ON CONDENSATIONS OF $C_p$-SPACES
ONTO COMPACTA

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Abstract. A condensation is a one-to-one onto mapping. It is established that, for each $\sigma$-compact metrizable space $X$, the space $C_p(X)$ of real-valued continuous functions on $X$ in the topology of pointwise convergence condenses onto a metrizable compactum. Note that not every Tychonoff space condenses onto a compactum.

The space $C_p(X)$ of all real-valued continuous functions on a Tychonoff space $X$ in the topology of pointwise convergence is $\sigma$-compact only in the trivial case when $X$ is finite (see Theorem 2 below). This makes natural the following question: when can $C_p(X)$ be condensed onto a compact space or onto a $\sigma$-compact space? A condensation is any one-to-one onto continuous mapping. A compactum is a compact Hausdorff space. We consider only Tychonoff spaces (observe that every $T_1$-space can be condensed onto a compact $T_1$-space). S. Banach was probably the first to ask when a (separable metrizable) space can be condensed onto a compact space (see [4]). E.G. Pytkeev solved one of Banach’s problems, proving that every separable Banach space can be condensed onto a (metrizable) compactum [7]. The question above appears as Problem 35 in [2], accompanied by several versions of it. In particular, Problem 39 in [2] runs as follows: Is it possible to condense $C_p(D^2)$ onto a compact space (where $D$ is the discrete two-point space, and $D^2$ is the Cantor set)? We answer this question below. The main result is:

1. Theorem. For any $\sigma$-compact metrizable space $X$, the space $C_p(X)$ condenses onto a metrizable compactum.

To prove Theorem 1, we need a result from [1]. If $X$ is a space and $Y$ is a subspace of $X$, then $C_p(Y, X)$ is the subspace of $C_p(X)$ consisting of restrictions to $Y$ of continuous real-valued functions on $X$. The next theorem was established in [1] (see Theorem 1.2.2):

2. Theorem. If $Y$ is dense in $X$, and $C_p(Y, X)$ is $\sigma$-countably compact, then $X$ is pseudocompact, and $Y$ is a $P$-space.

Recall that a $P$-space is a space in which every $G_\delta$-subset is open, and that a space is $\sigma$-countably compact, if it is the union of a countable family of countably compact subspaces.
Proof of Theorem 1. It suffices to establish that $C_p(X)$ condenses onto some compactum. Indeed, any compactum with a countable network is metrizable [3], and any network for $C_p(X)$ is a network for any coarser topology. Since $X$ has a countable base, $C_p(X)$ has a countable network [4].

If $X$ is discrete, then $C_p(X)$ is just $R^X$. Clearly, the space $R$ condenses onto a compactum, since, obviously, every locally compact space condenses onto a compactum [4]. Therefore, $R^X$ condenses onto a compactum.

It remains to consider the case when $X$ is not discrete. We can fix a countable subspace $Y$ dense in $X$ such that $C_p(Y, X)$ is not $\sigma$-compact. Indeed, assume the contrary, and take any countable $Y$ dense in $X$. Then, by Theorem 2, $Y$ is a $P$-space. Since $Y$ is countable, it follows that $Y$ is discrete. Therefore, $X$ is discrete, since $X$ is separable and every countable dense subspace of $X$ is discrete, a contradiction.

On the other hand, from Theorem 6.2 in [6] it follows that $C_p(Y, X)$ is a Borel subset of the separable complete metric space $R^Y$ (it is here that we use $\sigma$-compactness of $X$). Now, E.G. Pytkeev established in [2] that every non-$\sigma$-compact separable metrizable Borel space condenses onto a compactum. (Notice that the space $Q$ of rational numbers does not condense onto any compactum, since each non-empty countable compactum has an isolated point.)

Therefore, $C_p(Y, X)$ condenses onto a compactum $K$. Finally, since the natural restriction mapping of functions on $X$ to $Y$ is a condensation of $C_p(X)$ onto $C_p(Y, X)$, we conclude that $C_p(X)$ condenses onto the compactum $K$.

3. Remarks. A result similar to Theorem 1 holds for the space $C_b^b(X)$ of bounded continuous real-valued functions on $X$ in the topology of pointwise convergence: this space also condenses onto a compactum whenever $X$ is a $\sigma$-compact metrizable space. A minor change is needed in the proof: we should refer to Proposition 9.2 in [3]. Theorem 1 also implies that, under the same restrictions on $X$ as in Theorem 1, $C(X)$ condenses onto a compactum when $C(X)$ is endowed with any stronger topology than the topology of pointwise convergence (for example, with the compact-open topology).

4. Problem. Is it true that $C_p(X)$ condenses onto a compactum (onto a $\sigma$-compact space) for every separable metrizable space $X$?

I conjecture that it might be impossible to condense $C_p(J)$ onto any compactum, where $J$ is the space of irrational numbers.

5. Problem. Is it true that $C_p(X)$ condenses onto a $\sigma$-compact space for every compact space $X$?

References


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