PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 6, Pages 1703–1710
S 0002-9939(99)05161-8
Article electronically published on October 6, 1999

# INTERPOLATING SEQUENCES IN THE SPECTRUM OF $H^\infty$ I

RAYMOND MORTINI

(Communicated by Albert Baernstein II)

Abstract. We show that a sequence of trivial points in $M(H^\infty)$ is interpolating if and only if it is discrete. This answers a question of K. Izuchi. We also give a sufficient topological condition for a sequence of nontrivial points to be interpolating.

Let $H^\infty$ be the uniform algebra of all bounded analytic functions in the open unit disk $\mathbb{D}$. Its spectrum, or maximal ideal space, is the space $M(H^\infty)$ of all nonzero multiplicative linear functionals on $H^\infty$ endowed with the weak-$*$-topology. As usual, we identify a function $f$ in $H^\infty$ with its Gelfand transform, $\hat{f}$, defined by $\hat{f}(m) = m(f)$ for $m \in M(H^\infty)$. Moreover, we look upon $\mathbb{D}$ as a subset of $M(H^\infty)$. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in $M(H^\infty)$ is said to be *interpolating* if for any bounded sequence $(w_n)_{n\in\mathbb{N}}$ of complex numbers there exists a function $f \in H^\infty$ so that $f(x_n) = w_n$ for every $n$. Carleson's well known interpolation theorem describes the interpolating sequences in the unit disk (see the book [4] of Garnett for an extensive presentation of this beautiful theory). In fact, a sequence $(a_n)$ in $\mathbb{D}$ is interpolating if and only if

$$(C) \qquad \inf_{k\in\mathbb{N}} \prod_{j:j\neq k} \left| \frac{a_j - a_k}{1 - \overline{a}_j a_k} \right| \geq \delta > 0.$$

It is the aim of this paper to continue the study of the interpolating sequences in the whole spectrum of $H^\infty$. For previous results in this direction we refer the reader to [1], [2], [5], [6], [7], [9], [10], [11], [14], [15].

To proceed, we need to present a few definitions.

Let $x$ and $y$ be two points in $M(H^\infty)$. We define the pseudohyperbolic distance of $x$ to $y$ by

$$\rho(x, y) = \sup\{|f(x)| : f \in H^\infty, ||f||_\infty \leq 1, f(y) = 0\}.$$

Then for $a, b \in \mathbb{D}$ we have $\rho(a, b) = \left| \dfrac{a - b}{1 - \overline{a}b} \right|$.

It is well known that the relation defined on $M(H^\infty)$ by

$$x \sim y \quad \Leftrightarrow \quad \rho(x, y) < 1$$

defines an equivalence relation on $M(H^\infty)$. The equivalence class containing a point $m \in M(H^\infty)$ is called the Gleason part of $m$ and is denoted by $P(m)$. If the

Received by the editors March 6, 1998 and, in revised form, July 13, 1998.
1991 *Mathematics Subject Classification.* Primary 46J15.

part, $P(m)$, consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial. Hoffman's theory [8], [4] shows that for every Gleason part $P(m)$ there is a continuous map $L_m$ of $\mathbb{D}$ onto $P(m)$ with $L_m(0) = m$ such that $f \circ L_m$ is analytic on $\mathbb{D}$ for all $f \in H^\infty$. When the Gleason part of $m$ is trivial, $L_m$ is just a constant map. When $P(m)$ is nontrivial, the map $L_m$ is a bijection. The set of all nontrivial points in $M(H^\infty)$ is denoted by $G$, and the set of all trivial points is denoted by $\Gamma$. It is worth noting that there do exist nontrivial parts which are not homeomorphic to $\mathbb{D}$ (see [8]). A description of the homeomorphic parts has been given in [5]. Moreover, it is well known that there exists trivial points not belonging to the Shilov boundary (see e.g. [3], p. 162, [4], p. 438 or [8], p. 78).

Let us call a sequence of different points $(x_n)_{n\in\mathbb{N}}$ in a topological space $(X, \mathcal{T})$ *discrete*, if the restriction of the topology $\mathcal{T}$ to the set $\{x_n : n \in \mathbb{N}\}$ is the discrete topology. We have the following elementary and well known lemma.

**Lemma 0.1.** *Let $X$ be a compact Hausdorff space and let $(x_n)$ be a sequence of different points in $X$. Then the following assertions are equivalent:*

(1) *$(x_n)$ is discrete.*
(2) *There exists a sequence of open sets $U_n$ with $x_n \in U_n$ whose closures are pairwise disjoint.*
(3) *$x_j \notin \overline{\{x_n : n \neq j\}}$.*

*Remark.* If the topology is merely assumed to be a $T_1$ topology, then (1) and (3) above are equivalent.

As an example of a $T_1$-topology, we mention the hull-kernel topology in $M(H^\infty)$. Recall that a set in $M(H^\infty)$ is hull-kernel closed, if the hull of the ideal $I(E) = \{f \in H^\infty : f \equiv 0 \text{ on } E\}$ coincides with $E$ (see [3], p. 13).

We are now able to briefly review some of the results known so far on interpolating sequences in $M(H^\infty)$. Most of them have been obtained by S. Axler and P. Gorkin in [1], by K. Izuchi in [9], [10], [11] and by Gorkin, Lingenberg and the author of this paper in [5]. Some results have been reproven or/and slightly generalized by Sun Wei [14], [15].

First we note that every interpolating sequence is discrete (in the weak-$*$-topology). However, since the closure $E$ of an interpolating sequence is obviously an interpolation set (that is, $H^\infty|_E = C(E)$), hence hull-kernel closed, it is also clear that an interpolating sequence is discrete in the hull-kernel topology of $M(H^\infty)$. Of course, not every hull-kernel discrete sequence is interpolating (take any non-interpolating Blaschke sequence e.g.). It is useful to notice that a sequence $(x_n)$ is hull-kernel discrete if and only if the following condition $(B_2)$ holds:

$(B_2)$ There exist functions $f_n$ in $H^\infty$ such that

$$\|f_n\| \leq 1, f_n(x_k) = 0 \text{ for } k \neq n \text{ and } f_n(x_n) \neq 0.$$

In view of a generalization of Carleson's theorem above, Izuchi [11] introduced the following conditions for a sequence $(x_n)$ of points in $M(H^\infty)$.

$$(C_1) \qquad \inf_k \prod_{j:j\neq k} \rho(x_j, x_k) \geq \delta > 0.$$

$(C_2)$ There exist functions $f_n$ in $H^\infty$ such that

$$\|f_n\| \le 1, f_n(x_k) = 0 \text{ for } k \neq n \text{ and } \inf_n |f_n(x_n)| \ge \delta > 0.$$

$(C_3)$ There exist functions $f_n$ in $H^\infty$ such that

$$\|f_n\| \le 1, f_n(x_n) = 0, \prod_{n=1}^{\infty} f_n \in H^\infty \text{and}$$

$$\inf_k \Big| \Big( \prod_{n: n\neq k} f_n \Big)(x_k) \Big| \ge \delta > 0.$$

$(D_2)$ For every $\sigma \in ]0,1[$, there exist $f_n \in H^\infty$ such that

$$\|f_n\| \le 1, f_n(x_k) = 0 \text{ for } k \neq n \text{ and } \inf_n |f_n(x_n)| \ge \sigma > 0.$$

In the case of points in the unit disk, all the conditions $(C_1), (C_2)$ and $(C_3)$ are equivalent to Carleson's condition $(C)$. Izuchi showed in [11] that $(C_3) \Longrightarrow (C_2) \Longrightarrow (C_1)$. Obviously $(D_2) \Longrightarrow (C_2) \Longrightarrow (B_2)$.

Moreover, if $(x_n)$ is interpolating, then, by the open mapping theorem, $(x_n)$ satisfies $(C_2)$. If $(x_n)$ is a sequence of trivial points, then $(x_n)$ is interpolating if and only if $(x_n)$ satisfies $(C_2)$ (see [11], p. 224). Finally, if $(x_n)$ satisfies condition $(D_2)$, then $(x_n)$ is interpolating (see [11], p. 227).

In [5] it is shown that whenever a sequence $(x_n)$ is entirely contained in a Gleason part which is homeomorphic to $\mathbb{D}$, then $(x_n)$ is interpolating if and only if $(x_n)$ satisfies condition $(C_1)$, the direct analogue of Carleson's condition $(C)$ for the disk. On the other hand, condition $(C_1)$ is far from being sufficient for an arbitrary sequence to be interpolating (take for example a nondiscrete sequence of trivial points). In [11] Izuchi even gave an example of discrete sequence in $M(H^\infty)$, entirely contained in a nontrivial Gleason part, satisfying $(C_1)$, but which is not interpolating. (See also [14] for another example.) To get more insight into the structure of the interpolating sequences in $M(H^\infty)$, the following questions were asked by Izuchi ([11], p. 227 and 231):

(1) *Suppose that $x_n \in G$ for all $n$. Is condition $(C_2)$ equivalent to condition $(C_3)$?*

We note that one of the principal results of Izuchi in [11] was that whenever $x_n \in G$ satisfies condition $(C_3)$, then $(x_n)$ is interpolating.

(2) *Suppose that $x_n \in \Gamma$ for all $n$. Is condition $(C_2)$ in that case equivalent to $(C_3)$?*

(3) *Is every discrete sequence of trivial points interpolating?*

In this paper we shall give positive answers to questions (2) and (3). Moreover, we shall present another, more topological, sufficient condition for a sequence of nontrivial points to be interpolating.

## 1. Trivial points

In this section we give a complete topological characterization of the interpolating sequences consisting of trivial points. The proof is based on the following result of D. Suarez. To state it, we need to recall that a Blaschke product $B(z) = \prod_{n=0}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$ is said to be interpolating if its zero sequence $(a_n)$ is an interpolating sequence for $H^\infty$. Finite Blaschke products with distinct zeros are considered as interpolating. Moreover, we note that by Hoffman a point $m \in M(H^\infty)$

has a nontrivial Gleason part if and only if it lies in the closure of an interpolating sequence in $\mathbb{D}$.

**Lemma 1.1** ([13], p. 888)**.** *Let $E \subseteq M(H^\infty)$ be a closed set and $B$ a Blaschke product such that $|B| > 0$ on $E$. Let $0 < \sigma < 1$. Then there exists a finite factorization $B = B_0 B_1 \cdots B_N$, so that $B_0$ is a finite product of interpolating Blaschke products and $|B_j| > \sigma$ on $E$ for $1 \le j \le N$.*

**Theorem 1.2.** *Let $(x_n)_n$ be a sequence of trivial points in $M(H^\infty)$. Then $(x_n)_n$ is interpolating if and only if $(x_n)_n$ is discrete.*

*Proof.* We have already seen that any interpolating sequence is discrete. So, conversely, let $(x_n)$ be a discrete sequence of trivial points. By Izuchi's result mentioned in the introduction, it is sufficient to show that $(x_n)$ satisfies $(D_2)$; that is, for every $\varepsilon \in ]0,1[$ there exists a sequence of functions $f_n$ in the unit ball of $H^\infty$ so that $f_n(x_k) = 0$ for every $k \neq n$ and such that $\inf_{n\in\mathbb{N}} |f_n(x_n)| > \varepsilon$.

So, let $\varepsilon \in ]0,1[$. Choose $\varepsilon_n \in ]0,1[$, so that $\prod_{n=1}^\infty \varepsilon_n > \varepsilon$. According to Lemma 0.1 there exist open sets $U_j$ with $x_j \in U_j$ such that the closures of the $U_j$ are pairwise disjoint. Obviously, this implies that $x_n \notin \overline{\{x_k : k \neq n\}}$. So we may assume that $U_n \cap \overline{\bigcup_{k\neq n} U_k} = \emptyset$. Fix $n$. Using the fact proven by Suarez ([12], p. 242) that $H^\infty$ is separating, there exists a function $g_n \in H^\infty$ of norm less than one such that $g_n(x_n) = 0$, but $|g_n| > 0$ on $\overline{\bigcup_{k\neq n} U_k}$. Let $B$ be the Blaschke factor of $g_n$. Then $g_n = BG$ for some function $G \in H^\infty$ which has no zeros in $\mathbb{D}$. By Lemma 1.1, there exist factors $B_j$ of $B$ ($j = 1, \cdots, N$), such that $|B_j| > \sqrt{\varepsilon_n}$ on $\overline{\bigcup_{k\neq n} U_k}$ and $B = CB_1 \cdots B_N$, where $C$ is a finite product of interpolating Blaschke products. Since interpolating Blaschke products never vanish at a trivial point (see [8]), we see that at least one of the $B_j$ or $G$ vanishes at $x_n$. By taking roots of $G$, we may also assume that $|G| > \sqrt{\varepsilon_n}$ on $\overline{\bigcup_{k\neq n} U_k}$. We conclude that there exists a function $\tilde{g}_n$ of norm less than one vanishing at $x_n$ and which is bigger than $\varepsilon_n$ on $\overline{\bigcup_{k\neq n} U_k}$. Let $\tilde{g}_n = \tilde{B}_n \tilde{G}_n$, where $\tilde{B}_n$ is the Blaschke factor of $\tilde{g}_n$. By deleting, if necessary, a finite number of zeros of each of the Blaschke products $\tilde{B}_n$, we may assume that the product $\prod_{n=1}^\infty \tilde{B}_n$ converges locally uniformly on $\mathbb{D}$. Since for any bounded holomorphic function with no zeros in $\mathbb{D}$, the sequence of its $n$-th roots (main branches) converges in $\mathbb{D}$ locally uniformly to 1, we may assume that the series $\sum_{n=1}^\infty (|1-\tilde{G}_n^{1/N_n}|)$ converges locally uniformly on $\mathbb{D}$ for suitable chosen integers $N_n$. Hence the infinite product $\prod_{n=1}^\infty \tilde{G}_n^{1/N_n}$ converges to a bounded analytic function of norm less than one. Thus, we have constructed functions $\tilde{B}_n \tilde{G}_n^{1/N_n}$ which vanish at $x_n$, are bigger than $\varepsilon_n$ on $\overline{\bigcup_{k\neq n} U_k}$ and whose product

$$\prod_{n=1}^\infty \tilde{B}_n \tilde{G}_n^{1/N_n}$$

converges locally uniformly in $\mathbb{D}$. Let

$$f_n = \prod_{k:k\neq n} \tilde{B}_k \tilde{G}_k^{1/N_k}.$$

Then, of course, these products converge and $\|f_n\| \le 1$. Moreover, $f_n(x_k) = 0$ whenever $k \neq n$. Note also that for $k \neq n$, we have $|\tilde{B}_k \tilde{G}_k^{1/N_k}| > \varepsilon_k$ on $U_n \cap \mathbb{D}$.

Hence for every $z \in U_n \cap \mathbb{D}$

$$|f_n(z)| = \prod_{k \neq n} \left| \tilde{B}_k \tilde{G}_k^{1/N_k} \right|(z) \geq \prod_{k \neq n} \varepsilon_k > \varepsilon.$$

Since $x_n$ lies in the closure of $U_n \cap \mathbb{D}$, we conclude that $|f_n(x_n)| > \varepsilon$. □

As a corollary we obtain Hoffman's unpublished result that a sequence of points in the Shilov boundary of $H^\infty$ is interpolating if and only if it is a discrete sequence. His proof was entirely different and valid for a large class of strongly logmodular algebras with extremely disconnected Shilov boundary.

**Corollary 1.3.** *A sequence of trivial points in $M(H^\infty)$ is discrete if and only if it is hull-kernel discrete.*

*Proof.* Since the hull-kernel topology is weaker that the weak-$*$-topology, every hull-kernel discrete sequence obviously is discrete. Conversely, let $(x_n)$ be discrete. Then, by Theorem 1.2, $(x_n)$ is an interpolating sequence. Those, however, are, by the discussion in the introduction, hull-kernel discrete. □

The proof of Theorem 1.2 shows that the given discrete sequence actually satisfies $(C_3)$. Hence, in view of the aforementioned facts that $(C_3) \Rightarrow (C_2) \Rightarrow$ discrete, we obtain the following corollary:

**Corollary 1.4.** *If $(x_n)$ is a sequence of trivial points in $M(H^\infty)$, then the conditions $(C_2)$ and $(C_3)$ are equivalent.*

## 2. Nontrivial points

In this section we give a topological condition (in terms of the pseudohyperbolic metric) for a sequence of nontrivial points to be interpolating in $M(H^\infty)$. For the reader's convenience, we first recall two results of K. Hoffman.

**Lemma 2.1** ([8], [4], sec. 10)**.** *Let $m$ be a nontrivial point and $U$ a neighborhood of $m$ in $M(H^\infty)$. Then for every $0 < \delta < 1$ there exists an interpolating Blaschke product $B$ with zeros $(z_n)$ in $\mathbb{D} \cap U$ such that $B$ vanishes at $m$ and for which the uniform separating constant*

$$\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2)|B'(z_n)| > \delta.$$

Let $Z(f) = \{m \in M(H^\infty) : f(m) = 0\}$ be the zero set for $f \in H^\infty$ and let $\rho(m, E) = \inf\{\rho(m, x) : x \in E\}$ denote the pseudohyperbolic distance of a point $m \in M(H^\infty)$ to a subset $E$ of $M(H^\infty)$.

**Hoffman's Lemma 2.2** ([8], p. 89, 86, 106 and [4], p. 379, 404)**.** *Let $0 < \delta < 1$, $0 < \eta < (1 - \sqrt{1 - \delta^2})/\delta$ (that is, $0 < \eta < \rho(\delta, \eta)$) and let*

$$0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

*Then any interpolating Blaschke product $B$ with zeros $\{z_n\}$ such that $\delta(B) > \delta$ satisfies:*

*(1) $Z(B)$ is the closure of the zero set of $B$ in $\mathbb{D}$,*
*(2) $\rho(x, y) \geq \delta$ for any $x, y \in Z(B), x \neq y$, and*
*(3)* $$\{m \in M(H^\infty) : |B(m)| < \varepsilon\} \subseteq \{m \in M(H^\infty) : \rho(m, Z(B)) < \eta\}$$
$$\subseteq \{m \in M(H^\infty) : |B(m)| < \eta\}.$$

*In particular, by choosing* $\varepsilon > 0$ *so that* $\eta^2 < \varepsilon < \eta\frac{\delta-\eta}{1-\delta\eta}$, *we obtain that* $|B(m)| > \eta^2$ *whenever* $\rho(m, Z(B)) > \eta$.

With the exception of (2), these results are stated in Hoffman's paper. Although (2) is not explicitly stated, one can prove it as follows.

Let $\{u_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ be disjoint subsets of the zeros of $B$ in $\mathbb{D}$ such that $x \in \overline{\{u_n\}}$ and $y \in \overline{\{v_n\}}$. Let $B_1$ denote the subproduct of $B$ associated with $\{u_n\}$, and let $B_2$ denote the subproduct of $B$ associated with $\{v_n\}$. Then, for each $n$, we see that

$$|B_2(u_n)| \geq \prod_{k:z_k\neq u_n} \rho(z_k, u_n) \geq \delta.$$

Since $B_2$ is continuous on the maximal ideal space, $|B_2(x)| \geq \delta$ and $B_2(y) = 0$. Thus

$$\rho(x, y) = \sup\{|f(x)| : f \in H^\infty, ||f||_\infty \leq 1, f(y) = 0\} \geq |B_2(x)| \geq \delta.$$

**Theorem 2.3.** *Let* $(x_n)$ *be a discrete sequence of nontrivial points in* $M(H^\infty)$. *Suppose that there exist pairwise disjoint open sets* $U_n$ *in* $M(H^\infty)$ *containing* $x_n$ *such that*

$$\delta = \inf_j \prod_{n:n\neq j} \rho(U_j, U_n) > 0.$$

*Then* $(x_n)$ *is an interpolating sequence satisfying* $(C_3)$.

*Proof.* Let $\varepsilon_k$, $0 < \varepsilon_k < 1$, satisfy $\prod_{k=1}^{\infty} \varepsilon_k > \delta$. Choose $\varepsilon_{j,k}$ with $0 < \varepsilon_{j,k} < 1$ and $\inf_j \prod_{k:k\neq j} \varepsilon_{j,k} = \delta_1 > 0$ such that

$$\rho(U_j, U_k) > \varepsilon_{j,k}\rho(x_j, x_k).$$

(For example $\varepsilon_{j,k} = \rho(U_j, U_k)^2/\rho(x_j, x_k)$.) Fix $k$. Since $\rho(U_k, U_j) \to 1$ for $j \to \infty$, there exists $\theta(k) \in \mathbb{N}$ such that $\rho(U_k, U_j) > \varepsilon_k$ for every $j \geq \theta(k)$. Without loss of generality $k < \theta(k) < \theta(k + 1)$. Put $\theta(0) = 1$.

We are now going to show that $(x_n)$ satisfies $(C_3)$.

Let

$$\eta_k = \max\{\varepsilon_k, \varepsilon_{1,k}\rho(x_1, x_k), \varepsilon_{2,k}\rho(x_2, x_k), \cdots, \varepsilon_{\theta(k),k}\rho(x_{\theta(k)}, x_k)\}.$$

Choose $\delta_k \in ]0, 1[$ so that $\eta_k < \frac{1-\sqrt{1-\delta_k^2}}{\delta_k}$. By Lemma 2.1 there exists an interpolating Blaschke product $b_k$ such that $b_k(x_k) = 0, \delta(b_k) > \delta_k$ and $Z(b_k) \subseteq U_k$. Let $j \geq \theta(k)$ and let $x \in U_j$. Since $\rho(U_k, U_j) > \varepsilon_k$, we have by Hoffman's Lemma that $|b_k(x)| > \varepsilon_k^2$. Now let $1 \leq j < \theta(k)$. Then $Z(b_k) \subseteq U_k$, $x \in U_j$ and $\rho(U_j, U_k) > \varepsilon_{j,k}\rho(x_j, x_k)$ imply by Hoffman's Lemma that $|b_k(x)| > \varepsilon_{j,k}^2\rho^2(x_j, x_k)$.

By cutting off, if necessary, for each $k$ a finite number of zeros (in $\mathbb{D}$) of $b_k$, we may assume that the infinite product $\prod_{k=1}^{\infty} b_k$ converges locally uniformly in $\mathbb{D}$ to a Blaschke product. Of course we should not remove the zero $x_k$ if $x_k$ happens to lie in the unit disk.

Now fix $j$. Choose $k \in \mathbb{N} \cup \{0\}$ so that $\theta(k) \leq j < \theta(k + 1)$. For every $z \in U_j \cap \mathbb{D}$ write

$$\Big|\Big(\prod_{n:n\neq j} b_n\Big)\Big|(z) = \prod_{n:n\neq j} |b_n(z)| = \prod_{n=1}^{k} |b_n(z)| \prod_{k+1\leq n<j} |b_n(z)| \prod_{n:n>j} |b_n(z)|.$$

To estimate the first product, we note that $n \le k < \theta(k) \le j$. Since $n \le k$ implies $\theta(n) \le \theta(k) \le j$, we obtain

$$\prod_{n=1}^{k} |b_n(z)| > \prod_{n=1}^{k} \varepsilon_n^2.$$

For the second product, we see that $k+1 \le n < j < \theta(k+1) \le \theta(n)$. Hence

$$\prod_{k+1 \le n < j} |b_n(z)| > \prod_{k+1 \le n < j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n).$$

Finally, for the last product we have $j < n < \theta(n)$. Hence

$$\prod_{n > j} |b_n(z)| > \prod_{n > j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n).$$

Therefore

$$\big|\big(\prod_{n:n\neq j} b_n\big)\big|(z) > \prod_{n=1}^{\infty} \varepsilon_n^2 \prod_{n:n\neq j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n) > \delta^4 \delta_1^2.$$

Since $x_j \in \overline{U_j \cap \mathbb{D}}$, we see that $\left|\left(\prod_{n:n\neq j} b_n\right)\right|(x_j) \ge \delta^4 \delta_1^2$. This shows that $(x_n)$ satisfies $(C_3)$. Since $x_n \in G$, we get by Izuchi's result [11] cited above that $(x_n)$ is an interpolating sequence. □

*Remark.* It is worth noting that the condition $\delta = \inf_j \prod_{n:n\neq j} \rho(U_j, U_n) > 0$ is in fact necessary and sufficient for $(x_n)$ to be interpolating whenever $x_n \in \mathbb{D}$ for every $n$. We doubt, however, whether it will be necessary in the general case. Evidence for the failure of necessity comes from the following facts.

Let $(x_j)$ be a sequence in a nontrivial Gleason part $P(m)$ such that for an open set $U$ in $M(H^\infty)$ with $m \in U$ we have $\rho(U, x_j) \to 1$ for $j \to \infty$; then $(x_j)$ cannot cluster back into the part itself, that is,

$$(*) \qquad \left[\overline{\{x_j : j \in \mathbb{N}\}} \setminus \{x_j : j \in \mathbb{N}\}\right] \cap P(m) = \emptyset$$

and $x_n \notin \overline{\{x_j : j \neq n\}}$ for every $n$. This can be seen in the following way: Assume that there exists a cluster point $y$ of $(x_n)$ such that $\eta := \rho(m, y) < 1$. Choose $\delta \in ]0,1[$ so that

$$0 \le \sqrt[4]{\eta} < (1 - \sqrt{1-\delta^2})/\delta.$$

Let $b$ be any interpolating Blaschke product with $\delta(b) > \delta$, $Z(b) \subseteq U$ and $b(m) = 0$. Then by Schwarz's Lemma ([4], p. 401)

$$|b(y)| = \rho\big(b(y), b(m)\big) \le \rho(y, m) = \eta.$$

But $\rho(Z(b), x_n) > \sqrt[4]{\eta}$ for all $n \ge n_0$. Hence, by Hoffman's Lemma, $|b(x_n)| > \sqrt{\eta}$ for $n \ge n_0$. In particular, $|b(y)| \ge \sqrt{\eta}$, an obvious contradiction.

On the other hand, if $B$ is an interpolating Blaschke product which vanishes at a point $m \in G$ whose Gleason part is not homeomorphic to $\mathbb{D}$, then it is known from [5] that the sequence $(x_n)$ of its zeros in $P(m)$ does not contain any isolated point within the topological space $P(m)$; in other words $x_n \in \overline{\{x_j : j \neq n\}}$ for every $n$. Now choose an interpolating subsequence of $(x_n)$—the existence of which can be deduced from ([1], Theorem 3). We think that this sequence may have cluster points within the part itself, so violating $(*)$.

## References


[1] Axler, S., Gorkin, P.: Sequences in the maximal ideal space of $H^\infty$, Proc. Amer. Math. Soc. 108 (1990), 731-740. MR **90h:**46087

[2] Dufresnoy, A.: *Sur les compacts d'interpolation du spectre de* $H^\infty(\mathbb{D})$. Studia Math. 43 (1972), 235-244. MR **47:**838

[3] Gamelin, T.: *Uniform Algebras*, 2nd ed., Chelsea, New York, 1984.

[4] Garnett, J.B.: *Bounded Analytic Functions* , Academic Press, New York 1981. MR **83g:**30037

[5] Gorkin, P., Lingenberg, H.-M., Mortini, R.: *Homeomorphic disks in the spectrum of* $H^\infty$, Indiana Univ. Math. J. 39 (1990), 961-983. MR **92b:**46082

[6] Guo, Kunyu: *The interpolation problem on the maximal ideal space of* $H^\infty$. Northeast Math. J. 10 (1994), 208-214. MR **95k:**46085

[7] Guo, Kunyu: *The Earl theorem on the maximal ideal space of* $H^\infty$. J. Fudan Univ. 35 (1996), 393-396. MR **98c:**46105

[8] Hoffman, K.: *Bounded analytic functions and Gleason parts*, Ann. Math. 86 (1967), 74-111. MR **35:**5945

[9] Izuchi, Keiji: *Interpolating sequences in a homeomorphic part of* $H^\infty$, Proc. Amer. Math. Soc. 111 (1991), 1057-1065. MR **91i:**46056

[10] Izuchi, Keiji: *Interpolating sequences in the maximal ideal space of* $H^\infty$, J. Math. Soc. Japan 43 (1991), 721-731. MR **92m:**30071

[11] Izuchi, Keiji: *Interpolating sequences in the maximal ideal space of* $H^\infty$ *II*, Operator Theory, Advances and Applications, Vol. 59 (1992), 221-233. MR **94k:**30088

[12] Suarez, D.: *Cech cohomology and covering dimension for the* $H^\infty$ *maximal ideal space*, J. Funct. Anal. 123 (1994), 233-263. MR **95g:**46100

[13] Suarez, D.: *Trivial Gleason parts and the topological stable rank of* $H^\infty$, Amer. J. Math. **118** (1996), 879-904. MR **97i:**46092

[14] Wei, Sun: *An interpolating theorem on the maximal ideal space of* $H^\infty$, J. Sichuan Univ. 32 (1995), 117-120 (in Chinese). MR **96f:**46100

[15] Wei, Sun: *Countably compact Property of* $G(H^\infty)$, Northeast Math. J. 12 (1996), 421-426. MR **98g:**46071



Département de Mathématiques, Université de Metz, Ile du Saulcy, F-57045 Metz, France

*E-mail address*: mortini@poncelet.univ-metz.fr