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AN L^p DEFINITION OF INTERPOLATING BLASCHKE PRODUCTS

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ABSTRACT. We give a new characterization of interpolating Blaschke products in terms of L^p-norms of their reciprocals. We also obtain a characterization of finite unions of interpolating sequences.

1. INTRODUCTION

A sequence $\{z_k\}$ in the unit disk $\mathbb{D}$ of the complex plane is called an interpolating sequence if for every bounded sequence $\{w_k\}$ there exists a bounded analytic function f in $\mathbb{D}$ such that $f(z_k) = w_k$ for all k. In 1958 Carleson gave a geometrical characterization of interpolating sequences as those that are uniformly separated (see (3.1)) [C]. Interpolating Blaschke products are those whose zero set is an interpolating sequence. These Blaschke products play an expanding role in the study of H^∞, the bounded analytic functions in $\mathbb{D}$. The main result of this paper, Theorem C, characterizes these products in terms of L^p-norms of reciprocals of them. Since, in view of (2.3), condition (3.1) is conformally invariant, the characterization must reflect this invariance. We also obtain a characterization of finite unions of interpolating sequences from Theorem C.

2. THE HYPERBOLIC METRIC

The hyperbolic metric in $\mathbb{D}$ is given by $h(z, w) = \tanh^{-1}(2\rho(z, w))$ where

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|$$

for $z, w \in \mathbb{D}$. The ρ-metric has the following properties:

$$0 \leq \rho(z, w) < 1 \text{ for } z, w \in \mathbb{D}, \tag{2.1}$$

$$\rho(z, w) = 1 \text{ if } z \in \partial\mathbb{D},\ w \in \bar{\mathbb{D}}, \tag{2.2}$$

$$\rho(\tau(z), \tau(w)) = \rho(z, w) \text{ whenever } \tau \text{ is a conformal self-map of } \mathbb{D} \text{ and} \tag{2.3}$$

$$\rho(f(z), f(w)) \leq \rho(z, w) \text{ whenever } f : \mathbb{D} \to \bar{\mathbb{D}} \text{ is analytic.} \tag{2.4}$$

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Moreover we define the noneuclidean disk

$$\Delta(z_0, r) = \{z|\rho(z, z_0) < r\}.$$

The disk $\Delta(z_0, r)$ is the image of $\{w \big| |w| < r\}$ under $w(z) = (z + z_0)/(1 + \bar{z}_0 z)$. As such it is also a euclidean disk $B(c, R)$ where

$$c = \frac{1 - r^2}{1 - r^2|z_0|^2} z_0 \tag{2.5}$$

is the center and

$$R = r\frac{1 - |z_0|^2}{1 - r^2|z_0|^2} \tag{2.6}$$

is the radius. See [G].

A sequence $\{z_k\}$ is separated by $\delta > 0$ if

$$\inf_{j \neq k} \rho(z_k, z_j) \geq \delta > 0. \tag{2.7}$$

3. Characterization of interpolating sequences

We assume that the points of the sequence $\{z_k\}$ are the zeros of a convergent Blaschke product

$$B(z) = \prod_{j=1}^{\infty} \frac{-\bar{z}_j}{|z_j|} \left(\frac{z - \bar{z}_j}{1 - \bar{z}_j z} \right).$$

We write

$$B_k(z) = B(z) \frac{1 - \bar{z}_k z}{z - z_k}.$$

Theorem A. *Assume that* $0 < p < 2$. *If* $\{z_k\} \subset \mathbb{D}$ *is uniformly separated by* δ, *that is, if*

$$\inf_k \prod_{j \neq k} \rho(z_j, z_k) = \delta > 0, \tag{3.1}$$

then

$$\sup_k \frac{1}{|\Delta(z_k, \delta)|} \iint_{\Delta(z_k, \delta/2)} \frac{1}{|B(z)|^p} dA \tag{3.2}$$

is finite and depends only on δ *and* p.

Proof. Notice that (3.1) implies that $\{z_k\}$ is separated by δ. The triangle inequality and (2.4) give

$$\begin{aligned} \rho(B_k(z), 0) &\geq \rho(B_k(z_k), 0) - \rho(B_k(z_k), B_k(z)) \\ &\geq \rho(B_k(z_k), 0) - \rho(z_k, z) \\ &= |B_k(z_k)| - \rho(z_k, z). \end{aligned}$$

Hence when $z \in \Delta(z_k, \delta/2)$ we get, using (3.1),

$$|B_k(z)| \geq \delta - \delta/2 = \delta/2.$$

Whence

$$\iint_{\Delta(z_k, \delta/2)} \frac{1}{|B(z)|^p} dA \leq \left(\frac{2}{\delta}\right)^p \iint_{\Delta(z_k, \delta/2)} \left| \frac{1 - \bar{z}_k z}{z - z_k} \right|^p dA.$$

Changing variables this becomes

$$\left(\frac{2}{\delta}\right)^p \iint\limits_{|w|<\delta/2} \frac{1}{|w|^p} \frac{(1-|z_k|^2)^2}{|1+\bar{z}_k w|^4} dA,$$

which is

$$\begin{aligned} &\leq \left(\frac{2}{\delta}\right)^p (1-\delta/2)^{-4}(1-|z_k|^2)^2 \int_0^{2\pi}\int_0^{\delta/2} r^{(1-p)} dr\, d\theta \\ &\leq C(\delta,p)(1-|z_k|)^2. \end{aligned}$$

From (2.6) we see that

$$|\Delta(z_k,\delta/2)| = C(\delta)(1-|z_k|)^2$$

and so (3.2) follows. □

Theorem B. *If $\{z_k\}$ is separated by δ and (3.2) holds for some $0<p<2$, then the sequence $\{z_k\}$ is uniformly separated by a constant depending only on δ and p.*

Proof. Fix k and let $D_k = B(z_k, r_k)$, where $r_k = R_k - |c_k - z_k|$ with R_k and c_k given by (2.5) and (2.6) with $z_0 = z_k$ and $r = \delta/2$. Then $D_k \subset \Delta(z_k,\delta/2)$.

Next let $A_k(z) = 1/B_k(z)$. Choosing an analytic branch of $A_k^p(z)$ we have, for

$$\gamma_r = \{z | z = z_k + re^{i\theta},\ 0<r<r_k, 0\leq\theta<2\pi\},$$

$$\begin{aligned} 2\pi i A_k^p(z_k) &= \int_{\gamma_r} \frac{A_k^p(z)}{z-z_k}\, dz \\ &= \int_{\gamma_r} \frac{1}{|B(z)|^p} \frac{(z-z_k)^{(p-1)}}{(1-\bar{z}_k z)^p}\, dz. \end{aligned}$$

When $z \in D_k$, $|z-z_k|/|1-\bar{z}_k z| < \delta/2$ and so we obtain

$$\begin{aligned} 2\pi|A_k(z_k)|^p &\leq \left(\frac{\delta}{2}\right)^p \int_{\gamma_r} \frac{1}{|B(z)|^p} \frac{1}{|z-z_k|}\, |dz| \\ &= \left(\frac{\delta}{2}\right)^p \int_0^{2\pi} \frac{1}{|B(z_k+re^{i\theta})|^p}\, d\theta. \end{aligned}$$

Hence

$$\pi r_k^2 |A_k(z_k)|^p \leq \left(\frac{\delta}{2}\right)^p \iint\limits_{D_k} \frac{1}{|B(z)|^p}\, dA$$

and using (3.2) we have

$$|A_k(z_k)| \le \frac{\delta}{2}\left(\frac{1}{|D_k|}\iint\limits_{D_k}\frac{1}{|B(z)|^p}\,dA\right)^{1/p}$$
$$\le \left(\frac{C(\delta)}{|\Delta(z_k,\delta/2)|}\iint\limits_{\Delta(z_k,\delta/2)}\frac{1}{|B(z)|^p}\,dA\right)^{1/p}$$
$$< \infty.$$

Taking the supremum over $k = 1, 2, \dots$, it follows that $\{z_k\}$ is uniformly separated. □

Theorem C. *Assume that $\{z_k\}$ is separated by δ. If, for some p, $0 < p < 2$,*

$$\sup_\tau \iint\limits_{\mathbb{D}} \frac{1}{|B(\tau(z))|^p}\,dA < \infty, \tag{3.3}$$

where $\tau : \mathbb{D} \to \mathbb{D}$ is any conformal self-map of $\mathbb{D}$, then the sequence $\{z_k\}$ is uniformly separated.

Conversely, if $\{z_k\}$ is uniformly separated, then (3.3) holds for all p, $0 < p < 2$.

Proof. Assume that (3.3) holds for some p. Let $w = \tau(z) = (z + z_k)/(1 + \bar{z}_k z)$. Then

$$\iint\limits_{\mathbb{D}} \frac{1}{|B(\tau(z))|^p}\,dA = \iint\limits_{\mathbb{D}} \frac{1}{|B(w)|^p}\frac{(1-|z_k|^2)^2}{|1-\bar{z}_k w|^4}\,dA = M < \infty.$$

Next, when $w \in \Delta(z_k, \delta/2)$,

$$|1-\bar{z}_k w| = \left|1-|z_k|^2 - \bar{z}_k(w-z_k)\right|$$
$$\le 1-|z_k|^2 + |w-z_k|$$
$$\le C(\delta)(1-|z_k|^2)$$

by use of (2.5) and (2.6).

Hence

$$\iint\limits_{\Delta(z_k,\delta/2)} \frac{1}{|B(w)|^p}\,dA \le C(\delta)^4 M(1-|z_k|^2)^2.$$

Again, by (2.6), it follows that

$$(1-|z_k|^2)^2 \le \frac{4}{\pi\delta^2}|\Delta(z_k,\delta/2)|.$$

As such (3.2) follows for this p and by Theorem B the sequence $\{z_k\}$ is uniformly separated.

Our proof of the converse requires the following lemma, which follows from results in [KL]. We offer here a simple proof of the implication we will need. □

Lemma 3.4. *If $\{z_k\}$ is uniformly separated by δ, then*

$$\frac{1}{|B(z)|} \le \frac{4}{\delta^2} \quad \text{for } z \in \mathbb{D}\setminus\left(\bigcup_{k=1}^{\infty}\Delta(z_k,\delta/2)\right). \tag{3.5}$$

Proof. We define the finite Blaschke products

$$\begin{aligned} B^{(n)}(z) &= \prod_{j=1}^{n} \frac{-\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z} \\ &= \frac{-\bar{z}_k}{|z_k|} \frac{z - z_k}{1 - \bar{z}_k z} B_k^{(n)}(z). \end{aligned}$$

As such $1/B^{(n)}(z)$ is analytic in $D = \mathbb{D} \backslash (\bigcup_{j=1}^{n} \Delta(z_j, \delta/2))$, continuous across the boundary of $\mathbb{D}$, $\partial\mathbb{D}$, and $1/|B^{(n)}(z)| = 1$ on $\partial\mathbb{D}$. We may assume that the subharmonic function $1/|B^{(n)}(z)|$ in D attains its maximum at $z_0 \in \partial\Delta(z_k, \delta/2)$. As in the proof of Theorem A we see that (3.1) implies that $|B_k^{(n)}(z)| \geq \delta/2$ for $z \in \overline{\Delta(z_k, \delta/2)}$ and we obtain

$$\begin{aligned} \frac{1}{|B^{(n)}(z_0)|} &= \frac{1}{\rho(z_0, z_k)} \frac{1}{|B_k^{(n)}(z_0)|} \\ &\leq 4/\delta^2. \end{aligned}$$

Since this holds for all n it must hold for the infinite Blaschke product and (3.5) follows.

We now continue with the proof of Theorem C and assume that the sequence $\{z_k\}$ is uniformly separated by δ. Notice to obtain (3.3) that it is enough to show that

$$\iint\limits_{\mathbb{D}} \frac{1}{|B(z)|^p} \, dA \leq M(\delta) < \infty$$

where $M(\delta)$ depends only on δ. This is because the Blaschke product $B(\tau(w))$ has zeros $\{\tau^{-1}(z_k)\}$ and if $\{z_k\}$ is uniformly separated, then by (2.3) so is $\{\tau^{-1}(z_k)\}$ (with the same δ). To this purpose we write, with D as above,

$$\iint\limits_{\mathbb{D}} \frac{1}{|B(z)|^p} \, dA = \iint\limits_{\mathbb{D} \backslash D} \frac{1}{|B(z)|^p} \, dA + \iint\limits_{D} \frac{1}{|B(z)|^p} \, dA.$$

Using (3.5) we estimate the second integral by $(4/\delta^2)^p \pi$. Furthermore using (3.2)

$$\begin{aligned} \iint\limits_{\mathbb{D} \backslash D} \frac{1}{|B(z)|^p} \, dA &= \sum_{k=1}^{\infty} \iint\limits_{D_k} \frac{1}{|B(z)|^p} \, dA \\ &\leq C_1(\delta) \sum_{k=1}^{\infty} |\Delta\left(z_k, \delta/2\right)| \\ &\leq C_1(\delta)\pi. \end{aligned}$$

This concludes the proof of Theorem C. □

Corollary 3.6 appears throughout the literature. We give a new proof using Theorem C.

Corollary 3.6. *Suppose that $\{z_k\}$ and $\{\tilde{z}_k\}$ are uniformly separated sequences. If their union $\{z_k\} \cup \{\tilde{z}_k\}$ is separated, then it is also uniformly separated.*

Proof. Let B and $\tilde{B}$ be the Blaschke products with zeros $\{z_k\}$ and $\{\tilde{z}_k\}$. If $B^\star$ is the Blaschke product with zeros $\{z_k\} \cup \{\tilde{z}_k\}$, then $|B^\star| = |B|\ |\tilde{B}|$. By Hölder's inequality,

$$\left(\iint\limits_{\mathbb{D}} \frac{1}{|B^\star|^{1/2}}\, dA \right)^2 \leq \iint\limits_{\mathbb{D}} \frac{1}{|B|}\, dA \ \iint\limits_{\mathbb{D}} \frac{1}{|\tilde{B}|}\, dA.$$

It follows from Theorem C that the right-hand side is uniformly bounded under compositions with conformal $\tau : \mathbb{D} \to \mathbb{D}$. As such so is the left-hand side and Theorem C shows that $B^\star$ is an interpolating Blaschke product. □

We next give a characterization of finite unions of interpolating sequences. The result follows from Theorems A and C.

Theorem D. *The following statements are equivalent.*

(3.7) *The sequence $\{z_k\} \subset \mathbb{D}$ is the union of finitely many uniformly separated sequences.*

(3.8) *For some $0 < q < 2$,*

$$\sup_\tau \iint\limits_{\mathbb{D}} \frac{1}{|B(\tau(z))|^q}\, dA < \infty.$$

(3.9) *The Blaschke sum is conformally invariant, that is,*

$$\sup_\tau \sum_k (1 - |\tau(z_k)|) < \infty.$$

Here the supremum in both cases is over all conformal self-maps τ of $\mathbb{D}$.

The equivalence of (3.7) and (3.9) is proved in [Ho]. Also it is referred to in [N] and [MS]. We include this result for two reasons. First, we offer an alternative proof that (3.7) implies (3.9). The proof uses Lemma 3.4 instead of Hoffman's result [H]. Second, our proof that (3.8) implies (3.7) is geometrically similar to techniques in [Ho].

Proof that (3.7) implies (3.9). Assume that $|z_k| \leq \delta/2$ for some k. Then, using (3.1) and (2.4),

$$\begin{aligned}\prod_{j\neq k} |z_j| &= |B_k(0)| \\ &= \rho(B_k(0), 0) \\ &\geq \rho(B_k(z_k), 0) - \rho(B_k(z_k), B_k(0)) \\ &\geq \delta - \delta/2 = \delta/2.\end{aligned}$$

From the inequality

$$(1 - x) \leq -\log x, \tag{3.10}$$

we obtain

$$\sum_{j\neq k} (1 - |z_j|) \leq \log\left(\frac{2}{\delta}\right)$$

and

$$\sum_{j=1}^{\infty}(1-|z_j|) \le 1+\log\left(\frac{2}{\delta}\right).$$

Otherwise

$$\rho(z_j,0)=|z_j|>\frac{\delta}{2}$$

for all j. By (3.5) we get

$$|B(0)|=\prod_{j=1}^{\infty}|z_j| \ge \delta^2/4$$

and (3.9) follows by applying inequality (3.10) again. □

Proof that (3.7) implies (3.8). Assume that $\{z_k\}$ is the union of m uniformly separated sequences and $B^i(z)$, $i=1,\dots,m$, are the Blaschke products with those sequences as their zeros. As such $|B(z)|=\prod_{i=1}^{m}|B^i(z)|$ and by Hölder's inequality we have

$$\left(\iint_{\mathbb{D}}\frac{1}{|B(z)|^{p/m}}dA\right)^{m/p} \le \prod_{i=1}^{m}\left(\iint_{\mathbb{D}}\frac{1}{|B^i(z)|^{p}}dA\right)^{1/p}. \tag{3.11}$$

Now Theorem C shows that the right-hand side of (3.11) is uniformly bounded, for $0<p<2$, under compositions with conformal $\tau:\mathbb{D}\to\mathbb{D}$. As such (3.8) follows.

Next, as in [Ho], we decompose the disk $\mathbb{D}$ into polar rectangles. Let W be the collection of rectangles of the form

$$\left\{z\in\mathbb{D}\big|1-2^{-\alpha}\le|z|<1-2^{-\alpha-1}, 2\pi\beta 2^{-\alpha}\le \arg z<2\pi(\beta+1)2^{-\alpha}\right\}$$

where $\alpha=0,1,2,\dots$ and $\beta=0,1,2,\dots,2^{\alpha}-1$.

We write $\nu(R)$ for the number of zeros of $\{z_k\}$ that lie in the rectangle $R\in W$. □

Lemma 3.12. *The sequence $\{z_k\}$ is a finite union of separated sequences if and only if $M=\sup\{\nu(R)|R\in W\}$ is finite.*

Proof. One calculates that there exists $\eta<1$ such that

$$\rho(z,w)<\eta \tag{3.13}$$

when $z,w\in R\in W$, and

$$\rho(z,w)>1/\eta \tag{3.14}$$

when z and w lie in noncontiguous rectangles in W. If M is finite, then by (3.14) the sequence $\{z_k\}$ can be divided into $4M$ separated sequences. (It is in this way that the implication, (3.9) implies (3.7), is proven in [Ho].) Conversely, suppose that $\{z_k\}$ is the union of N separated sequences, each separated by σ. In view of (3.13) each rectangle in W can be divided into a fixed number of subrectangles with ρ-diameters less than or equal to σ. As such no subrectangle contains more than N points of the sequence $\{z_k\}$ and so $M<\infty$.

We now prove that (3.8) implies (3.7). First we show that (3.8) implies $M=\sup\{\nu(R)|R\in W\}$ is finite. Let $z_0\in R\in W$ and assume there are ν zeros of $B(z)$ in R. If $\tau_0(z)=(z-z_0)/(1-\bar{z}_0 z)$, then $\tau_0(R)$ is contained in the closed disk about the origin of diameter η. This follows from (3.13) and (2.3). As such $B(\tau_0^{-1})$ has ν zeros, $z_1, z_2, \dots, z_\nu$, with $|z_j| \le \eta$ for $j = 1, 2, \dots, \nu$. Hence

$$\iint_{\mathbb{D}} \frac{1}{|B(\tau_0^{-1}(z))|^q} dA \ge \iint_{B(0,\eta)} \prod_{j=1}^{\nu} \rho(z, z_j)^{-q} dA$$
$$\ge \pi\eta^{2-\nu q}.$$

Whence if $\{R_i\}$ is a sequence in W with $\nu(R_i) \to \infty$, then (3.8) fails.

Next it follows from Lemma 3.12 that, if (3.8) holds, then $\{z_k\}$ is a finite union of separated sequences. Let $B^1, B^2, \dots, B^n$ be the Blaschke products with these sequences as their zeros. We have, for each $j = 1, 2, \dots, N$,

$$\iint_{\mathbb{D}} \frac{1}{|B^j(z)|^q} dA < \iint_{\mathbb{D}} \frac{1}{|B(z)|^q} dA. \tag{3.15}$$

Replacing z with $\tau(z)$ in (3.15), we see that (3.8) implies, by Theorem C, that each $B^j(z)$ is an interpolating Blaschke product. □

4. Remarks

In view of the substantial use of conformal invariance, one might wonder if a Blaschke product exists for which (3.3) fails for $\tau(z) = z$ alone. However we note that the above constructions give

$$\iint_{\mathbb{D}} \frac{1}{|B(z)|^p} dA \ge \sum_{R \in W} \eta^{-\nu(R)p} m(R).$$

Hence if $\nu(R)$ grows sufficiently fast the sum diverges, for example if $z_k = 1 - k^{-2}$.

We also remark that the exponent, p/m, in (3.11), is best possible in the case of a zero of order m.

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