INTERPOLATING SEQUENCES IN THE SPECTRUM OF $H^\infty$ I

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Abstract. We show that a sequence of trivial points in $M(H^\infty)$ is interpolating if and only if it is discrete. This answers a question of K. Izuchi. We also give a sufficient topological condition for a sequence of nontrivial points to be interpolating.

Let $H^\infty$ be the uniform algebra of all bounded analytic functions in the open unit disk $\mathbb{D}$. Its spectrum, or maximal ideal space, is the space $M(H^\infty)$ of all nonzero multiplicative linear functionals on $H^\infty$ endowed with the weak-$\ast$-topology. As usual, we identify a function $f$ in $H^\infty$ with its Gelfand transform, $\widehat{f}$, defined by $\widehat{f}(m) = m(f)$ for $m \in M(H^\infty)$. Moreover, we look upon $\mathbb{D}$ as a subset of $M(H^\infty)$. A sequence $(x_n)_{n \in \mathbb{N}}$ of points in $M(H^\infty)$ is said to be interpolating if for any bounded sequence $(w_n)_{n \in \mathbb{N}}$ of complex numbers there exists a function $f \in H^\infty$ so that $f(x_n) = w_n$ for every $n$. Carleson’s well known interpolation theorem describes the interpolating sequences in the unit disk (see the book [4] of Garnett for an extensive presentation of this beautiful theory). In fact, a sequence $(a_n)$ in $\mathbb{D}$ is interpolating if and only if

\[ (C) \quad \inf_{k \in \mathbb{N}} \prod_{j: j \neq k} \left| \frac{a_j - a_k}{1 - \overline{a_j}a_k} \right| \geq \delta > 0. \]

It is the aim of this paper to continue the study of the interpolating sequences in the whole spectrum of $H^\infty$. For previous results in this direction we refer the reader to [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13].

To proceed, we need to present a few definitions.

Let $x$ and $y$ be two points in $M(H^\infty)$. We define the pseudohyperbolic distance of $x$ to $y$ by

\[ \rho(x, y) = \sup\{|f(x)| : f \in H^\infty, \|f\|_\infty \leq 1, f(y) = 0\}. \]

Then for $a, b \in \mathbb{D}$ we have $\rho(a, b) = \left| \frac{a - b}{1 - \overline{a}b} \right|$. It is well known that the relation defined on $M(H^\infty)$ by

\[ x \sim y \quad \Leftrightarrow \quad \rho(x, y) < 1 \]

defines an equivalence relation on $M(H^\infty)$. The equivalence class containing a point $m \in M(H^\infty)$ is called the Gleason part of $m$ and is denoted by $P(m)$. If the
part, \( P(m) \), consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial. Hoffman’s theory \([8]\), \([4]\) shows that for every Gleason part \( P(m) \) there is a continuous map \( L_m \) of \( \mathbb{D} \) onto \( P(m) \) with \( L_m(0) = m \) such that \( f \circ L_m \) is analytic on \( \mathbb{D} \) for all \( f \in H^\infty \). When the Gleason part of \( m \) is trivial, \( L_m \) is just a constant map. When \( P(m) \) is nontrivial, the map \( L_m \) is a bijection. The set of all nontrivial points in \( M(H^\infty) \) is denoted by \( G \), and the set of all trivial points is denoted by \( \mathcal{T} \). It is worth noting that there do exist nontrivial parts which are not homeomorphic to \( \mathbb{D} \) (see \([8]\)). A description of the homeomorphic parts has been given in \([5]\). Moreover, it is well known that there exists trivial points not belonging to the Shilov boundary (see e.g. \([3]\), p. 162, \([4]\), p. 438 or \([8]\), p. 78).

Let us call a sequence of different points \((x_n)_{n \in \mathbb{N}}\) in a topological space \((X, T)\) **discrete**, if the restriction of the topology \( T \) to the set \( \{x_n : n \in \mathbb{N}\} \) is the discrete topology. We have the following elementary and well known lemma.

**Lemma 0.1.** Let \( X \) be a compact Hausdorff space and let \((x_n)\) be a sequence of different points in \( X \). Then the following assertions are equivalent:

1. \((x_n)\) is discrete.
2. There exists a sequence of open sets \( U_n \) with \( x_n \in U_n \) whose closures are pairwise disjoint.
3. \( x_j \notin \{x_n : n \neq j\} \).

**Remark.** If the topology is merely assumed to be a \( T_1 \)-topology, then (1) and (3) above are equivalent.

As an example of a \( T_1 \)-topology, we mention the hull-kernel topology in \( M(H^\infty) \). Recall that a set in \( M(H^\infty) \) is hull-kernel closed, if the hull of the ideal \( I(E) = \{ f \in H^\infty : f \equiv 0 \text{ on } E\} \) coincides with \( E \) (see \([3]\), p. 13).

We are now able to briefly review some of the results known so far on interpolating sequences in \( M(H^\infty) \). Most of them have been obtained by S. Axler and P. Gorkin in \([1]\), by K. Izuchi in \([9]\), \([10]\), \([11]\) and by Gorkin, Lingenberg and the author of this paper in \([5]\). Some results have been reproven or/and slightly generalized by Sun Wei \([14]\), \([15]\).

First we note that every interpolating sequence is discrete (in the weak-* topology). However, since the closure \( E \) of an interpolating sequence is obviously an interpolation set (that is, \( H^\infty|_E = C(E) \)), hence hull-kernel closed, it is also clear that an interpolating sequence is discrete in the hull-kernel topology of \( M(H^\infty) \). Of course, not every hull-kernel discrete sequence is interpolating (take any non-interpolating Blaschke sequence e.g.). It is useful to notice that a sequence \((x_n)\) is hull-kernel discrete if and only if the following condition \((B_2)\) holds:

\[(B_2) \text{ There exist functions } f_n \text{ in } H^\infty \text{ such that } \|f_n\| \leq 1, f_n(x_k) = 0 \text{ for } k \neq n \text{ and } f_n(x_n) \neq 0.\]

In view of a generalization of Carleson’s theorem above, Izuchi \([11]\) introduced the following conditions for a sequence \((x_n)\) of points in \( M(H^\infty) \).

\[(C_1) \text{ } \inf_k \prod_{j:j \neq k} \rho(x_j, x_k) \geq \delta > 0.\]
entirely contained in a nontrivial Gleason part, satisfying (C2) satisﬁes condition (C), then (xk) is interpolating.

Moreover, if (xn) is interpolating, then, by the open mapping theorem, (xn) satisfies (C2). If (xn) is a sequence of trivial points, then (xn) is interpolating if and only if (xn) satisﬁes (C2) (see [11], p. 224). Finally, if (xn) satisﬁes condition (D2), then (xn) is interpolating (see [11], p. 227).

In [5] it is shown that whenever a sequence (xn) is entirely contained in a Gleason part which is homeomorphic to \( \mathbb{D} \), then (xn) is interpolating if and only if (xn) satisﬁes condition (C1), the direct analogue of Carleson’s condition (C) for the disk. On the other hand, condition (C1) is far from being suﬃcient for an arbitrary sequence to be interpolating (take for example a nondiscrete sequence of trivial points). In [11] Izuchi even gave an example of discrete sequence in \( M(H^\infty) \), entirely contained in a nontrivial Gleason part, satisfying (C1), but which is not interpolating. (See also [14] for another example.) To get more insight into the structure of the interpolating sequences in \( M(H^\infty) \), the following questions were asked by Izuchi ([11], p. 227 and 231):

1. Suppose that \( x_n \in G \) for all \( n \). Is condition (C2) equivalent to condition (C3)?

   We note that one of the principal results of Izuchi in [11] was that whenever \( x_n \in G \) satisﬁes condition (C3), then (xn) is interpolating.

2. Suppose that \( x_n \in \Gamma \) for all \( n \). Is condition (C2) in that case equivalent to (C3)?

3. Is every discrete sequence of trivial points interpolating?

   In this paper we shall give positive answers to questions (2) and (3). Moreover, we shall present another, more topological, suﬃcient condition for a sequence of nontrivial points to be interpolating.

### 1. Trivial points

In this section we give a complete topological characterization of the interpolating sequences consisting of trivial points. The proof is based on the following result of D. Suarez. To state it, we need to recall that a Blaschke product

\[ B(z) = \prod_{n=0}^{\infty} \frac{a_{n}z}{a_{n}z - 1} \]

is said to be interpolating if its zero sequence \( (a_n) \) is an interpolating sequence for \( H^\infty \). Finite Blaschke products with distinct zeros are considered as interpolating. Moreover, we note that by Hoffman a point \( m \in M(H^\infty) \)
has a nontrivial Gleason part if and only if it lies in the closure of an interpolating sequence in \( \mathbb{D} \).

**Lemma 1.1** ([13], p. 888). Let \( E \subseteq M(H^\infty) \) be a closed set and \( B \) a Blaschke product such that \( |B| > 0 \) on \( E \). Let \( 0 < \sigma < 1 \). Then there exists a finite factorization \( B = B_0B_1 \cdots B_N \), so that \( B_0 \) is a finite product of interpolating Blaschke products and \( |B_j| > \sigma \) on \( E \) for \( 1 \leq j \leq N \).

**Theorem 1.2.** Let \( (x_n) \) be a sequence of trivial points in \( M(H^\infty) \). Then \( (x_n) \) is interpolating if and only if \( (x_n) \) is discrete.

**Proof.** We have already seen that any interpolating sequence is discrete. So, conversely, let \( (x_n) \) be a discrete sequence of trivial points. By Izuchi’s result mentioned in the introduction, it is sufficient to show that \( (x_n) \) satisfies \( (D_2) \); that is, for every \( \varepsilon \in (0,1) \) there exists a sequence of functions \( f_n \) in the unit ball of \( H^\infty \) so that \( f_n(x_k) = 0 \) for every \( k \neq n \) and such that \( \inf_{n \in \mathbb{N}} |f_n(x_n)| > \varepsilon \).

So, let \( \varepsilon \in (0,1) \). Choose \( \varepsilon_n \in (0,1) \), so that \( \prod_{n=1}^{\infty} \varepsilon_n > \varepsilon \). According to Lemma 0.1 there exist open sets \( U_j \) with \( x_j \in U_j \) such that the closures of the \( U_j \) are pairwise disjoint. Obviously, this implies that \( x_n \notin \{x_k : k \neq n\} \). So we may assume that \( U_n \cap \bigcup_{k \neq n} U_k = \emptyset \). Fix \( n \). Using the fact proven by Suarez ([12], p. 242) that \( H^\infty \) is separating, there exists a function \( g_n \in H^\infty \) of norm less than one such that \( g_n(x_n) = 0 \), but \( |g_n| > 0 \) on \( \bigcup_{k \neq n} U_k \). Let \( B \) be the Blaschke factor of \( g_n \). Then \( g_n = BG \) for some function \( G \in H^\infty \) which has no zeros in \( \mathbb{D} \). By Lemma 1.1, there exist factors \( B_j \) of \( B \) (\( j = 1, \cdots, N \)), such that \( |B_j| > \sqrt{\varepsilon_n} \) on \( \bigcup_{k \neq n} U_k \) and \( B = CB_1 \cdots B_N \), where \( C \) is a finite product of interpolating Blaschke products. Since interpolating Blaschke products never vanish at a trivial point (see [3]), we see that at least one of the \( B_j \) or \( G \) vanishes at \( x_n \). By taking roots of \( G \), we may also assume that \( |G| > \sqrt{\varepsilon_n} \) on \( \bigcup_{k \neq n} U_k \). We conclude that there exists a function \( \tilde{g}_n \) of norm less than one vanishing at \( x_n \) and which is bigger than \( \varepsilon_n \) on \( \bigcup_{k \neq n} U_k \). Let \( \tilde{g}_n = \tilde{B}_n \tilde{G}_n \), where \( \tilde{B}_n \) is the Blaschke factor of \( \tilde{g}_n \). By deleting, if necessary, a finite number of zeros of each of the Blaschke products \( \tilde{B}_n \), we may assume that the product \( \prod_{n=1}^{\infty} \tilde{B}_n \) converges locally uniformly on \( \mathbb{D} \). Since for any bounded holomorphic function with no zeros in \( \mathbb{D} \), the sequence of its \( n \)-th roots (main branches) converges in \( \mathbb{D} \) locally uniformly to 1, we may assume that the series \( \sum_{n=1}^{\infty} (|1 - \tilde{G}_n^{1/N_n}|) \) converges locally uniformly on \( \mathbb{D} \) for suitable chosen integers \( N_n \). Hence the infinite product \( \prod_{n=1}^{\infty} \tilde{G}_n^{1/N_n} \) converges to a bounded analytic function of norm less than one. Thus, we have constructed functions \( \tilde{B}_n \tilde{G}_n^{1/N_n} \) which vanish at \( x_n \), are bigger than \( \varepsilon_n \) on \( \bigcup_{k \neq n} U_k \) and whose product

\[
\prod_{n=1}^{\infty} \tilde{B}_n \tilde{G}_n^{1/N_n}
\]

converges locally uniformly in \( \mathbb{D} \). Let

\[
f_n = \prod_{k: k \neq n} \tilde{B}_k \tilde{G}_k^{1/N_k}.
\]

Then, of course, these products converge and \( \|f_n\| \leq 1 \). Moreover, \( f_n(x_k) = 0 \) whenever \( k \neq n \). Note also that for \( k \neq n \), we have \( |\tilde{B}_k \tilde{G}_k^{1/N_k}| > \varepsilon_k \) on \( U_n \cap \mathbb{D} \).
Hence for every \( z \in U_n \cap \mathbb{D} \)
\[
\left|f_n(z)\right| = \prod_{k \neq n} \left|\hat{B}_k \hat{G}_k^{1/N_k}\right|(z) \geq \prod_{k \neq n} \varepsilon_k > \varepsilon.
\]

Since \( x_n \) lies in the closure of \( U_n \cap \mathbb{D} \), we conclude that \( |f_n(x_n)| > \varepsilon \). \( \square \)

As a corollary we obtain Hoffman’s unpublished result that a sequence of points in the Shilov boundary of \( H^\infty \) is interpolating if and only if it is a discrete sequence.

His proof was entirely different and valid for a large class of strongly logmodular algebras with extremely disconnected Shilov boundary.

**Corollary 1.3.** A sequence of trivial points in \( M(H^\infty) \) is discrete if and only if it is hull-kernel discrete.

**Proof.** Since the hull-kernel topology is weaker than the weak-* topology, every hull-kernel discrete sequence obviously is discrete. Conversely, let \( (x_n) \) be discrete.

Then, by Theorem 1.2, \( (x_n) \) is an interpolating sequence. Those, however, are, by the discussion in the introduction, hull-kernel discrete. \( \square \)

The proof of Theorem 1.2 shows that the given discrete sequence actually satisfies \( (C_3) \). Hence, in view of the aforementioned facts that \( (C_3) \Rightarrow (C_2) \Rightarrow \) discrete, we obtain the following corollary:

**Corollary 1.4.** If \( (x_n) \) is a sequence of trivial points in \( M(H^\infty) \), then the conditions \( (C_2) \) and \( (C_3) \) are equivalent.

2. **Nontrivial Points**

In this section we give a topological condition (in terms of the pseudohyperbolic metric) for a sequence of nontrivial points to be interpolating in \( M(H^\infty) \). For the reader’s convenience, we first recall two results of K. Hoffman.

**Lemma 2.1** (\([8], [4], \) sec. 10). Let \( m \) be a nontrivial point and \( U \) a neighborhood of \( m \) in \( M(H^\infty) \). Then for every \( 0 < \delta < 1 \) there exists an interpolating Blaschke product \( B \) with zeros \( \{z_n\} \) in \( \mathbb{D} \cap U \) such that \( B \) vanishes at \( m \) and for which the uniform separating constant
\[
\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2)|B'(z_n)| > \delta.
\]

Let \( Z(f) = \{m \in M(H^\infty) : f(m) = 0\} \) be the zero set for \( f \in H^\infty \) and let \( \rho(m, E) = \inf\{\rho(m, x) : x \in E\} \) denote the pseudohyperbolic distance of a point \( m \in M(H^\infty) \) to a subset \( E \) of \( M(H^\infty) \).

**Hoffman’s Lemma 2.2** (\([8], \) p. 89, 86, 106 and \([4], \) p. 379, 404). Let \( 0 < \delta < 1 \), \( 0 < \eta < (1 - \sqrt{1 - \delta^2})/\delta \) (that is, \( 0 < \eta < \rho(\delta, \eta) \)) and let
\[
0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta \eta}.
\]

Then any interpolating Blaschke product \( B \) with zeros \( \{z_n\} \) such that \( \delta(B) > \delta \) satisfies:

1. \( Z(B) \) is the closure of the zero set of \( B \) in \( \mathbb{D} \),
2. \( \rho(x, y) \geq \delta \) for any \( x, y \in Z(B), x \neq y \), and
3. \( \{m \in M(H^\infty) : |B(m)| < \varepsilon\} \subseteq \{m \in M(H^\infty) : \rho(m, Z(B)) < \eta\} \subseteq \{m \in M(H^\infty) : |B(m)| < \eta\} \).
In particular, by choosing $\varepsilon > 0$ so that $\eta^2 < \varepsilon < \eta^2 \frac{\delta - \varepsilon}{1 - \delta}$, we obtain that $|B(m)| > \eta^2$ whenever $\rho(m, Z(B)) > \eta$.

With the exception of (2), these results are stated in Hoffman’s paper. Although (2) is not explicitly stated, one can prove it as follows.

Let $\{u_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ be disjoint subsets of the zeros of $B$ in $\mathbb{D}$ such that $x \in \{u_n\}$ and $y \in \{v_n\}$. Let $B_1$ denote the subproduct of $B$ associated with $\{u_n\}$, and let $B_2$ denote the subproduct of $B$ associated with $\{v_n\}$. Then, for each $n$, we see that

$$|B_2(u_n)| \geq \prod_{k : z_k \neq u_n} \rho(z_k, u_n) \geq \delta.$$ 

Since $B_2$ is continuous on the maximal ideal space, $|B_2(x)| \geq \delta$ and $B_2(y) = 0$. Thus

$$\rho(x, y) = \sup \{|f(x)| : f \in H^\infty, ||f||_\infty \leq 1, f(y) = 0\} \geq |B_2(x)| \geq \delta.$$

**Theorem 2.3.** Let $(x_n)$ be a discrete sequence of nontrivial points in $M(H^\infty)$. Suppose that there exist pairwise disjoint open sets $U_n$ in $M(H^\infty)$ containing $x_n$ such that

$$\delta = \inf \prod_{i \neq j} \rho(U_i, U_n) > 0.$$ 

Then $(x_n)$ is an interpolating sequence satisfying $(C_3)$.

**Proof.** Let $\varepsilon_k, 0 < \varepsilon_k < 1$, satisfy $\prod_{k=1}^\infty \varepsilon_k > \delta$. Choose $\varepsilon_{j,k}$ with $0 < \varepsilon_{j,k} < 1$ and $\inf_j \prod_{k \neq j} \varepsilon_{j,k} = \delta_1 > 0$ such that

$$\rho(U_j, U_k) > \varepsilon_{j,k} \rho(x_j, x_k).$$

(For example $\varepsilon_{j,k} = \rho(U_j, U_k)^2/\rho(x_j, x_k)$.) Fix $k$. Since $\rho(U_k, U_j) \to 1$ for $j \to \infty$, there exists $\theta(k) \in \mathbb{N}$ such that $\rho(U_k, U_j) > \varepsilon_k$ for every $j \geq \theta(k)$. Without loss of generality $k < \theta(k) < \theta(k + 1)$. Put $\theta(0) = 1$.

We are now going to show that $(x_n)$ satisfies $(C_3)$.

Let

$$\eta_k = \max\{\varepsilon_k, \varepsilon_{1,k}, \rho(x_1, x_k), \varepsilon_{2,k}, \rho(x_2, x_k), \ldots, \varepsilon_{\theta(k),k}, \rho(x_{\theta(k)}, x_k)\}.$$ 

Choose $\delta_k \in [0, 1]$ so that $\eta_k < 1 - \sqrt{1 - \delta_k^2}$. By Lemma 2.1 there exists an interpolating Blaschke product $b_k$ such that $b_k(x_k) = 0, \delta(b_k) > \delta_k$ and $Z(b_k) \subseteq U_k$. Let $j \geq \theta(k)$ and $x \in U_j$. Since $\rho(U_k, U_j) > \varepsilon_k$, we have by Hoffman’s Lemma that $|b_k(x)| > \varepsilon_k^2$. Now let $1 \leq j < \theta(k)$. Then $Z(b_k) \subseteq U_k, x \in U_j$ and $\rho(U_j, U_k) > \varepsilon_{j,k} \rho(x_j, x_k)$ imply by Hoffman’s Lemma that $|b_k(x)| > \varepsilon_k^2 \rho(x_j, x_k)$.

By cutting off, if necessary, for each $k$ a finite number of zeros (in $\mathbb{D}$) of $b_k$, we may assume that the infinite product $\prod_{k=1}^\infty b_k$ converges locally uniformly in $\mathbb{D}$ to a Blaschke product. Of course we should not remove the zero $x_k$ if $x_k$ happens to lie in the unit disk.

Now fix $j$. Choose $k \in \mathbb{N} \cup \{0\}$ so that $\theta(k) \leq j < \theta(k+1)$. For every $z \in U_j \cap \mathbb{D}$ write

$$|\left(\prod_{n \neq j} b_n\right)(z)| = \prod_{n \neq j} |b_n(z)| = \prod_{n=1}^k |b_n(z)| \prod_{k+1 \leq n < j} |b_n(z)| \prod_{n > j} |b_n(z)|.$$
To estimate the first product, we note that \( n \leq k < \theta(k) \leq j \). Since \( n \leq k \) implies \( \theta(n) \leq \theta(k) \leq j \), we obtain
\[
\prod_{n=1}^{k} |b_n(z)| > \prod_{n=1}^{k} \varepsilon_n^2.
\]

For the second product, we see that \( k + 1 \leq n \leq j \) and \( \theta(k + 1) \leq \theta(n) \). Hence
\[
\prod_{k+1 \leq n < j} |b_n(z)| > \prod_{k+1 \leq n < j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n).
\]

Finally, for the last product we have \( j < n \leq \theta(n) \). Hence
\[
\prod_{n > j} |b_n(z)| > \prod_{n > j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n).
\]

Therefore
\[
\left| \left( \prod_{n:n \neq j} b_n \right)(z) \right| > \prod_{n=1}^{\infty} \varepsilon_n^2 \prod_{n:n \neq j} \varepsilon_{j,n}^2 \rho^2(x_j, x_n) > \delta^4 \delta_1^2.
\]

Since \( x_j \in U \cap \mathbb{D} \), we see that \( \left| \left( \prod_{n:n \neq j} b_n \right)(x_j) \right| \geq \delta^4 \delta_1^2 \). This shows that \( (x_n) \) satisfies \( (C_3) \). Since \( x_n \in G \), we get by Izuchi’s result [11] cited above that \( (x_n) \) is an interpolating sequence.

**Remark.** It is worth noting that the condition \( \delta = \inf_j \prod_{n:n \neq j} \rho(U_j, U_n) > 0 \) is in fact necessary and sufficient for \( (x_n) \) to be interpolating whenever \( x_n \in \mathbb{D} \) for every \( n \). We doubt, however, whether it will be necessary in the general case. Evidence for the failure of necessity comes from the following facts.

Let \( (x_j) \) be a sequence in a nontrivial Gleason part \( P(m) \) such that for an open set \( U \) in \( M(H^\infty) \) with \( m \in U \) we have \( \rho(U, x_j) \to 1 \) for \( j \to \infty \); then \( (x_j) \) cannot cluster back into the part itself, that is,
\[
\left( \{x_j : j \in \mathbb{N} \} \setminus \{x_j : j \in \mathbb{N} \} \right) \cap P(m) = \emptyset
\]
and \( x_n \notin \{x_j : j \neq n\} \) for every \( n \). This can be seen in the following way: Assume that there exists a cluster point \( y \) of \( (x_n) \) such that \( \eta := \rho(m, y) < 1 \). Choose \( \delta \in [0, 1] \) so that
\[
0 \leq \sqrt[3]{\eta} < (1 - \sqrt{1 - \delta^2})/\delta.
\]

Let \( b \) be any interpolating Blaschke product with \( \delta(b) > \delta \), \( Z(b) \subseteq U \) and \( \rho(U) = 0 \). Then by Schwarz’s Lemma ([4], p. 401)
\[
|b(y)| = \rho(y, b(m)) \leq \rho(y, m) = \eta.
\]

But \( \rho(Z(b), x_n) > \sqrt[3]{\eta} \) for all \( n \geq n_0 \). Hence, by Hoffman’s Lemma, \( |b(x_n)| > \sqrt[3]{\eta} \) for \( n \geq n_0 \). In particular, \( |b(y)| \geq \sqrt[3]{\eta} \), an obvious contradiction.

On the other hand, if \( B \) is an interpolating Blaschke product which vanishes at a point \( m \in G \) whose Gleason part is not homeomorphic to \( \mathbb{D} \), then it is known from [5] that the sequence \( (x_n) \) of its zeros in \( P(m) \) does not contain any isolated point within the topological space \( P(m) \); in other words \( x_n \notin \{x_j : j \neq n\} \) for every \( n \). Now choose an interpolating subsequence of \( (x_n) \)—the existence of which can be deduced from ([4], Theorem 3). We think that this sequence may have cluster points within the part itself, so violating \( (\ast) \).
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