PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 6, Pages 1627–1632
S 0002-9939(99)05182-5
Article electronically published on October 29, 1999

NORMAL SUBGROUPS OF $GL_n(D)$ ARE NOT FINITELY GENERATED

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(Communicated by Ken Goodearl)

In memory of M. L. Mehrabadi

Abstract. As a generalization of Wedderburn's classic theorem, it is shown that the multiplicative group of a noncommutative finite dimensional division algebra cannot be finitely generated. Also, the following conjecture is investigated: An infinite non-central normal subgroup of $GL_n(D)$ cannot be finitely generated.

1. Introduction

The general question of what groups can occur as multiplicative groups of a noncommutative division algebra is an unsolved problem, but some progress has been made along this line. For instance, Hua has proved that the multiplicative group of a noncommutative division algebra cannot be solvable. Also, in Chapter 3 of [4], there are several group theoretic conditions whose occurrence in the multiplicative group of a division algebra entails the commutativity of the algebra. Much research in recent years has been focused on the study of the structure of the multiplicative subgroups of division algebras; for example see [1], [3], [6], [7], [8], [10], [11], and [12] for an introduction. Before stating our results, we fix some notation. Throughout, D is a division algebra with centre F; we shall denote their multiplicative subgroups by D^*, F^*, respectively.

Our first result is a generalization of a beautiful commutativity theorem which was discovered by Wedderburn in 1905, that is:

"*If D^* is a finite group, then D is commutative.*"

Here for finite dimensional division algebras over their centres, we show that

"*If D^* is a finitely generated group, then D is commutative.*"

It is believed that the assumption for D being finite dimensional is superfluous in the above statement, and so the following conjecture will arise naturally:

Conjecture 1. *If D^* is finitely generated, then D is commutative.*

The above result may be extended to general linear groups over D. To be more precise, we prove that if D is an infinite division algebra with $[D:F]<\infty$, then $GL_n(D)$, $n\geq 2$, contains no non-central finitely generated normal subgroups. But

Received by the editors January 14, 1998 and, in revised form, July 29, 1998.
1991 *Mathematics Subject Classification.* Primary 15A33, 16K20.
Key words and phrases. Division ring, normal subgroup, finitely generated.

in the infinite dimensional case one does not know if the same conclusion holds. So, it seems natural to pose the following conjecture:

Conjecture 2. *If D is a division algebra and n is a natural number, then the infinite non-central normal subgroups of $GL_n(D)$ are not finitely generated.*

In [14, p. 429], it has been proved that if $x \in D\backslash F$, then $|cl(x)| = |D|$, where $cl(x)$ denotes the set of all conjugates of x. This implies that the multiplicative subgroup of any uncountable division algebra cannot contain any non-central finitely generated normal subgroup. For if $a \in N\backslash F$, then $cl(a) \subset N$ and N must be uncountable. In the general case, it is not known whether a multiplicative group of a division algebra can contain a non-central finitely generated normal subgroup. We recall that the finite subgroups of the multiplicative group of a division algebra have been determined by Amitsur in [2]. It is easily seen that if $N \lhd D^*$ and N is finite, then $N \subset F$. We begin the material of this note with the following

Theorem 1. *Let D be a division algebra of finite dimension over its centre F. If D^* is finitely generated, then D is commutative.*

Proof. Since D^* is finitely generated, we conclude that the Whitehead group $K_1(D) := D^*/D'$ is also finitely generated. Therefore every subgroup of $K_1(D)$ must be finitely generated (cf. [14]). In particular, the group F^*D'/D' and consequently $F^*/F^* \cap D'$ is also finitely generated. First, we claim that $F^* \cap D'$ is finite. To see this, it is enough to show that any $x \in F^* \cap D'$ is a root of a polynomial $x^n - 1$, where n is the index of D. Let $x \in F^* \cap D'$. Since $x \in D'$, there exist commutators $c_i \in D'$ such that $x = c_1c_2 \dots c_r$. Now, taking the reduced norm from both sides of the equality, we obtain $x^n = 1$ and the claim is established. Now, the group $F^*/F^* \cap D'$ is finitely generated and $F^* \cap D'$ is finite. Thus F^* must be finitely generated. Finally, we show that F^* is finite and so D is finite and commutative. To prove this, we consider two cases.

i) $Char\, F = 0$. Denote the rational numbers by Q. In this case Q^* is a subgroup of F^*. But F^* is a finitely generated abelian group. This implies that Q^* is finitely generated which is nonsense.

ii) $Char\, F = p > 0$. First we assume that F is algebraic over its prime subfield Z_p, say. It follows that F^* is a torsion group and since F^* is finitely generated, we conclude that F^* is finite as desired.

Thus, without loss of generality, we may assume that there exists an element $x \in F$ which is transcendental over Z_p. So the field $Z_p(x)$ is a subfield of F. We claim that for any natural number m, there exists an irreducible polynomial of degree m in $Z_p[x]$. Consider the finite field of p^m elements and suppose that α is a generator for its cyclic group. It is now easy to see that the minimal polynomial of α over Z_p is an irreducible polynomial of degree m and so the claim is established. We denote this polynomial by $f_m(x)$. Since F^* is a finitely generated abelian group we may consider it as a finitely generated Z-module. It follows that F^* is Noetherian as Z-module. Let $G_i, i = 1, 2, \dots,$ be the group generated by $f_1(x), f_2(x), \dots, f_i(x)$, respectively. Now the following ascending chain of subgroups of F^* does not stop,

$$G_1 \subset G_2 \subset G_3 \subset \dots .$$

This contradicts the Noetherian condition. □

Before stating the next theorem we need the following lemma (cf. [9] or [10]).

Lemma. *Let D be an algebraic division algebra over its centre F. Then for any $x \in D$ there exists natural number $n(x)$ such that $x^{n(x)} = rc$, where $r \in F$ and $c \in D'$.*

Theorem 2. *Let D be a division algebra algebraic over its centre F. If D^* is finitely generated, then so is its derived group D'.*

Proof. Suppose that D^* is the group generated by the elements $x_1, \dots, x_m$. By the Lemma, there exist natural numbers n_i $(1 \leq i \leq m)$, such that $x_i^{n_i} = r_i c_i$, where $r_i \in F$ and $c_i \in D'$. Assume that F_1 is the group generated by the elements $r_1, \dots, r_m$. Now, put $G(D) := D^*/F_1 D'$. It is easily checked that $G(D)$ is an abelian torsion group whose exponent divides $\prod_{i=1}^{m} n_i$. Furthermore, $G(D)$ is finitely generated since D^* is finitely generated. This implies that $G(D)$ is finite. Thus $F_1 D'$ is finitely generated (cf. [13, p. 298]). Using the isomorphism theorems for groups, we conclude that $D'/(F_1 \cap D')$ is finitely generated. On the other hand, $F_1 \cap D'$ is finitely generated because F_1 is an abelian finitely generated group. Since both groups $D'/(F_1 \cap D')$ and $F_1 \cap D'$ are finitely generated, we conclude that D' is finitely generated. □

Theorem 3. *Let D be a finite dimensional division algebra over its centre F. If N is a non-central finitely generated normal subgroup of D^*, then there exists a finite set $\Lambda \subset F$ such that $F = P(\Lambda)$, where P is the prime subfield of F.*

Proof. Assume that L is the division algebra generated by all elements of N. Since L is invariant under all inner automorphisms of D, by Cartan-Brauer-Hua's Theorem (cf. [13, p. 272]), $L = D$ or L is central. If L is central, then N is central, a contradiction. Thus, we may assume that $L = D$. Suppose that $[D : F] = n$ and consider the regular matrix representation of D in $GL_n(F)$. Since N is finitely generated, there exist matrices $A_1, \dots, A_k \in GL_n(F)$, such that $N = \langle A_1, \dots, A_k \rangle$. Let Λ be the set of all elements in F occurring as the entries of A_i and $A_i^{-1}, i = 1, \dots, k$. If H is the subring generated by N, then H is a subring of $M_n(P(\Lambda))$, the n by n matrix ring over $P(\Lambda)$, where $P(\Lambda)$ is the subfield of F generated by Λ. Now, since $L \subset M_n(P(\Lambda))$ we conclude that $aI \in M_n(P(\Lambda))$, for any $a \in F^*$ and so $a \in P(\Lambda)$. Hence, $F = P(\Lambda)$ and the proof is complete. □

Now, we investigate the structure of normal subgroups of D^* to see if they can be finitely generated. As we have seen before, if D is uncountable, then there is no non-central finitely generated normal subgroup in D^*. Therefore the multiplicative group of real quaternions does not contain any non-central finitely generated normal subgroups. The next result shows that for a finite dimensional division ring whose centre is a subset of algebraic numbers, the same conclusion holds and so the rational quaternions does not contain any non-central finitely generated normal subgroups.

Theorem 4. *Let F be an algebraic extension of Q, and D be a finite dimensional division algebra over its centre F. Then D^* contains no non-central finitely generated normal subgroups.*

Proof. If N is a non-central finitely generated normal subgroup of D^*, then by Theorem 3 there exist elements $r_1, \dots, r_s \in F$ such that $F = Q(r_1, \dots, r_s)$, and so $[F : Q] = m < \infty$. Assume now that $[D : F] = n$ and put $k = mn$. Thus D^* has a matrix representation in $GL_k(Q)$. Let $1, \alpha_2, \dots, \alpha_n$ be an F-basis of D.

Let us consider an element $a \in N\backslash F$. Since a is not in the centre we conclude that a does not commute with all α_i's. Without loss of generality assume that $a\alpha_2 \neq \alpha_2 a$. Assume also that $A, B \in GL_k(Q)$ be the matrix representations of a, α_2, respectively. It is clear that for each $x \in Q$, the matrix representation of $x+\alpha_2$ is $B_x = xI + B$. Since N is finitely generated, by the argument used in the proof of Theorem 3, we conclude that there is a set $\Lambda = \{m_1/n_1, \dots, m_t/n_t\} \subset Q$ such that each element of N has a matrix representation in $GL_k(Z[\Lambda])$. On the other hand, $N \lhd D^*$ and so for each element $x \in Q$ we have $B_x A B_x^{-1} \in GL_k(Z[\Lambda])$. Since $det\, B_x$ is a polynomial in x of degree k, and for each $1 \leq i, j \leq k$, the (i,j)-th entry of B_x^{-1} is of the form $f_{ij}(x)/g(x) \in Q(x)$, where $deg\, g(x) = k$, $deg\, f_{ij}(x) \leq k-1$, we conclude that the (i,j)-th entry of the matrix $B_x A B_x^{-1}$ is of the form $f_{ij}(x)/g(x)$, where for each $1 \leq i, j \leq k$, we have $deg\, f_{ij}(x) \leq k$. If for each $1 \leq i, j \leq k$, there are rational numbers q_{ij} such that for any $x \in Q$, $f_{ij}(x)/g(x) = q_{ij}$, then for any $x_1, x_2 \in F$ with $x_1 \neq x_2$, we have $(x_1+\alpha_2)a(x_1+\alpha_2)^{-1} = (x_2+\alpha_2)a(x_2+\alpha_2)^{-1}$. This implies that $x_1(a\alpha_2 - \alpha_2 a) = x_2(a\alpha_2 - \alpha_2 a)$ and since $a\alpha_2 - \alpha_2 a \neq 0$ we conclude that $x_1 = x_2$, a contradiction. Thus there exists an entry of $B_x A B_x^{-1}$, say (i,j)-th which depends on x. Put $f_{ij}(x) = \sum_{i=0}^{k} a_i x^i$, $g(x) = x^k + \sum_{i=0}^{k-1} b_i x^i$. Thus for each $x \in Q$ we have $f_{ij}(x)/g(x) \in Z[\Lambda]$. If $a_k = m_{t+1}/n_{t+1}$, then for each $x \in Q$ we obtain

$$\frac{f_{ij}(x)}{g(x)} - a_k \in Z[\Lambda \cup \{m_{t+1}/n_{t+1}\}].$$

So there exists a polynomial $f(x) \in Q[x]$ such that $deg\, f(x) \leq k-1$ and for each $x \in Q$ we have $f(x)/g(x) \in Z[\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. Multiplying $f(x)$ and $g(x)$ by suitable scalars, we may assume that $f(x), g(x) \in Z[x]$. Put $f(x) = \sum_{i=0}^{k-1} a'_i x^i \in Z[x]$, $g(x) = \sum_{i=0}^{k} b'_i x^i$. Since $det\, B \neq 0$, we may assume that $b'_0 \neq 0$. Now, change the variable x to $b'_0 x$ to obtain $f_1(x), g_1(x) \in Z[x]$, such that $deg\, g_1 = k$, $deg f_1 \leq k-1$, where the constant term of $g_1(x)$ is 1, and for each $x \in Q$ we have

$$f_1(x)/g_1(x) \in Z[\Lambda \cup \{m_{t+1}/n_{t+1}\}].$$

Assume that $S = \{p_1, \dots, p_l\}$ is the set of all primes occurring in the factorizations of $n_1, \dots, n_{t+1}$ into prime numbers. For each natural number r, put $x_r = (p_1 p_2 \dots p_l)^r$. Since $deg\, f_1 < deg\, g_1$, for a large enough number r, we obtain that $f_1(x_r)/g_1(x_r) < 1$. On the other hand, for each $r \geq 1$, and each $1 \leq i \leq l$, $g_1(x_r)$ and p_i are coprime, that is, $(g_1(x_r), p_i) = 1$. It is not hard to see that if $u/v \in Z[m_1/n_1, \dots, m_{t+1}/n_{t+1}]$ with $(u,v) = 1$, then each prime factor of v belongs to S. Now since $f_1(x_r)/g_1(x_r) \in Z[m_1/n_1, \dots, m_{t+1}/n_{t+1}]$ and for each $1 \leq i \leq l$, $r \geq 1$, $(g_1(x_r), p_i) = 1$, we reach a contradiction, and so the result follows. □

To prove our next result, we need the following

Theorem A (cf. [15, p. 56])**.** *Let R be a finitely generated integral domain. Then $GL_n(R)$ contains a normal subgroup T of finite index such that all the elements of finite order in T are unipotent (so if $CharR = 0$, T is torsion-free).*

The $Char\, R = 0$ case of the above theorem was proved by A. Selberg and independently by M. I. Kargapolov. Selberg's method of proof can be extended to give a proof of the whole of the theorem. We are now in the position to prove the following theorem.

Theorem 5. *Let D be a division algebra of finite dimension over its centre F. If N is an infinite non-central normal subgroup of $GL_n(D)$, $n \geq 2$, then N cannot be finitely generated.*

Proof. Assume that N is an infinite non-central finitely generated normal subgroup of $GL_n(D)$, $n \geq 2$. Since D is of finite dimension over F, we may view N as a subgroup $GL_{nm}(R)$, where R is a finitely generated integral domain and $m = [D : F]$. Furthermore, since $N \lhd GL_n(D)$ and N is non-central, it is well known that $SL_n(D) \subset N$. By Theorem A, $GL_{nm}(R)$ contains a normal subgroup T of finite index such that all the elements of finite order in T are unipotent. Now, we have $[N : N \cap T] = [TN : T] \leq [GL_{nm}(R) : T] \leq \infty$. But $N \cap T$ is a subnormal subgroup of $GL_n(D)$. If $N \cap T$ is non-central, then we find $SL_n(D) \subset N \cap T$. Let us consider the element

$$A = \begin{bmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} & 0 \\ 0 & I \end{bmatrix} \in T,$$

where I is identity matrix of order $n-2$. It is easily checked that A is of finite order but it is not unipotent, a contradiction. Therefore assume that $N \cap T \subset FI$. Since $SL_n(D) \subset N$, for any $0 \neq x \in D$, we have

$$\begin{bmatrix} \begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix} & 0 \\ 0 & I \end{bmatrix} \in N.$$

If $[N : N \cap T] = r$, then for any $x \in D$ we find $x^{2r} = 1$. This implies that F is finite and consequently $D = F$, which contradicts the fact that N is infinite. □

If G is any group, $\Phi_1(G)$ denotes the intersection of all the maximal subgroups of finite index in G, or G itself if none such exist. We use the following theorems to prove our last result.

Theorem B (cf. [15, p. 63])**.** *If R is a finitely generated integral domain and G is a subgroup of $GL_n(R)$, then $\Phi_1(G)$ is nilpotent.*

Theorem C (cf. [12])**.** *Let G be a finitely generated linear group that has no solvable subgroups of finite index. Then G has maximal subgroup of infinite index. Moreover the set of such subgroups is uncountable.*

Theorem D (cf. [14, p. 440])**.** *If D is a division algebra over its centre F, G a non-central subnormal subgroup of D^*, then G is not solvable.*

Theorem 6. *Let D be a division algebra of finite dimension over its centre F. If N is a non-central finitely generated normal subgroup of D^*, then N contains maximal subgroups of finite as well as infinite index. Moreover, the set of all maximal subgroups of infinite index is uncountable.*

Proof. If N is a non-central normal subgroup of D^*, then, as we saw before, there is a finitely generated integral domain R such that $N < GL_n(R)$, where n is the dimension of D over F. First we show that N has maximal subgroups of finite index. Indeed, if there is no such maximal subgroup, by Theorem B, we have that $\Phi_1(N) = N$ is nilpotent. But, by Theorem D, N cannot be nilpotent which is a contradiction. Now, we claim that N has maximal subgroups of infinite index. To do this, by Theorem C, it is enough to show that N has no solvable subgroups of finite index. Suppose that H is such a subgroup of N. Thus we have $K = \bigcap_{x \in N} xHx^{-1}$ is a normal subgroup of finite index in N. Furthermore, since H is solvable we conclude that K is also solvable. Now, Theorem D implies that K is central. Assume that $[N : K] = m$; then for any $x \in N$, we have $x^m \in F$. Now, consider the element $u \in N \cap D'$. We have $u^m = r \in F$ and so $(N(u))^m = r^n$, where N is the norm function of D to F. Since $u \in D'$ we conclude that $u^{mn} = 1$. Thus, by Theorem 8 of [5], we obtain $N \cap D' \subset F$. Now, a result of [14, p. 440] implies that either N or D' is central. Our hypothesis forces that $D' \subset F$, i.e., D' is radical over F. Therefore, by Lemma 2 of [10], D is commutative which contradicts the fact that N is non-central. □

Acknowledgement

The authors are indebted to the Research Council of the Sharif University of Technology and Institute for Studies in Theoretical Physics and Mathematics for their support. They wish to thank M. L. Mehrabadi for his helpful discussion about Theorem 1.

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