
\|f(tx) - T(tx)\| \leq a^{-1}\psi(tN)(1 - \frac{\psi(a)}{a})\left[\theta_{(q/p)x, bx}(\psi(\|\frac{q}{pN}x\|) + \psi(\|\frac{b}{N}x\|))
\right.
\]
\[\left. + \frac{q}{p}\sum_{i=2}^{p}\theta_{(1/p)x, (i-1)x/p}(\psi(\|\frac{1}{pN}x\|) + \psi(\|\frac{i-1}{pN}x\|))
\right.
\]
\[\left. + \sum_{i=2}^{q}\theta_{(1/p)x, (i-1)x/p}(\psi(\|\frac{1}{pN}x\|) + \psi(\|\frac{i-1}{pN}x\|))
\right.
\]
\[\left. + \sum_{i=2}^{b}\theta_{x, (i-1)x}(\psi(\|\frac{1}{N}x\|) + \psi(\|\frac{i-1}{N}x\|))\right].
\]

(18)

Since \(\psi(Nt) < Nt\) for all \(t \in (c_x, d_x)\), the right-hand side of the inequality of (18) is bounded for \(t \in (c_x, d_x)\). From \(A_x = \{\|f(ux)\| : u \in (c_x, d_x)\}\) is bounded, \(C_x = \{\|T(ux)\| : u \in (c_x, d_x)\}\) is bounded. Applying Lemma 2.2, \(T\) is linear. \(\square\)

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