ON THE COMPLEXITY OF DESCRIPTION OF REPRESENTATIONS OF \( \star \)-ALGEBRAS GENERATED BY IDEMPOTENTS

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Abstract. In this paper, we introduce a quasiorder (majorization) on \( \star \)-algebras with respect to the complexity of description of their representations. We show that \( C^*(F_2) \succ A \) for any finitely generated \( \star \)-algebra \( A \) (algebras \( B \) such that \( B \succ C^*(F_2) \) are called \( \star \)-wild). We show that the \( \star \)-algebra generated by orthogonal projections \( p, p_1, p_2, \ldots, p_n \) is \( \star \)-wild if \( n \geq 2 \). We also prove that \( \star \)-algebras generated by a pair of idempotents and an orthogonal projection, or by a pair of idempotents \( q_1, q_2 \) (\( q_1 q_2 = q_2 q_1 = 0 \)), etc., are \( \star \)-wild.

Introduction

In Section 2 of this article, following [1–4], we introduce a quasiorder (majorization) on \( \star \)-algebras with respect to the complexity of the structure of their \( \star \)-representations by bounded operators. If \( A \) and \( B \) are \( C^\ast \)-algebras, then \( A \succ B \) means (see Section 2) that there is a \( C^\ast \)-ideal \( J \) in \( A \) such that \( A/J \approx M_n(\mathbb{C}) \otimes B \) for some \( n \in \mathbb{N} \cup \{\infty\} \) (\( M_\infty(\mathbb{C}) = K \) is the \( C^\ast \)-algebra of compact operators).

In Section 2 following [1], we show that for any finitely generated \( \star \)-algebra \( A \), we have that \( C^*(F_2) \succ A \) (here \( F_2 \) is the free group with two generators), and so that if \( B \succ C^*(F_2) \), then the problem of describing, up to a unitary equivalence, all \( \star \)-representations of the algebra is extremely difficult (in the article such problems are called \( \star \)-wild).

Then in Sections 3 and 4, we study, from the point of view developed in Section 2, the complexity of representations of some \( \star \)-algebras generated by idempotents or, which is the same thing, the complexity of a unitary description of some families of idempotents \( \{Q_i\}_{i=1}^n \), \( Q_i \in \mathcal{L}(H) \) (\( i = 1, \ldots, n \)), on a separable Hilbert space \( H \) (for the utility of solution of such a problem see, for example, [5] and the bibliography therein). In particular, in Section 3 we give a simpler proof of the fact that the \( \star \)-algebra generated by orthogonal projections (self-adjoint idempotents) \( p, q_1, \ldots, q_n \) such that \( q_i q_j = 0, i \neq j \) ("all but one" mutually orthogonal projections), is \( \star \)-wild if \( n \geq 2 \) (see also [8]).

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In Section 1, we prove that the \( * \)-algebra is \( * \)-wild if it is generated by one of the following families of operators:

(i) an idempotent and an orthogonal projection (the problem of unitary description of pairs consisting of an idempotent and an orthogonal projection);

(ii) a pair of idempotents \( q_1, q_2 \) such that \( q_1q_2 = q_2q_1 = 0 \);

(iii) a resolution of the identity into a sum of idempotents \( \{q_k\}_{k=1}^n, q_1 + \cdots + q_n = e \) (\( n \geq 3 \)), or a resolution of the identity into a weighted sum of orthogonal projections \( \{p_k\}_{k=1}^n, \frac{1}{2}(p_1 + \cdots + p_n) = e (n \geq 5) \).

For corresponding enveloping \( C^* \)-algebras, if they exist, this means that their factor algebras are isomorphic to \( M_n(C^*(\mathcal{F}_2)) \) for some \( n \in \mathbb{N} \).

Complexity of description of \( * \)-representations of group \( C^* \)-algebras, their generalizations, \( * \)-algebras that correspond to classes of not self-adjoint operators, Wick \( * \)-algebras, etc., will be studied in a forthcoming paper.

It is noted that the problems close to the subject of this article are considered in [1].

1. Definitions and notations

In this article, we consider the problem of giving a description, up to a unitary equivalence, of families of idempotents \( Q_1, \ldots, Q_n \). As usual, two families of operators \( \{X_{\alpha}\}_{\alpha \in \Lambda} \) in \( H \) and \( \{\hat{X}_{\alpha}\}_{\alpha \in \Lambda} \) in \( \hat{H} \) are unitarily equivalent if there exists a unitary operator \( U : H \rightarrow \hat{H} \) such that

\[
UX_{\alpha} = \hat{X}_{\alpha}U \quad (\alpha \in \Lambda).
\]

It is natural to make such a description within the framework of the theory of representations of \( * \)-algebras so as to associate to idempotents \( \{Q_k\}_{k=1}^n \) a representation of the \( * \)-algebra \( \mathfrak{Q}_n \). This is the factor algebra of the free \( * \)-algebra, with generators \( q_1, \ldots, q_n, q_1^*, \ldots, q_n^* \), with respect to the two-sided \( * \)-ideal, generated by the relations \( q_k^2 = q_k, \quad (q_k^*)^2 = q_k^* \quad (k = 1, \ldots, n) \). In the sequel, the \( * \)-algebra \( \mathfrak{A} \) defined by generators \( x_1, x_2, \ldots, x_n, x_1^*, x_2^*, \ldots, x_n^* \) and relations \( P_l(x_1, x_2, \ldots, x_n, x_1^*, x_2^*, \ldots, x_n^*) = 0, l = 1, 2, \ldots, m \), will be denoted by \( C(x_1, \ldots, x_n | P_l(x_1, x_2, \ldots, x_n, x_1^*, x_2^*, \ldots, x_n^*) = 0, l = 1, \ldots, m) \). It will also be assumed that, besides these relations, all relations obtained from them by taking * are valid as well.

A representation of a \( * \)-algebra \( \mathfrak{A} \) is a \( * \)-homomorphism \( \pi : \mathfrak{A} \rightarrow L(H) \) into the \( * \)-algebra \( L(H) \) of bounded operators in a complex separable Hilbert space \( H \). By \( \text{Rep} \mathfrak{A} \) we denote the category, objects of which are representations of the algebra \( \mathfrak{A} \) and morphisms (intertwining operators). Each representation \( \pi \) of the \( * \)-algebra \( \mathfrak{A} = C(x_1, \ldots, x_n | P_l(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*) = 0, l = 1, \ldots, m) \) determines the family of bounded operators \( \{X_k\} = \{\pi(x_k)\}, k = 1, \ldots, n \), such that

\[
P_l(X_1, \ldots, X_n, X_1^*, \ldots, X_n^*) = 0, \quad l = 1, \ldots, m.
\]

Conversely, a given family of operators \( \{X_k\}, k = 1, \ldots, n \), such that

\[
P_l(X_1, \ldots, X_n, X_1^*, \ldots, X_n^*) = 0, \quad l = 1, \ldots, m,
\]

uniquely defines a representation of the whole \( * \)-algebra \( \mathfrak{A} \). Thus the problem of a unitary description of families of operators \( \{X_k\}, k = 1, \ldots, n \), satisfying relations [1] is a problem of description, up to a unitary equivalence, of representations of the \( * \)-algebra \( \mathfrak{A} \).
In the sequel, we will be considering the unitary classification problems for representations of the following \(*\)-algebras (and, correspondingly, the unitary classification problems for the following families of operators):

1) \(\mathfrak{S}_n = \mathbb{C}\langle a_1, \ldots, a_n \mid a_i = a_i^*, i = 1, \ldots, n \rangle\) (classification problem for \(n\) self-adjoint operators);

2) \(\mathfrak{U}_n = \mathbb{C}\langle u_1, \ldots, u_n \mid u_iu_i^* = u_i^*u_i = e, i = 1, \ldots, n \rangle\) (classification problem for \(n\) unitary operators);

3) \(\mathfrak{P}_n = \mathbb{C}\langle p_1, \ldots, p_n \mid p_i^2 = p_i = p_i^*, i = 1, \ldots, n \rangle\) (classification problem for \(n\) orthogonal projections);

4) \(\mathfrak{C} = \mathbb{C}\langle p_1, p_2, p_3 \mid p_i^2 = p_i = p_i^*, i = 1, 2, 3; p_1p_2 = p_2p_1 = 0 \rangle\) (classification problem for a triple of orthogonal projections, two of which are orthogonal);

5) \(\mathfrak{O}_n = \mathbb{C}\langle q_1, \ldots, q_n \mid q_i^2 = q_i, i = 1, \ldots, n \rangle\) (classification problem for \(n\) idempotents);

6) \(\mathfrak{D} = \mathbb{C}\langle q, p \mid q^2 = q, p^2 = p = p^* \rangle\) (classification problem for a pair of operators, one of which is idempotent and the other is an orthogonal projection);

7) \(\mathfrak{O}_{n,\perp} = \mathbb{C}\langle q_1, \ldots, q_n \mid q_i^2 = q_i, i = 1, \ldots, n; q_iq_j = 0 \text{ for } i \neq j \rangle\) (classification problem for \(n\) mutually orthogonal idempotents).

2. Majorization of \(C^*\)-algebras and \(*\)-algebras with respect to complexity of their representations

Before passing to the problem of unitary description of representations of \(*\)-algebras generated by idempotents, we give definitions and some results concerning the ideology and methodology of \(*\)-wildness. In the theory of representations of \(C^*\)-algebras, it was suggested (\[S\]) that the representation problem be considered wild if it contains a standard difficult problem of the representation theory, e. g. the problem to describe, up to similarity, a pair of matrices without relations. To define an analogue of wildness for \(*\)-algebras (\(*\)-wildness), it was suggested in \[I\] to choose, for a standard difficult problem in the theory of \(*\)-representations, the problem of describing pairs of self-adjoint (or unitary) operators up to a unitary equivalence (free \(*\)-algebra \(\mathfrak{S}_2\) (or \(I_2\)) generated by a pair of self-adjoint (or unitary) generators); and there were indications that suggested that the problems, which contain the standard \(*\)-wild problems, be regarded as \(*\)-wild. One can prove that these problems contain as a subproblem the problem of describing \(*\)-representations of any affine \(*\)-algebra.

We give exact definitions, examples, and statements necessary for what follows.

**Definition 1.** Let \(\mathfrak{A}\) be a \(*\)-algebra. A pair \((\hat{\mathfrak{A}}; \hat{\phi} : \mathfrak{A} \to \hat{\mathfrak{A}})\), where \(\hat{\mathfrak{A}}\) is a \(*\)-algebra and \(\hat{\phi}\) is a \(*\)-homomorphism, is called an enveloping \(*\)-algebra of the algebra \(\mathfrak{A}\) if, for any \(*\)-representation \(\pi : \mathfrak{A} \to L(H)\) of the algebra \(\mathfrak{A}\), there exists a unique \(*\)-representation \(\hat{\pi} : \hat{\mathfrak{A}} \to L(H)\) such that the diagram

\[
\begin{array}{ccc}
\hat{\mathfrak{A}} & \xrightarrow{\hat{\phi}} & \hat{\mathfrak{A}} \\
\downarrow{\pi} & & \downarrow{\hat{\pi}} \\
\mathfrak{A} & \xrightarrow{\pi} & L(H)
\end{array}
\]

is commutative, and any operator \(X : H_1 \to H_2\) which intertwines two representations \(\pi_1 : \mathfrak{A} \to L(H_1), \pi_2 : \mathfrak{A} \to L(H_2)\) of the algebra \(\mathfrak{A}\) is also an intertwining operator for the representations \(\hat{\pi}_1, \hat{\pi}_2\) of the algebra \(\hat{\mathfrak{A}}\).
The following examples of enveloping $*$-algebras will be used in the sequel.

1) $\tilde{A} = A$, $\phi$ is the identity mapping;

2) Let $\Sigma$ be any set of elements of an algebra $A$, the images of which are invertible operators for any representation $\pi: A \to L(H)$. Let $\tilde{A} = A[\Sigma^{-1}]$ be the quotient algebra (see [9]) of the algebra $A$ with respect to the set $\Sigma$, and let $\phi$ be the natural imbedding of $A$ into $A[\Sigma^{-1}]$;

3) Let $A$ be a star-bounded $*$-algebra, $\hat{A}$ its enveloping $C^*$-algebra, and $\phi$ its canonical $*$-homomorphism of $A$ into $\hat{A}$, defined by a faithful representation (see, for example, [10]).

Let $M_n(\hat{A})$ be the matrix algebra over $\hat{A}$ with the naturally given $*$-structure. Any representation $\pi: A \to L(H)$ induces the representation

$$\pi_n: M_n(\hat{A}) \to L(H \oplus H \oplus \cdots \oplus H)$$

and, hence, the representation

$$\tilde{\pi}_n: M_n(\tilde{A}) \to L(H \oplus H \oplus \cdots \oplus H)$$

of the algebra $M_n(A)$ enveloping the algebra $M_n(A)$. If $\psi: B \to M_n(\hat{A})$ is a $*$-homomorphism of the algebras, then there is a natural way to construct the functor $F_\psi: \text{Rep}(\hat{A}) \to \text{Rep}(B)$. By definition, $F_\psi(\pi) = \tilde{\pi}_n \circ \psi$ and, if $\alpha: \pi_1 \to \pi_1$ is a morphism of representations, then $F_\psi(\alpha) = \text{diag}(\alpha, \alpha, \ldots, \alpha)$.

**Definition 2.** A $*$-algebra $B$ majorizes a $*$-algebra $A$ ($B \succ A$) if there exist $n \in \mathbb{N}$, an enveloping algebra $\hat{A}$, and a $*$-homomorphism $\psi: B \to M_n(\hat{A})$ such that the functor $F_\psi: \text{Rep}(\hat{A}) \to \text{Rep}(B)$ is full and faithful.

**Remark 1.** For a class of $C^*$-algebras, it is possible to consider only homomorphisms $\psi: B \to M_n(\hat{A})$ in Definition 2 since for a $C^*$-algebra $A$, $M_n(A)$ is also a $C^*$-algebra, and the unique enveloping algebra of a $C^*$-algebra is the algebra itself. Moreover, we have the following. A $C^*$-algebra $B$ majorizes a $C^*$-algebra $A$ if and only if $B$ contains an ideal $I$ such that $B/I \cong M_n(A)$. This isomorphism is defined precisely by the mapping $\psi$. A proof of this claim is not difficult and will be given elsewhere.

**Remark 2.** It is also possible to show that the majorization is a quasiorder relation: if $C \succ B$ and $B \succ A$, then $C \succ A$.

**Remark 3.** One should also note that if $B \succ A$, and $\tilde{A}$, $\hat{A}$ are enveloping algebras of $A$, $B$ correspondingly, then $B \succ \hat{A}$, $\hat{B} \succ A$, $B \succ \hat{A}$, and $M_n(\hat{A})$ is an enveloping algebra of the algebra $M_n(A)$ with the natural embedding of the algebra $M_n(A)$ into $M_n(\hat{A})$.

**Theorem 1.** $\mathcal{S}_2 \succ \mathcal{S}_m$ for any $m = 1, 2, \ldots$.

**Proof.** For the algebra $\mathcal{S}_m$, take the algebra

$$\mathcal{S}_m = \mathbb{C}\langle b_1, \ldots, b_m \mid b_i = b_i^*, i = 1, \ldots, m \rangle$$
itself, \( n = m + 2 \). Define a homomorphism \( \psi : \mathfrak{S}_2 \to M_{m+2}(\mathfrak{S}_m) \) as follows:

\[
\psi(a_1) = \begin{bmatrix}
  e & \frac{1}{2}e & 0 \\
  \frac{1}{2}e & \frac{1}{3}e & \ddots & \ddots \\
  0 & \frac{1}{m}e & \frac{1}{m+1}e & \frac{1}{m+2}e
\end{bmatrix},
\]

\[
\psi(a_2) = \begin{bmatrix}
  0 & e & b_1 \\
  e & 0 & e & b_2 & 0 \\
  b_1 & e & 0 & e & \ddots \\
  b_2 & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & b_{m-1} & e & 0 & e \\
  & & b_m & e & 0
\end{bmatrix}.
\]

One can directly check that the functor \( F_\psi \) is full and faithful.

Theorem 1 permits us to say that the problem of unitary classification of pairs of self-adjoint operators contains, as a subproblem, the problem of unitary classification of representation of any \(*\)-algebra with a countable number of generators (because it is always possible to choose these generators to be self-adjoint).

**Corollary 1.** \( \mathfrak{S}_2 \simeq \mathcal{U}_m \) for any \( m = 1, 2, 3, \ldots \).

**Theorem 2.** \( \mathcal{U}_2 \simeq \mathfrak{S}_2 \).

**Proof.** Choose the enveloping algebra \( \mathfrak{S}_2 \) to be the quotient algebra of the algebra \( \mathfrak{S}_2 \) with respect to the set

\[
\Sigma = \{ a - ie, a + ie, b - ie, b + ie \}, \quad n = 1,
\]

\[
\psi(u_1) = (a - ie)(a + ie)^{-1}, \quad \psi(u_2) = (b - ie)(b + ie)^{-1}
\]

(the Cayley transformation). The rest is obvious.

Theorems 1 and 2 allow us, as a model of complexity for problems of unitary classification of representations of \(*\)-algebras, to choose the problem of unitary classification of representations of the algebra \( \mathcal{U}_2 \) or, which is the same thing, its enveloping \( C^* \)-algebra \( C^*(\mathcal{F}_2) \), where \( \mathcal{F}_2 \) is a free group with two generators.

**Definition 3.** A \(*\)-algebra \( \mathfrak{A} \) is called \(*\)-wild if \( \mathfrak{A} \simeq C^*(\mathcal{F}_2) \).

**Remark 4.** A \( C^* \)-algebra \( \mathfrak{A} \) is \(*\)-wild if and only if there exist \( n \in \mathbb{N} \cup \{ \infty \} \) and a \( C^* \)-ideal \( \mathcal{I} \subset \mathfrak{A} \) such that \( \mathfrak{A}/\mathcal{I} \approx M_n(C^*(\mathcal{F}_2)) \).

Nuclear \( C^* \)-algebras have only hyperfinite factor representations and, consequently, they cannot be \(*\)-wild. There also exist non-nuclear \( C^* \)-algebras which are not wild. For example, the group \( C^* \)-algebra \( C^*(B(m, 2)) \) of the Burnside group with two generators and sufficiently large odd \( n \) is not nuclear (\([11]\) and \([12]\)) and it is also not wild \([13]\).
3. A UNITARY DESCRIPTION OF ORTHOGONAL PROJECTIONS

Following the general ideology of representation theory for \(*\)-algebras, we will try to solve the problem of unitary description of \(*\)-representations of the algebra \(\mathfrak{A}\) by using a description of its irreducible representations and of all its representations as an integral of irreducible ones; a representation \(\pi\) of a \(*\)-algebra \(\mathfrak{A}\) is irreducible if there is no nontrivial subspace in \(H\), invariant with respect to the operators \(\pi(x)\) for all \(x \in \mathfrak{A}\).

For representations of the \(*\)-algebra \(\mathfrak{P}_2 = \mathbb{C}\langle p_1, p_2 \mid p_1^* = p_1 = p_2^2, p_2 = p_2^2 \rangle\) (a pair of orthogonal projections \(P_1, P_2\)), there is a structure theorem (see, for example, [14] and others) that gives a decomposition of representations into a direct sum (or integral) of irreducible representations which are either one-dimensional or two-dimensional, and, up to a unitary equivalence, they coincide with one of the following:

1) four one-dimensional: \(\pi_{(0,0)}(p_1) = \pi_{(0,0)}(p_2) = 0; \pi_{(1,1)}(p_1) = 0, \pi_{(0,1)}(p_2) = 1; \pi_{(1,0)}(p_1) = 1, \pi_{(1,0)}(p_2) = 0; \pi_{(1,1)}(p_1) = \pi_{(1,1)}(p_2) = 1;\)
2) the family, parametrized by \(\phi \in (0, \pi/2)\), of two-dimensional

\[
\pi_\phi(p_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_\phi(p_2) = \begin{pmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{pmatrix}.
\]

One possible way to prove this theorem is to directly verify that

\[
\mathfrak{P}_2 = \mathbb{C}\langle a, b \mid a = a^*, b = b^*; \{a, b\} = ab + ba = 0; a^2 + b^2 = e \rangle,
\]
where \(a = p_1 - p_2, b = e - p_1 - p_2\), and then to apply results from [15] about the structure of a pair of anticommuting self-adjoint operators.

The problem is to describe, up to a unitary equivalence, a family of orthogonal projections \(P_1, P_2, \ldots, P_n\) for \(n \geq 3\) that is \(*\)-wild. We give a fairly simple proof that this problem is \(*\)-wild for \(n = 3\).

**Theorem 3.** Let \(\mathcal{P} = \mathbb{C}\langle p_1, p_2, p_3 \mid p_i^2 = p_i^* = p_i (i = 1, 2, 3)\rangle\). Then \(\mathcal{P}_3 \succ C^*(\mathcal{F}_2)\), i.e. \(\mathcal{P}_3\) is \(*\)-wild.

**Proof.** Let us define the homomorphism \(\psi: \mathcal{P}_3 \to M_4(\mathcal{F}_2)\) as follows:

\[
\psi(p_1) = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\psi(p_2) = \begin{pmatrix} \frac{1}{2}e & 0 & \frac{1}{2}e & 0 \\ 0 & \frac{1}{2}e & 0 & \frac{1}{2}e \\ \frac{1}{2}e & 0 & \frac{1}{2}e & 0 \\ 0 & \frac{1}{2}e & 0 & \frac{1}{2}e \end{pmatrix},
\]

\[
\psi(p_3) = \begin{pmatrix} \frac{3}{8}e & \frac{\sqrt{3}}{8}u^* & \frac{\sqrt{3}}{8}u & \frac{3}{8}e \\ \frac{\sqrt{3}}{8}u^* & \frac{5}{8}e & \frac{3}{8}v & -\frac{\sqrt{3}}{8}vu^* \\ \frac{\sqrt{3}}{8}u^* & \frac{3}{8}v^* & \frac{1}{4}e & 0 \\ \frac{3}{8}e & -\frac{\sqrt{3}}{8}uv^* & 0 & \frac{3}{4}e \end{pmatrix}.
\]
It is easy to check that the corresponding functor $F_\psi: \text{Rep} C^*(\mathcal{F}_2) \to \text{RepP}_3$ is full and faithful.

Moreover, the following theorem holds ([1]).

**Theorem 4.** Let

$$\mathcal{C} = \mathbb{C}(p_1, p_2, p_3 \mid p_i^2 = p_i, p_ip_j = p_jp_i = 0).$$

Then $\mathcal{C} \sim C^*(\mathcal{F}_2)$, i.e. $\mathcal{C}$ is $\ast$-wild.

**Proof.** The proof given here is simpler than the proof in [1].

Let

$$E_k = \begin{bmatrix} e & 0 \\ \vdots & \ddots \\ 0 & \cdots & e \end{bmatrix}, \quad e \text{ is the identity in the algebra } C^*(\mathcal{F}_2),$$

where

$$J_1 = \begin{bmatrix} E_4 \\ 0_{3 \times 4} \\ 0_{5 \times 4} \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0_{4 \times 3} \\ E_3 \\ 0_{5 \times 3} \end{bmatrix}, \quad J_3 = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix},$$

and

$$J_1J_1^* = J_2J_2^* = J_3J_3^* = 0.$$

Set $\psi(p_i) = J_iJ_i^*$; $\psi$ defines a homomorphism of the $\ast$-algebra $\mathcal{C}$ into $M_{14}(C^*(\mathcal{F}_2))$. One can directly check that the functor $F_\psi: \text{Rep}(\mathcal{C}) \to \text{Rep}(C^*(\mathcal{F}_2))$ is full and faithful.

**Corollary 2.** The problem of a unitary classification of “all but one” orthogonal projections $p, p_1, \ldots, p_n$ ($p_ip_j = 0$ for $i \neq j$) is $\ast$-wild if $n \geq 2$.

**Corollary 3.** The problem of unitary classification of quadruples of orthogonal projections $p_1, p_2, p_3, p_4$ such that

$$\alpha(p_1 + p_2 + p_3 + p_4) = I, \quad 0 < \alpha < 1,$$

for a fixed $\alpha \neq \frac{1}{2}$, has only a finite number of irreducible representations, the dimension of which depends on the parameter $\alpha$ ([16]). If $\alpha = \frac{1}{2}$, then irreducible representations are only in dimensions one and two.

It directly follows from Theorem 4 that the problem of unitary classification of five orthogonal projections $p_1, p_2, p_3, p_4, p_5$ such that $p_1 + p_2 + p_3 + p_4 + p_5 = 2I$ is $\ast$-wild.
For a single idempotent, the situation is similar to the situation for two orthogonal projections. Consider the *-algebra $Q_1$ generated by a pair of an idempotent and its adjoint $q_1, q_1^*$. Let $q_1 = a_1 + ib_1, q_1^* = a_1 - ib_1$, where $a_1^2 = a_1, b_1^2 = b_1$. The *-algebra $Q_1$ coincides with the algebra $$\mathbb{C}\langle a, b \mid a = a^*, b = b^*; \{a, b\} = ab + ba = 0, a^2 - b^2 = e\rangle,$$
where $a = 2(a_1 - \frac{1}{2}e), b = 2b_1$.

Irreducible representations of the algebra $Q_1$, up to a unitary equivalence, coincide with one of the following:

1) two one-dimensional representations given by $\pi_0(q_1) = 0$ and $\pi_1(q_1) = 1$;
2) a family, depending on a parameter $y > 0$, of two-dimensional representations:

$$\pi_{(x,y)}(q_1) = Q_1 = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}.$$

By decomposing a representation of the algebra $Q_1$ into the direct sum of irreducible representations on a finite-dimensional space $H$, we obtain a structure theorem (see [17] and [18]) for the unitary description of idempotents in the finite-dimensional case.

There is a structure theorem that gives a description of any bounded idempotent on any separable Hilbert space in the form of an integral of irreducible representations [12].

Consider the problem of a unitary description of pairs of idempotents $Q_1, Q_2$ ($Q_1^2 = Q_1, Q_2^2 = Q_2$). The fact that the problem of a unitary description of pairs of idempotents is difficult is just mathematical folklore. We will prove a corresponding theorem and show that, even if an additional restriction of self-adjointness is imposed on one of the idempotents (one of the idempotents is an orthogonal projection), the problem does not become easier.

**Theorem 5.** Let $Q_2 = \mathbb{C}\langle q_1, q_2 \mid q_1^2 = q_1, q_2^2 = q_2\rangle, D = \mathbb{C}\langle q, p \mid q^2 = q, p^2 = p = p^*\rangle, S_2 = \mathbb{C}\langle a_1, a_2 \mid a_1 = a_1^*, a_2 = a_2^*\rangle$. Then $Q_2 \succ D \succ S_2$, so that the *-algebras $Q_2, D$ are *-wild.

**Proof.** Because $D$ is a factor algebra of the algebra $Q_2$, we have that $Q_2 \succ D$ (we choose an enveloping algebra for $D$ to be the algebra $D$ itself, $n = 1, \psi: Q_2 \rightarrow D$ is the natural epimorphism of the algebra onto the factor algebra).

Let us show that $D \succ S_2$. Construct the homomorphism $\psi: D \rightarrow M_2(S_2)$:

$$\psi(q) = \begin{pmatrix} e & a_1 + ia_2 \\ 0 & e \end{pmatrix}, \quad \psi(p) = \frac{1}{2} \begin{pmatrix} e & e \\ e & e \end{pmatrix}.$$ 

It is easy to check that the corresponding functor $F_\psi: \text{Rep} S_2 \rightarrow \text{Rep} D$ is full and faithful.

**Corollary 4.** The algebra $Q_n$, for $n \geq 2$ (the problem of unitary description of $n$ idempotents if $n \geq 2$) is *-wild.

Finally, we show that the *-algebra $Q_{n, I}$ (the problem of unitary classification of a family of pairwise orthogonal idempotents $Q_1, Q_2, \ldots, Q_n, Q_i Q_j = 0$ for $i \neq j$) is *-wild for $n \geq 2$. 

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Theorem 6. Let
\[ \Omega_{2,\perp} = \mathbb{C}\langle q_1, q_2 \mid q_1^2 = q_1, q_2^2 = q_2, q_1q_2 = q_2q_1 = 0 \rangle. \]
Then \( \Omega_{2,\perp} \cong \mathcal{S}_2 \), i.e. \( \Omega_{2,\perp} \) is a wild \(*\)-algebra.

Proof. Let us define a homomorphism \( \psi : \Omega_{2,\perp} \to M_3(\mathcal{S}_2) \) as follows:
\[
\psi(q_1) = \begin{bmatrix} e & e & a_1 + ia_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \psi(q_2) = \begin{bmatrix} 0 & -e & -e \\ 0 & e & e \\ 0 & 0 & 0 \end{bmatrix}.
\]
One can directly check that \( \|\psi(q_k)\|^2 = \psi(q_k), k = 1, 2, \psi(q_1)\psi(q_2) = \psi(q_2)\psi(q_1) = 0 \), and that the functor \( F_\psi : \text{Rep}\mathcal{S}_2 \to \text{Rep}\Omega_{2,\perp} \) is full and faithful.

Corollary 5. The problem of unitary classification of pairs of commuting idempotents is \(*\)-wild.

Corollary 6. The \(*\)-algebra \( \Omega_{n,\perp} = \mathbb{C}\langle q_1, \ldots, q_n \mid q_i^2 = q_i, i = 1, \ldots, n; q_iq_j = 0 \text{ for } i \neq j \rangle \) (the problem of unitary classification of \( n \) pairwise orthogonal idempotents) is \(*\)-wild for \( n \geq 2 \).

Corollary 7. The \(*\)-algebra \( \mathbb{C}\langle q_1, \ldots, q_n \mid q_i^2 = q_i, i = 1, \ldots, n; q_1 + q_2 + \cdots + q_n = e \rangle \) (the problem of unitary classification of \( n \) idempotents \( Q_1, \ldots, Q_n \) such that \( Q_1 + \cdots + Q_n = I \)) is \(*\)-wild for \( n \geq 3 \).

Proof. If \( m = 3 \), the condition \( q_1 + q_2 + q_3 = e \) implies that the idempotents \( q_1, q_2, q_3 \) are pairwise orthogonal. Then the algebra under consideration coincides with the algebra \( \Omega_{2,\perp} \).

Corollary 8. Let \( \mathfrak{A}_{R_3} = \mathbb{C}\langle x \mid R_3(x) \overset{\text{def}}{=} (x - \alpha_1 e)(x - \alpha_2 e)(x - \alpha_3 e) = 0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \alpha_k \neq \alpha_l \text{ for } k \neq l \rangle \). Then \( \mathfrak{A}_{R_3} \cong \Omega_{2,\perp} \), and consequently, the \(*\)-algebra \( \mathfrak{A}_{R_3} \) is \(*\)-wild.

Proof. Define the homomorphism \( \psi : \mathfrak{A}_{R_3} \to \Omega_{2,\perp} \) as follows:
\[
\psi(a) = \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 (e - q_1 - q_2).
\]
It is easy to check that the functor \( F_\psi \) is full and faithful.

Remark 5. Corollary 8 is given in [19]. The proof in [19] actually uses the fact that the problem of unitary classification of two orthogonal idempotents is \(*\)-wild, and implicitly contains this proof.

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