SPLITTINGS OF BANACH SPACES
INDUCED BY CLIFFORD ALGEBRAS

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Abstract. Let $H$ be an infinite-dimensional Hilbert space of density character $m$. By representing $H$ as a module over an appropriate Clifford algebra, it is proved that $H$ possesses a family $\{A_\alpha\}_{\alpha \in m}$ of proper closed nonzero subspaces such that

$$d(S_{A_\alpha}, S_{A_\beta}) = d(S_{A_\alpha^+}, S_{A_\beta}) = d(S_{A_\alpha^-}, S_{A_\beta^+}) = \sqrt{2 - \sqrt{2}} \quad (\alpha \neq \beta).$$

Analogous results are proved for $L_p$ spaces and for $c_0(X)$ and $\ell_p(X)$ ($1 \leq p \leq \infty$) when $X$ is an arbitrary nonzero Banach space.

1. Introduction and notation

Let us begin with some notation. Let $C_1$ and $C_2$ be nonempty subsets of a Banach space $(X, \| \cdot \|)$. The distance between $C_1$ and $C_2$, denoted $d(C_1, C_2)$, is defined as follows:

$$d(C_1, C_2) = \inf\{ \| c_1 - c_2 \| : c_1 \in C_1, c_2 \in C_2 \}.$$

For a closed subspace $A$ of $X$, its unit sphere, denoted $S_A$, is the set $\{ a \in A : \| a \| = 1 \}$.

A decomposition of $X$ into the Banach space direct sum $X = A \oplus B$ of two nonzero closed subspaces $A$ and $B$ will be called a splitting of $X$, denoted $(A, B)$. For a given family $\{(A_\gamma, B_\gamma) : \gamma \in \Gamma\}$ of splittings of $X$, a convenient measure of the extent to which the splittings in this family differ from one another is afforded by the quantity

$$\delta = \inf\{ d(S_{A_\alpha}, S_{A_\beta}), d(S_{B_\alpha}, S_{B_\beta}), d(S_{A_\alpha}, S_{B_\beta}) : \alpha, \beta, \gamma \in \Gamma \quad (\alpha \neq \beta) \}.$$ 

If $\delta > 0$, then the unit spheres of all the subspaces occurring in the splittings are separated from each other by a distance $\delta$. Such a family of splittings will be said to be well-separated.

This paper proves the existence of infinite well-separated families of splittings for certain Banach spaces. First the case of an infinite-dimensional Hilbert space $H$ is considered; here it is more natural to consider only orthogonal splittings (i.e. orthogonal decompositions) of $H$. It is proved that if $H$ has density character $m$, then there exists a family of orthogonal splittings of $H$ of cardinality $m$ for which $\delta = \sqrt{2 - \sqrt{2}}$, which is best possible. The Hilbert space argument is then
generalized to prove that if $X$ is an arbitrary nonzero Banach space, then $c_0(X)$ and $\ell_p(X)$ $(1 \leq p \leq \infty)$ admit infinite well-separated families of contractively complemented splittings.

The main idea in the proof is to represent $H$ (or $\ell_p(X)$) as a module over an appropriate infinite-dimensional Clifford algebra. The existence of the required splittings is then a consequence of algebraic identities in the Clifford algebra. The proof is self-contained and presupposes only an acquaintance with the terminology of elementary abstract algebra.

The Banach space notation and terminology employed throughout are standard. Let us only recall that the space $\ell_p(X)$ $(1 \leq p \leq \infty)$ is the space of sequences $\langle x_n \rangle$ $(x_n \in X)$ equipped with the norm $\| \langle x_n \rangle \| = \left( \sum \| x_n \|^p \right)^{1/p}$ for $p < \infty$; $\sup_n \| x_n \|$ for $p = \infty$.

The space $c_0(X)$ is the subspace of $\ell_\infty(X)$ whose elements consist of sequences which tend to zero in norm.

The proofs are valid for both real and complex Banach spaces: the underlying field of scalars (either $\mathbb{R}$ or $\mathbb{C}$) will be denoted by $F$.

2. SPLITTINGS OF HILBERT SPACES

Let $H$ be a separable infinite-dimensional Hilbert space. For $2 \leq n \leq \infty$, define $\delta_n$ as follows:

$$\delta_n = \sup \{ \inf \{ d(S_{A_j}, S_{A_k}), d(S_{A_j'}, S_{A_k}), d(S_{A_j'}, S_{A_k'}) : 0 \leq j, k < n, j \neq k \} \},$$

where the supremum is taken over all $n$-tuples $\{A_j\}_{0 \leq j < n}$ of proper closed nonzero subspaces of $H$. Clearly, $\{\delta_n\}_{n \geq 1}$ is a decreasing sequence of nonnegative numbers with $0 \leq \delta_\infty \leq \lim \delta_n$.

We prove below (Theorem 3) that $\delta_n = \sqrt{2 - \sqrt{2}}$ for all $2 \leq n \leq \infty$.

**Proposition 1.** Let $A$ be a subspace of $H$ and let $P$ be the orthogonal projection onto $A$.

(a) If $x \in S_H$, then

$$\min \{d(x, S_A), d(x, S_{A'})\} \leq \sqrt{2 - \sqrt{2}}$$

with equality if and only if $\|Px\| = 1/\sqrt{2}$ (in which case $d(x, S_A) = d(x, S_{A'}) = \sqrt{2 - \sqrt{2}}$).

(b) Let $B$ be the closed linear span of an orthonormal sequence $\{e_n\}$. Then

$$d(S_A, S_B) = d(S_{A'}, S_B) = \sqrt{2 - \sqrt{2}}$$

if and only if $\{\sqrt{2}P\}e_n$ is an orthonormal sequence in $H$.

**Proof.** (a) Let $a = Px$ and $a' = (I - P)x$. Then $1 = \|x\|^2 = \|a\|^2 + \|a'\|^2$, and so $\max(\|a\|, \|a'\|) \geq 1/\sqrt{2}$. Without loss of generality, we shall assume that $\|a\| \geq 1/\sqrt{2}$. Let $y \in S_A$. Then

$$\|y - x\|^2 = \|y - a\|^2 + \|a'\|^2.$$
The distance \( \|y-a\| \) is minimized when \( y = a/\|a\| \); in this case, \( \|y-a\| = 1 - \|a\| \), and so
\[
\|y-x\|^2 = (1 - \|a\|)^2 + \|a\|^2
= (1 - \|a\|)^2 + (1 - \|a\|^2)
= 2 - 2\|a\|
\leq 2 - \sqrt{2},
\]
since we are assuming that \( \|a\| \geq 1/\sqrt{2} \). It follows that \( d(x,S_A) \leq \sqrt{2 - \sqrt{2}} \), with equality if and only if \( \|a\| = \|a'| = 1/\sqrt{2} \).

(b) Suppose that (2) holds. Then from (a) we deduce that \( d(x,S_A) = d(x,S_{A^\perp}) = \sqrt{2 - \sqrt{2}} \) for all \( x \in S_B \), whence \( \|Px\| = 1/\sqrt{2} \) (by (a) again) for all \( x \in S_B \). It follows that \( \sqrt{2}P \) is an isometry from \( B \) into \( A \), and hence that \( \{\sqrt{2}P(e_n)\} \) is an orthonormal sequence in \( A \). Conversely, if \( \{\sqrt{2}P(e_n)\} \) is an orthonormal sequence in \( A \), then \( \|Px\| = 1/\sqrt{2} \) for all \( x \in S_B \), and so (by (a)) \( d(x,S_A) = d(x,S_{A^\perp}) = \sqrt{2 - \sqrt{2}} \) for all \( x \in S_B \), which gives (2).

Let \( \{e_n\} \) be an orthonormal basis for \( H \), let \( A \) be the closed linear span of the orthonormal sequence \( \{e_{2n}\}_{n \geq 1} \), and let \( B \) be the closed linear span of the orthonormal sequence \( \{(1/\sqrt{2})(e_{2n} + e_{2n-1})\} \). It follows from Proposition 1 that
\[
d(S_A,S_B) = d(S_{A^\perp},S_B) = d(S_{A^\perp},S_{B^\perp}) = \sqrt{2 - \sqrt{2}},
\]
and also that this pair of splittings is the best possible in the sense that the constant \( \sqrt{2 - \sqrt{2}} \) cannot be improved (i.e., increased). This proves that \( \delta_2 = \sqrt{2 - \sqrt{2}} \).

In fact, (3) determines \( A \) and \( B \) uniquely up to an isomorphism of \( H \), as the following result shows.

**Corollary 2.** Suppose that \( H \) is a separable infinite-dimensional Hilbert space. Let \( A \) and \( B \) be closed subspaces of \( H \) which satisfy equality (3). Then there exist orthonormal bases \( \{e_n\} \) and \( \{f_n\} \) of \( A \) and \( A^\perp \) such that \( \{(1/\sqrt{2})(e_n + f_n)\} \) and \( \{(1/\sqrt{2})(e_n - f_n)\} \) are orthonormal bases of \( B \) and \( B^\perp \).

**Proof.** Let \( P \) be the orthogonal projection of \( H \) onto \( A \) and let \( Q = I - P \). Let \( \{g_n\} \) be an orthonormal basis for \( B \). Then, from Proposition 1 and (3), it follows that \( \{e_n\} \) and \( \{f_n\} \) are orthonormal sequences in \( A \) and \( A^\perp \), where \( e_n = (\sqrt{2})P(g_n) \) and \( f_n = (\sqrt{2})Q(g_n) \). Let us show that \( \{e_n\} \) is in fact an orthonormal basis of \( A \). So suppose that \( e \in A \) and that \( e,e_n \) = 0 for all \( n \). Then \( \langle e,P(y) \rangle = 0 \) for all \( y \in B \). From Proposition 1 and (3) we have \( e = y + z \) where \( y \in B, \ z \in B^\perp \), and \( \|y\| = \|z\| = (1/\sqrt{2})\|e\| \). Hence \( e = P(e) = P(y) + P(z) \), and so
\[
\frac{\|e\|^2}{4} = \|P(z)\|^2 = \|e\|^2 + \|P(y)\|^2 = \|e\|^2 + \|z\|^2.
\]
Thus \( e = 0 \), which implies that \( \{e_n\} \) is an orthonormal basis of \( A \). Similarly \( \{f_n\} \) is an orthonormal basis of \( A^\perp \). Note that \( g_n = (1/\sqrt{2})(e_n + f_n) \) and recall that \( \{g_n\} \) is an orthonormal basis of \( B \). Clearly, \( \{(1/\sqrt{2})(e_n - f_n)\} \) is an orthonormal basis of \( B^\perp \).
Theorem 3. Let $H$ be a separable infinite-dimensional Hilbert space. There exists a sequence $\{A_n\}$ of subspaces of $H$ such that

$$d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{A_m}^\perp) = d(S_{A_m}^\perp, S_{A_m}^\perp) = \sqrt{2 - \sqrt{2}} \quad (n \neq m).$$

In particular, $\delta_n = \sqrt{2 - \sqrt{2}}$ for all $2 \leq n \leq \infty$.

Proof. We begin our construction by defining a group $G$ which is generated by an element $-1$ and by the sequence of symbols $\{e_n\}_{n \geq 1}$ satisfying the generating relations:

$$e_n^2 = -1 \quad \text{and} \quad (-1)^2 = 1.$$ 

The element $-1$ belongs to the center of $G$ and we define for any $g \in G$:

$$-g = (-1)(g) = (g)(-1).$$

Multiplication among the $e_n$’s is defined to be anti-commutative:

$$e_m e_n = -e_n e_m \quad (n \neq m).$$

The elements of $G$ consist of finite strings of the generators. For any finite subset $I \subset \mathbb{N}$, let $e_I = e_{n_1} e_{n_2} \cdots e_{n_k}$, where $I = \{n_1, n_2, \ldots, n_k\}$ and $1 \leq n_1 < n_2 < \cdots < n_k$, and let $e_I = 1$ for $I = \emptyset$. Finally, setting $W = \{e_I : I \text{ is a finite subset of } \mathbb{N}\}$, we can list the group elements thus:

$$G = \{w, -w : w \in W\}.$$ 

Consider the collection $C$ of all finite sums of the form $\sum_{w \in W} \lambda_w w$, where $\lambda_w \in \mathbb{F}$. Next define scalar multiplication on $G \setminus W = \{-w : w \in W\}$ as one would expect:

$$\lambda(-w) = (-\lambda)w \quad (w \in W, \lambda \in \mathbb{F}).$$

Then $C$ becomes an algebra over $\mathbb{F}$ with multiplication defined in the obvious way using the group multiplication of $G$ and the distributive law. In fact, if we denote by $V$ the inner product space over $\mathbb{F}$ which has orthonormal basis $\{e_n\}$, then $C$ is the universal Clifford algebra associated to $V$. We refer the reader to [3] for a discussion of Clifford algebras and for a proof of the associativity of the algebra multiplication.

Let the symbols $x$ and $y$ be the generators of a free left-module over $C$, which we shall denote by $M$. It follows that every $m \in M$ is uniquely expressible in the form $c_1 x + c_2 y$, where $c_1, c_2 \in C$. Note that the elements of $M$ are of the form

$$m = \left(\sum_{w \in W} \lambda_w w\right)x + \left(\sum_{w \in W} \mu_w w\right)y = \sum_{w \in W} (\lambda_w)w x + \sum_{w \in W} (\mu_w)w y,$$

where the sums are finite and the $\lambda$’s and $\mu$’s belong to $\mathbb{F}$. Next we turn the module $M$ into an inner product space over $\mathbb{F}$ by taking the set $\Xi = \{wx, wy : w \in W\}$ to be an orthonormal spanning set. Let $H$ be the completion of this inner product space, so that $H$ is a separable infinite-dimensional Hilbert space with orthonormal basis $\Xi$. Note that the inner product $\langle \cdot, \cdot \rangle$ on $H$ is defined by the relations

$$\langle wx, w'y \rangle = 0 \quad (w, w' \in W)$$

and

$$\langle wx, w'x \rangle = \begin{cases} 1, & \text{for } w = w', \\ 0, & \text{for } w \neq w'. \end{cases}$$
For each \( w' \in W \) there exists a permutation \( \pi_{w'} \) of \( W \) such that the mapping \( w \mapsto w'w \) is given by
\[
 w'w = \pm \pi_{w'}(w),
\]
where the choice of signs depends, of course, on the element \( w \). For convenience, let us call such a mapping a \textit{sign-changing permutation}. It follows that the linear operator corresponding to the action of \( w' \in W \) on \( H \) given by
\[
 (4) \quad w' \left\{ \sum_{w} (\lambda_w w x + \gamma_w w y) \right\} = \sum_{w} \left\{ \lambda_w (w'wx) + \gamma_w (w'wy) \right\}
\]
is in fact an isomorphism of \( H \), as it is the linear extension of a sign-changing permutation of the orthonormal basis \( \Xi \). We may now regard \( H \) as a left \( C \)-module by extending the action of (4) to the whole of \( C \) by linearity.

Let \( A_n \) be the norm-closed left \( C \)-submodule generated by the element \( x + e_n y \). One easily sees that
\[
 \Xi_n = \left\{ \frac{wx + (we_n)y}{\sqrt{2}} : w \in W \right\}
\]
is an orthonormal basis for \( A_n \), and that
\[
 \Xi'_n = \left\{ \frac{wx - (we_n)y}{\sqrt{2}} : w \in W \right\}
\]
is an orthonormal basis for \( A^+_n \). Having defined the subspaces that make up our sequence of splittings, it remains to show that the distances \( d(S_{A_n}, S_{A_m}) \) and \( d(S^+_n, S^+_m) \) are all equal to \( \sqrt{2} - \sqrt{2} \) for \( n \neq m \).

Let \( P_n \) denote the orthogonal projection from \( H \) onto \( A_n \). To prove that \( d(S_{A_n}, S_{A_m}) = \sqrt{2} - \sqrt{2} \), we shall invoke Proposition 1. It suffices to prove that \( P_n \) maps the basis \( \Xi_m \) of \( A_m \) onto an orthonormal sequence of vectors in \( A_n \), all having norm \( \sqrt{2} \). To this end, let us express the vector \((x + e_m y)/\sqrt{2}\) in the form
\[
 \frac{x + e_m y}{\sqrt{2}} = \left( \frac{1 - e_m e_n}{2} \right) \left( \frac{x + e_n y}{\sqrt{2}} \right) + \left( \frac{1 + e_m e_n}{2} \right) \left( \frac{x - e_n y}{\sqrt{2}} \right).
\]
Then, for \( w \in W \), we have
\[
 \frac{wx + (we_m)y}{\sqrt{2}} = \left( \frac{w(1 - e_m e_n)}{2} \right) \left( \frac{x + e_n y}{\sqrt{2}} \right) + \left( \frac{w(1 + e_m e_n)}{2} \right) \left( \frac{x - e_n y}{\sqrt{2}} \right).
\]
Note that in this form the vector \( (wx + (we_m)y)/\sqrt{2} \) is written as the sum of vectors from \( A_n \) and \( A^+_n \), and so it is easy to compute its projection onto \( A_n \):
\[
 (5) \quad P_n \left( \frac{wx + (we_m)y}{\sqrt{2}} \right) = \frac{w(1 - e_m e_n)}{2} \left( \frac{x + e_n y}{\sqrt{2}} \right)
 = \frac{1}{2\sqrt{2}} \left\{ wx - (we_m e_n)x + (we_n) y - (we_m e_n) y \right\}
 = \frac{1}{2\sqrt{2}} \left\{ wx - (we_m e_n)x + (we_n) y + (we_m) y \right\}.
\]
First observe that from (5)
\[ \left\| P_n \left( \frac{wx + (we_m)y}{\sqrt{2}} \right) \right\| = \frac{1}{2\sqrt{2}} (1^2 + 1^2 + 1^2 + 1^2)^{1/2} = \frac{1}{\sqrt{2}}, \]
which proves that \( P_n \) maps each member of \( \Xi_m \) onto a vector of norm \( 1/\sqrt{2} \).

Now let us show that \( P_n \) maps \( \Xi_m \) onto an orthogonal sequence; that is, that
\[ \left\langle P_n \left( \frac{wx + (we_m)y}{\sqrt{2}} \right), P_n \left( \frac{w'x + (w'e_m)y}{\sqrt{2}} \right) \right\rangle = 0 \]
for all \( w \neq w' \).

Replacing \( w \) by \( w' \) in (5), we get
\[ P_n \left( \frac{w'x + (w'e_m)y}{\sqrt{2}} \right) = \frac{1}{2\sqrt{2}} \left\{ w'x - (w'e_m)e_n \right\} x + (w'e_n) y + (w'e_m)y \}.

The four vectors occurring on the right-hand side of (5) and the four on the right-hand side of (6) belong to the orthonormal basis \( \Xi \) of \( H \). If these sets of basis vectors are disjoint, then clearly the inner product is zero. These sets of basis vectors will be disjoint unless
\[ w' = \pm we_m e_n \quad \text{or} \quad w = \pm w'e_m e_n. \]

It remains to show that the inner product is zero if one of the two conditions of (7) holds. Without loss of generality, let us assume that \( w' = we_m e_n \). In this case, we have
\[
w'x - (w'e_m e_n)x + (w'e_n) y + (w'e_m)y
= (we_m e_n)x - (we_m e_n e_m e_n)x + (we_m e_n y) + (we_m e_n e_m)y
= (we_m e_n)x - (we_m (-e_m e_n) e_n)x + (we_m (-1))y + (we_m (-e_m e_n))y
= (we_m e_n)x + (we_m (-1))x - (we_m)y - (we_m(-1)e_n)y
= (we_m e_n)x + wx - (we_m)y + (we_n)y,
\]
and it follows from (6) that
\[ P_n \left( \frac{w'x + (w'e_m)y}{\sqrt{2}} \right) = \frac{1}{2\sqrt{2}} \left\{ wx + (we_m e_n)x - (we_m)y + (we_n)y \right\}. \]

Computing the inner product, we get
\[
\left\langle P_n \left( \frac{wx + (we_m)y}{\sqrt{2}} \right), P_n \left( \frac{w'x + (w'e_m)y}{\sqrt{2}} \right) \right\rangle
= \frac{1}{8} \left\{ \frac{wx - (we_m e_n)x + (we_m)y + (we_n)y}{\sqrt{2}} \right\}
= \frac{1}{8} \left\{ (wx, wx) - \left( (we_m e_n)x, (we_m e_n)x \right) - \left( (we_m y, (we_m)y \right) + \left( (we_n y, (we_n) y \right) \right\}
= \frac{1}{8} \left\{ \|wx\|^2 - \| (we_m e_n)x \|^2 - \| (we_m)y \|^2 + \| (we_n)y \|^2 \right\}
= \frac{1}{8} (1 - 1 - 1 + 1) = 0.
\]
Thus, $P_n$ maps $\Xi_m$ onto an orthogonal sequence of vectors in $A_n$, which completes the proof that $d(S_{A_n}, S_{A_n}) = \sqrt{2 - \sqrt{2}}$. Similar computations prove that $d(S_{A_n}, S_{A_n^2})$ and $d(S_{A_n^2}, S_{A_n^2})$ are also equal to $\sqrt{2 - \sqrt{2}}$.  

The proof of Theorem 3 readily extends to nonseparable Hilbert spaces to yield the result stated in the abstract (or a paraphrase thereof).

**Theorem 4.** Let $\Gamma$ be an infinite set. Then there exists a family $\{A_\gamma\}_{\gamma \in \Gamma}$ of subspaces of $\ell_2(\Gamma)$ such that

$$d(S_{A_\gamma}, S_{A_\delta}) = d(S_{A_\gamma}, S_{A_\delta^2}) = d(S_{A_\delta^2}, S_{A_\delta}) = \sqrt{2 - \sqrt{2}}$$

for all $\gamma, \delta \in \Gamma$ with $\gamma \neq \delta$.

**Proof.** The proof is analogous to the proof of Theorem 3, with the modification that the group $G$ is now generated by the (possibly uncountable) set $\{e_{\gamma}\}_{\gamma \in \Gamma}$. We construct the Clifford algebra $C$ that is associated to the inner product space which has $\{e_{\gamma}\}_{\gamma \in \Gamma}$ as an orthonormal basis. Next we define the $C$-module $M$ and the Hilbert space $H$, as in the proof of Theorem 3, and one sees that $H$ has an orthonormal spanning set of the same cardinality as $\Gamma$. Then we define a collection of subspaces $A_{\gamma}$ ($\gamma \in \Gamma$), as in the proof of Theorem 3. The distance calculations are identical.

**Remark 5.** Obviously, the cardinality of $\Gamma$ is the largest possible cardinality of any well-separated family of splittings of $\ell_2(\Gamma)$.

### 3. Splittings of Banach spaces

Recall that there exists an infinite-dimensional Banach space $X$ [1] which is hereditarily indecomposable; that is, $X$ has the property that no closed subspace of $X$ can be expressed as a direct sum of two further infinite-dimensional closed subspaces. Let $X = A_1 \oplus B_1$ and $X = A_2 \oplus B_2$ be a pair of splittings of $X$ such that both $A_1$ and $A_2$ are infinite-dimensional. Since $X$ is indecomposable, both $B_1$ and $B_2$ are finite-dimensional, whence $A_1 \cap A_2$ is nonzero, and in particular $d(S_{A_1}, S_{A_2}) = 0$. This shows that Theorem 3 cannot be extended to the category of Banach spaces simply by replacing orthogonal projections by bounded projections.

However, we are able to modify our construction in order to obtain a result for $\ell_p(X)$, when $X$ is an arbitrary nonzero Banach space. Regarding $X$ and $M$ as vector spaces over $\mathbb{F}$, we may form the vector space tensor product $X \otimes M$. A typical element of $X \otimes M$ may be expressed uniquely as a finite sum of the form:

$$a = \sum_{w \in W} \{a_w \otimes (wx) + a'_w \otimes (wy)\} \quad (a_w, a'_w \in X).$$

We now equip $X \otimes M$ with the norm

$$\|a\|_p = \begin{cases} \left(\sum \|a_w\|^p + \|a'_w\|^p\right)^{1/p} & \text{for } p < \infty, \\ \sup \{\|a_w\|, \|a'_w\| : w \in W\} & \text{for } p = \infty. \end{cases}$$

Taking the completion of $\|\cdot\|_p$ we obtain, for $p < \infty$, a Banach space $X_p$ that is isometrically isomorphic to $\ell_p(X)$, and, for $p = \infty$, a space $X_\infty^0$ that is isometrically isomorphic to $c_0(X)$. The elements of $X_p$ have the unique representation given by $(8)$ when the sums are allowed to be infinite.
Similarly, one defines the space $X_\infty$, which is isometrically isomorphic to $\ell_\infty(X)$, in the obvious fashion.

The action of the Clifford algebra $C$ on $M$ extends to $X \otimes M$ by means of the following definition:

$$c \left( \sum \{a_w \otimes (wx) + a'_w \otimes (wy)\} \right) = \sum \{a_w \otimes (cwx) + a'_w \otimes (cwy)\}.$$

Extending this action to $X_p$ (by continuity) turns these Banach spaces into $C$-modules for which each $w \in W$ acts as an isometric isomorphism of $X_p$.

We can now define, for each $n \geq 1$, a pair of complementary subspaces in $X_p$.

Let $A_n$ consist of all vectors $a \in X_p$ of the form

$$a = \sum_{w \in W} a_w \otimes (wx + we_n y),$$

and let $B_n$ consist of all vectors $b$ of the form

$$b = \sum_{w \in W} b_w \otimes (wx - we_n y).$$

**Proposition 6.** Let $1 \leq p \leq \infty$. Then $A_n$ and $B_n$ are contractively complemented in $X_p$ and

$$d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{B_m}) = d(S_{B_n}, S_{B_m}) \geq \frac{1}{2} \quad (m \neq n).$$

The same result holds for $X^0_p$.

**Proof.** We give the proof only for $p < \infty$. Fix $m, n \in \mathbb{N}$. Consider the projection $P_n$ on $X_p$ given by

$$P_n(a) = \sum_{w \in W} \{(1/2)a_w + (1/2)a'_w\} \otimes (wx + we_n y),$$

when $a$ is represented (uniquely) as:

$$a = \sum_{w \in W} \{a_w \otimes (wx) + a'_w \otimes (we_n y)\} \quad (a_w, a'_w \in X).$$

Clearly $A_n$ is the range of $P_n$, and $P_n$ is contractive by convexity of the norm in $X$. The complementary projection $Q_n = I - P_n$ is also contractive and has range $B_n$. Suppose that $m \neq n$. Observe that $W$ can be expressed as the disjoint union $W = W_1 \cup W_2$, in which each $w \in W_1$ corresponds to a unique element $w' = \pm we_m e_n$ in $W_2$. Thus each $a \in A_m$ can be written uniquely in the form

$$a = \sum_{w \in W_1} \{a_w \otimes (w(x + e_m y)) + b_w \otimes ((we_m e_n)(x + e_m y))\}.$$

From the identity

$$wx + (we_m)y = \frac{w(1 - e_m e_n)}{2} (x + e_n y) + \frac{w(1 + e_m e_n)}{2} (x - e_n y),$$

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we obtain
\[ P_n(a) = \sum_{w \in W_1} a_w \otimes \left( \frac{w(1-e_m e_n)}{2} \right) (x + e_n y) \]
\[ + \sum_{w \in W_1} b_w \otimes \left( \frac{w e_m e_n (1-e_m e_n)}{2} \right) (x + e_n y) \]
\[ = \sum_{w \in W_1} \left\{ \left( \frac{a_w + b_w}{2} \right) \otimes w(x + e_n y) + \left( \frac{b_w - a_w}{2} \right) \otimes w e_m e_n (x + e_n y) \right\}, \]
whence
\[ \| P_n(a) \|^p = \sum_{w \in W_1} \{ \| (1/2)(a_w + b_w) \|^p + \| (1/2)(a_w - b_w) \|^p \} \]
\[ \geq 2^{-p} \sum_{w \in W_1} \{ \| a_w \|^p + \| b_w \|^p \} \]
\[ = 2^{-p} \| a \|^p, \]
and so \( \| P_n(a) \| \geq (1/2) \| a \| \). A similar calculation shows that \( \| Q_n(a) \| \geq (1/2) \| a \| \). Finally,
\[ d(a, S_{A_n}) \geq \| Q_n(a) \| \geq (1/2) \| a \|, \]
which implies that \( d(S_{A_m}, S_{A_n}) \geq 1/2 \). By symmetry, it follows that
\[ d(S_{B_m}, S_{A_n}) = d(S_{B_m}, S_{B_n}) = d(S_{A_m}, S_{A_n}). \]

Since \( X_p \) is isometrically isomorphic to \( \ell_p(X) \), we obtain the following theorem.

**Theorem 7.** Let \( 1 \leq p < \infty \) and let \( X \) be an arbitrary nonzero Banach space. For each \( n \geq 1 \), there exist closed subspaces \( A_n \) and \( B_n \) of \( \ell_p(X) \) such that the following hold:

(a) \( \ell_p(X) = A_n \oplus B_n \) and the corresponding projections are contractions;
(b) \( d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{B_m}) = d(S_{B_m}, S_{B_n}) \geq 1/2 \) for \( m \neq n \).

The same result holds for \( c_0(X) \) and \( \ell_\infty(X) \).

Next we consider splittings of Lebesgue \( L_p \) spaces for which we require the following lemma, whose proof is an easy deduction from Clarkson’s inequalities (see e.g. [4]), which we leave to the reader.

**Lemma 8.** Let \( (X, \Sigma, \mu) \) be a measure space and let \( 1 < p < \infty \). For all \( f, g \in L_p(\mu) \), we have
\[ \left( \left\| \frac{f+g}{2} \right\|^p + \left\| \frac{f-g}{2} \right\|^p \right) \geq 2^{-p/p^*} (\| f \|^p + \| g \|^p), \]
where \( p^* = \min(p, p/(p-1)) \).

**Theorem 9.** Let \( (X, \Sigma, \mu) \) be a separable \( \sigma \)-finite measure space which has either no atoms or infinitely many atoms and let \( 1 < p < \infty \). For each \( n \geq 1 \), there exist closed subspaces \( A_n \) and \( B_n \) of \( L_p(\mu) \) such that the following hold:

(a) \( L_p(\mu) = A_n \oplus B_n \) and the corresponding projections are contractions;
(b) \( d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{B_m}) = d(S_{B_m}, S_{B_n}) \geq 2^{-1/p^*} \) for \( m \neq n \), where
\[ p^* = \min(p, p/(p-1)). \]

The same result holds for \( L_1(\mu) \) and \( L_\infty(\mu) \) with constant \( 1/2 \).
Proof. The assumption on \((S, \Sigma, \mu)\) implies that \(L_p(\mu)\) is isometrically isomorphic to \(\ell_p(L_p(\mu))\). (In fact \(L_p(\mu)\) will be isometrically isomorphic to either \(\ell_p\), \(L_p(0, 1)\) or \(\ell_p \oplus_p L_p(0, 1)\).) Using Lemma 8 we can replace the factor of \(2^{-\frac{1}{p}}\) in equation (9) of Proposition 6 by the larger constant of \(2^{-\frac{p}{p'}}\). With this change the argument of Proposition 6 yields the desired result.

Remark 10. The constant \(2^{-\frac{1}{p'}}\) is probably not best possible. In fact, we have already seen that for \(p = 2\) the best constant is \(\sqrt{2} - \sqrt{2} = 0.7653\ldots\) (as opposed to \(1/\sqrt{2} = 0.7071\ldots\)). Finally, let us observe one further consequence.

Corollary 11. Let \(K\) be an infinite compact metric space. Then \(C(K)\) admits an infinite well-separated family of splittings.

Proof. This follows from Theorem 7 since \(C(K)\) is linearly isomorphic to \(c_0(C(K))\) whenever \(K\) is an infinite compact metric space (see e.g. [4]).

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