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AN OBSTRUCTION TO THE CONFORMAL COMPACTIFICATION OF RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study noncompact complete Riemannian n-manifolds with $n \geq 3$ which are not pointwise conformal to subdomains of any compact Riemannian n-manifold. For this, we compare the Sobolev Quotient at infinity of a noncompact complete Riemannian manifold with that of the singular set in a compact Riemannian manifold using the method for the Yamabe problem.

1. Introduction

In this paper, we study noncompact complete Riemannian n-manifolds with $n \geq 3$ which are not pointwise conformal to subdomains of any compact Riemannian n-manifold. We show that the conformal structure at infinity provides an obstruction to the conformal compactification of certain noncompact complete Riemannian manifolds. This obstruction is interesting because Nomizu and Ozeki [6] showed that any Riemannian manifold admits a complete metric by the conformal change of metric.

First, we calculate the Sobolev Quotient of a singular set in a given compact Riemannian manifold using the method which was developed for the Yamabe problem. Second, we compare the Sobolev Quotient at infinity of a given noncompact complete Riemannian manifold with the Sobolev Quotient of a singular set in a compact Riemannian manifold. To do this we introduce some notation:

$$Q(M,g) \equiv \inf_{u \in C_0^\infty(M)} \frac{\int_M |\nabla u|^2 + C_n S\, u^2 \; dV_g}{(\int_M u^{2n/(n-2)} \; dV_g)^{(n-2)/n}}$$

and

$$\overline{Q(M,g)} \equiv \lim_{r \to \infty} \inf_{u \in C_0^\infty(M-B_r)} \frac{\int_{M-B_r} |\nabla u|^2 + C_n S\, u^2 \; dV_g}{(\int_{M-B_r} u^{2n/(n-2)} \; dV_g)^{(n-2)/n}},$$

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where $C_n = \frac{n-2}{4(n-1)}$, S is the scalar curvature of (M,g) and B_r is the ball of radius r with center at a fixed point x_0.

We also use the notation $Q(M)$ for $Q(M,g)$ and $Q(S^n)$ for $Q(S^n, g_0)$, where g_0 is the standard metric. Note that $\overline{Q(M,g)}$ is well defined since

$$\inf_{u \in C_0^\infty(M-B_r)} \frac{\int\limits_{M-B_r} |\nabla u|^2 + C_n S u^2 dV_g}{\left(\int\limits_{M-B_r} u^{2n/(n-2)}\, dV_g\right)^{(n-2)/n}}$$

$$\leq \inf_{u \in C_0^\infty(M-B_{r+1})} \frac{\int\limits_{M-B_{r+1}} |\nabla u|^2 + C_n S u^2 dV_g}{\left(\int\limits_{M-B_{r+1}} u^{2n/(n-2)}\, dV_g\right)^{(n-2)/n}}.$$

$\overline{Q(M,g)}$ was used for a study of the Yamabe problem on noncompact complete Riemannian manifolds by Kim [4]. Obstructions for the existence of a complete conformal metric with prescribed scalar curvature on a subdomain $K - \Gamma$ of a compact Riemannian manifold (K,h) were studied by Aviles and McOwen [2], Schoen and Yau [8], Delanoe [3] and McOwen [5] using the dimension of Γ. In this paper, we have an obstruction for the conformal compactification of a noncompact complete Riemannian manifold using the conformal structure at infinity.

2. The Sobolev Quotient of a singular set

In this section, we show that the Sobolev Quotient of a singular set is the same as that of the standard sphere. Let us denote the n dimensional Riemannian volume of a subset Ω of (M,g) by $|\Omega|$. We state our theorem.

Theorem 2.1. *Let (M,g) be a Riemannian manifold of dimension $n \geq 3$. Assume that there exists a sequence $\{\Gamma_i\}$ of smooth bounded open subsets of (M,g) with $|\Gamma_i| \to 0$ and $\Gamma_i \subset \Gamma_1$ for $i = 1, 2, \cdots$. Then we have $\lim_{i\to\infty} Q(\Gamma_i) = Q(S^n)$.*

Proof. Assume that there exists a subsequence $\{\Gamma_{i_k}\}$ of $\{\Gamma_i\}$ such that

$$\lim_{i\to\infty} Q(\Gamma_{i_k}) < Q(S^n). \tag{2.1}$$

We use the same notation $\{\Gamma_i\}$ for this subsequence. By the work on the Yamabe problem in the compact case (see [9], [1], [7], and [8]), there exists a positive solution $u_i \in C^\infty(\Gamma_i)$ with

$$-\Delta u_i + C_n S u_i = q_i u_i^{(n+2)/(n-2)} \text{ on } \Gamma_i, \tag{2.2}$$

$$\int\limits_{\Gamma_i} u_i^{2n/(n-2)}\, dV_g = 1 \quad \text{and} \quad u_i = 0 \quad \text{on} \quad \partial\Gamma_i,$$

where $q_i = Q(\Gamma_i)$. We extend the domain of u_i by defining $u_i = 0$ on the outside of Γ_i. By multiplying (2.2) by u_i^{1+2b} where $b > 0$ and integrating over Γ_i,

$$q_i \int\limits_{\Gamma_i} u_i^{2n/(n-2)+2b}\, dV_g = \int\limits_{\Gamma_i} \frac{1+2b}{(1+b)^2} |\nabla u_i^{1+b}|^2 + C_n S u_i^{2+2b}\, dV_g. \tag{2.3}$$

Using Hölder's inequality,

$$\begin{aligned}\int\limits_{\Gamma_i} u_i^{2n/(n-2)+2b}\, dV_g \\ \leq (\int\limits_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\, dV_g)^{\frac{n-2}{n}} (\int\limits_{\Gamma_i} u_i^{(2n/(n-2)+2b-(1+b)2)\frac{n}{2}}\, dV_g)^{\frac{2}{n}} \\ \leq (\int\limits_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\, dV_g)^{\frac{n-2}{n}}.\end{aligned} \tag{2.4}$$

By the Sobolev Embedding Theorem on Riemannian manifolds for u_i on Γ_i, $\Gamma_i \subset \Gamma_1 \subset M$ (note $\overline{\Gamma_1}$ is compact), and (2.3), for any given $\epsilon > 0$, there exists $C(\epsilon)$ with

$$\begin{aligned}(\int\limits_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\, dV_g)^{\frac{n-2}{n}} \\ \leq (1+\epsilon)\frac{1}{Q(S^n)} \int\limits_{\Gamma_i} |\nabla u_i^{1+b}|^2\, dV_g + C(\epsilon) \int\limits_{\Gamma_i} u_i^{2+2b}\, dV_g \\ \leq (1+\epsilon)\frac{1}{Q(S^n)} \frac{(1+b)^2}{(1+2b)} \Big(\int\limits_{\Gamma_i} q_i u_i^{2n/(n-2)+2b} - C_n \int\limits_{\Gamma_i} S\, u_i^{2+2b}\, dV_g \Big) \\ +C(\epsilon) \int\limits_{\Gamma_i} u_i^{2+2b}\, dV_g.\end{aligned} \tag{2.5}$$

From (2.4),

$$\begin{aligned}(\int_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\, dV_g)^{\frac{n-2}{n}} \\ \leq (1+\epsilon)\frac{1}{Q(S^n)} \frac{(1+b)^2}{(1+2b)} \Big(q_i (\int_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\, dV_g)^{\frac{n-2}{n}} \\ -C_n \int_{\Gamma_i} S\, u_i^{2+2b}\, dV_g \Big) + C(\epsilon) \int_{\Gamma_i} u_i^{2+2b}\, dV_g.\end{aligned} \tag{2.6}$$

Since $q_i < c < \ Q(S^n)$ for some c, we can take $\epsilon > 0$ and $0 < b < \frac{2}{n-2}$ so that

$$(1+\epsilon)\frac{q_i}{Q(S^n)} \frac{(1+b)^2}{(1+2b)} < c_0 < 1 \tag{2.7}$$

for some constant c_0. Therefore, we have

$$\begin{aligned}\Big(1-(1+\epsilon)\frac{q_i}{Q(S^n)} \frac{(1+b)^2}{(1+2b)}\Big) \Big(\int_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\, dV_g\Big)^{\frac{n-2}{n}} \\ +C \int_{\Gamma_i} S\, u_i^{2+2b}\, dV_g \\ \leq C(\epsilon) \int_{\Gamma_i} u_i^{2+2b}\, dV_g,\end{aligned} \tag{2.8}$$

where C is a general positive constant independent of i. We may assume that $|\Gamma_i| < 1$. Using Hölder's inequality,

$$\begin{aligned}\int_{\Gamma_i} u_i^{2+2b}\,dV_g &\le \Big(\int_{\Gamma_i} u_i^{2n/(n-2)}\,dV_g\Big)^{t_1}|\Gamma_i|^{1-t_1}\\ &\le |\Gamma_i|^{1-t_1} < 1,\end{aligned}$$

where $t_1 = \frac{(1+b)(n-2)}{n}$.

Finally, we have

$$\Big(\int_M u_i^{(1+b)2n/(n-2)}\,dV_g\Big)^{\frac{n-2}{n}} = \Big(\int_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\,dV_g\Big)^{\frac{n-2}{n}} \le A_1, \tag{2.9}$$

where A_1 does not depend on i.

From (2.9), we have

$$\begin{aligned}1 = \int_{\Gamma_i} u_i^{2n/(n-2)}\,dV_g &\le \Big(\int_{\Gamma_i} u_i^{(1+b)2n/(n-2)}\,dV_g\Big)^t|\Gamma_i|^s\\ &\le A_1^{nt/(n-2)}|\Gamma_i|^s,\end{aligned} \tag{2.10}$$

where $t = \frac{1}{1+b}$ and $s = \frac{b}{1+b}$.

Since $|\Gamma_i| \to 0$, (2.10) does not hold. This contradiction comes from (2.1). Thus Theorem 2.1 is proved. □

3. Applications on noncompact complete Riemannian manifolds

Using Theorem 2.1, we show that there exists an obstruction to the conformal compactification of certain noncompact complete Riemannian manifolds.

Theorem 3.1. *Let $\overline{(M,g)}$ be a noncompact complete Riemannian manifold of dimension $n \ge 3$ with $\overline{Q(M,g)} < Q(S^n)$. Then (M,g) is not pointwise conformal to a subdomain of any compact Riemannian n-manifold.*

Proof. Assume that (M,g) is pointwise conformal to a subdomain $(M, u^{4/(n-2)}g)$ in (K,h), where (K,h) is a compact Riemannian n-manifold. Take smooth compact domain X_i in M with $X_i \subset \overline{X_i} \subset X_{i+1}$ so that $\mathrm{Vol}(M - X_i, u^{4/(n-2)}g) \to 0$ as $i \to \infty$, where $\mathrm{Vol}(M - X_i, u^{4/(n-2)}g)$ is the volume of $M - X_i$ with the metric $u^{4/(n-2)}g$. We let $|\cdot|_h$ ($|\cdot|_g$) be the length of the tangent vector and S_h (S_g) be the scalar curvature with the metric h (respectively g).

Since we assume that $\overline{Q(M,g)} < Q(S^n)$, there exists a smooth function $f_i \in C_0^\infty(M - \overline{X_i})$ with

$$\frac{\int_{M-\overline{X_i}} |\nabla f_i|_g^2 + C_n S_g {f_i}^2 dV_g}{\Big(\int_{M-\overline{X_i}} {f_i}^{2n/(n-2)}\,dV_g\Big)^{(n-2)/n}} < Q(S^n) - c_0$$

for a positive constant c_0 and $i = 1, 2, \cdots$.

We take a smooth bounded open subset Y_i of M with $X_i \subset \overline{X_i} \subset Y_i$ and

$$Z_i \subset Y_i - X_i \subset M - X_i,$$

where Z_i is the support of f_i. By the conformal invariance of $Q(Y_i - \overline{X_i}, h)$,

$$
\begin{aligned}
(3.1) \qquad Q(Y_i - \overline{X_i}, h) &= Q(Y_i - \overline{X_i}, u^{4/(n-2)} g) \\
&\le \frac{\int_{Y_i - \overline{X_i}} |\nabla (f_i/u)|_h^2 + C_n S_h (f_i/u)^2 dV_h}{\left(\int_{Y_i - \overline{X_i}} (f_i/u)^{2n/(n-2)} \, dV_h \right)^{(n-2)/n}} \\
&\le \frac{\int_{Y_i - \overline{X_i}} |\nabla f_i|_g^2 + C_n S_g {f_i}^2 dV_g}{\left(\int_{Y_i - \overline{X_i}} {f_i}^{2n/(n-2)} \, dV_g \right)^{(n-2)/n}} \\
&\le Q(S^n) - c_0.
\end{aligned}
$$

Since the $\mathrm{Vol}(Y_i - \overline{X_i}, u^{4/(n-2)} g) \to 0$, (3.1) contradicts Theorem 2.1. Theorem 3.1 is proved. □

Using Theorem 3.1, we have the following.

Corollary 3.2. *Let (M, g) be a noncompact complete Riemannian manifold of dimension $n \ge 3$. Assume that there exists a positive number c_0 and a sequence $\{\Omega_i\}$ of smooth open sets where $\Omega_i \subset M - B_i(x_0)$ for $i = 1, 2, \cdots$ and $Q(\Omega_i) < Q(S^n) - c_0$. Then (M, g) is not pointwise conformal to a subdomain of any compact Riemannian n-manifold.*

Example 3.3. Let $B(p, r)$ be the Euclidean ball of radius r with center p in (R^n, δ_{ij}) with $n \ge 3$. Take any smooth metric h on $B(p, 2)$ so that $h = \delta_{ij}$ on $B(p, 2) - B(p, 1)$ and h is not conformally flat at the center p. Then $Q(B(p, 2), h) < Q(S^n)$ by the resolution of the Yamabe problem. Consider (R^n, g) with $n \ge 3$, where $g = h$ on each Euclidean balls $B((5n, 0), 2)$ with center $(5n, 0) \in R \times R^{n-1}$ and radius 2 in the Euclidean norm, for $n = 1, 2, 3, \cdots$ and $g = \delta_{ij}$ on $R^n - \bigcup B((5n, 0), 2)$. Then (R^n, g) is not pointwise conformal to a subdomain of any compact Riemannian n-manifold, since $\overline{Q(R^n, g)} < Q(S^n)$.

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