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DIMENSION ZERO VS MEASURE ZERO

ONDŘEJ ZINDULKA

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Abstract. The following problem is discussed: If X is a topological space of universal measure zero, does it have also dimension zero? It is shown that in a model of set theory it is so for separable metric spaces and that under the Martin's Axiom there are separable metric spaces of positive dimension yet of universal measure zero. It is also shown that for each finite measure in a metric space there is a zero–dimensional subspace that has full measure. Similar questions concerning perfectly meager sets and other types of small sets are also discussed.

1. Introduction

If an analytic subspace of a Euclidean space has positive topological dimension, then it admits a Borel probability measure that vanishes on singletons, i.e. such a space is not of universal measure zero. In general, it is not that unreasonable to expect a separable metric space of positive topological dimension to be ample enough to admit a Borel probability measure vanishing on singletons.

So the question is: Is it so that a separable metric space of universal measure has topological dimension zero? Or can it be that there is a separable metric space, perhaps a subset of $\mathbb{R}^n$, that is of universal measure zero and yet has positive topological dimension?

It turns out that the problem is independent in that in some models of ZFC, the Zermelo–Fraenkel set theory including the axiom of choice, the answer is yes and in other models it is no.

Instead of universal measure zero one can pose analogous questions concerning perfectly meager sets, strong universal measure zero sets, Q-sets, etc.

Section 2 contains a result that is used in section 3 and that is of independent interest: In a metrizable space, each finite measure admits a zero–dimensional subset whose complement is of measure zero. A category analogue of this result is also established.

In section 3 we prove that under the Martin's Axiom, there are separable metric spaces that have universal measure zero and positive dimension. We also obtain a set in the plane that has universal measure zero and yet projects onto each line. On the other hand, in a random reals model this cannot happen. I am greatly indebted to Stevo Todorčević, who pointed me to a model in which Theorem 3.5 fails.

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In section 4 we discuss the situation for perfectly meager spaces. The results are to some extent, but not completely, similar. We construct, with the aid of the Continuum Hypothesis, a separable metric space that has infinite dimension and at the same time is perfectly meager and of universal measure zero.

In section 5 we consider strong measure zero, σ-sets and Q-sets, and list some unresolved problems.

We use ω and ω_1 to denote the first infinite and uncountable cardinal, respectively, and $\mathfrak{c}$ to denote the cardinal of continuum. $|A|$ denotes the cardinality of a set A. All topological spaces we work with are metrizable, so the term "space" refers to a metrizable topological space. A Borel measure (or just a measure) in a metric or topological space is a σ-additive $[0,\infty]$–valued measure on the algebra of Borel sets. For a Borel measure μ we denote by $\mu^\sharp$ its outer measure, i.e. a mapping $A \mapsto \inf\{\mu E : E \supseteq A \text{ Borel}\}$ defined for each $A \subseteq X$.

2. Measures are carried by zero–dimensional subsets

Recall that a space is zero–dimensional if it has a base consisting of clopen sets. In this section we ask if in a metric space each finite Borel measure μ is carried by a zero–dimensional subset, i.e. if there is a Borel, zero–dimensional set $B \subseteq X$ such that $\mu(X \setminus B) = 0$. An affirmative answer is given by Corollary 2.3 and Theorem 2.8.

The proofs actually rely only on combinatorial properties of the ideal of negligible sets. If X is a space and $\mathcal{J}$ an ideal on X, call $\mathcal{J}$ *proper* if it contains all singletons and $X \notin \mathcal{J}$. Differing slightly from the usual definitions, call $\mathcal{J}$ *σ-additive* if $\bigcup \mathcal{F} \in \mathcal{J}$ for each countable family $\mathcal{F} \subseteq \mathcal{J}$ of closed sets. Call $\mathcal{J}$ *saturated* if each uncountable disjoint family of closed sets contains an element of $\mathcal{J}$.

Theorem 2.1. *In each metric space X there exists a cover $\{G_\alpha : \alpha < \omega_1\}$ of X by G_δ–subsets such that*

(i) *for each $\alpha < \omega_1$, the set G_α is zero–dimensional,*
(ii) *for each σ-additive saturated ideal $\mathcal{J}$ in X there exists $\alpha < \omega_1$ such that $X \setminus G_\alpha \in \mathcal{J}$.*

Proof. By the Bing's metrization theorem, X has a σ-discrete base. Let $\langle \mathcal{B}_n : n \in \omega\rangle$ be a sequence of discrete open families such that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a base of X. Since metric spaces are perfectly normal, for each $B \in \mathcal{B}$ there is a continuous function $\phi_B : X \to [0,1]$ such that $B = \phi_B^{-1}(0,1]$. As $\mathcal{B}_n$'s are discrete, the formula

$$f_n = \sum_{B\in\mathcal{B}_n} \phi_B \tag{2.1}$$

defines for each $n \in \omega$ a continuous function $f_n : X \to [0,1]$. For each $n, j \in \omega$ and $x \in X$ set

$$g_{nj}(x) = \min\big(1,\, j\cdot f_n(x)\big). \tag{2.2}$$

Then g_{nj}'s are obviously continuous. Pick a set $E \subseteq (0,1)$ such that $|E| = \omega_1$ and enumerate it as $\{r_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$ set

$$\begin{aligned} F_\alpha &= \bigcup \{g_{nj}^{-1}(r_\alpha) : n, j \in \omega\},\\ G_\alpha &= X \setminus F_\alpha,\\ \mathcal{D}_\alpha &= \{g_{nj}^{-1}(r_\alpha, 1] \cap B : B \in \mathcal{B}_n \,\&\, n, j \in \omega\}. \end{aligned}$$

Each G_α is obviously a G_δ–set. We show that $\{G_\alpha : \alpha < \omega_1\}$ is the required family.

a) $\{G_\alpha : \alpha < \omega_1\}$ *is a cover of* X. We have to show that $\bigcap_{\alpha<\omega_1} F_\alpha = \emptyset$, so assume for the contrary that there is $x \in X$ such that $x \in F_\alpha$ for each $\alpha < \omega_1$. Then for each $\alpha < \omega_1$ there are n_α and j_α such that $g_{n_\alpha j_\alpha}(x) = r_\alpha$. The mapping $\alpha \mapsto \langle n_\alpha, j_\alpha\rangle$ maps ω_1 into $\omega \times \omega$, therefore there is a pair $\langle n, j\rangle$ that is an image of two distinct ordinals $\alpha < \beta < \omega_1$. Thus $g_{nj}(x) = r_\alpha$ and $g_{nj}(x) = r_\beta$, hence $r_\alpha = r_\beta$, and hence $\alpha = \beta$, a contradiction.

b) For each $\alpha < \omega_1$, $\mathcal{D}_\alpha$ *is a* σ*-discrete base for* X. $\mathcal{D}_\alpha$ is obviously open. It is also σ-discrete: for the family $\{g_{nj}^{-1}(r_\alpha, 1] \cap B : B \in \mathcal{B}_n\}$ is discrete for each $n, j \in \omega$. To show that it is a base, consider a point $x \in X$ and a neighborhood U of x. Since $\mathcal{B}$ is a base, there is $n \in \omega$ and $B \in \mathcal{B}_n$ such that $x \in B \subseteq U$. Since $\phi_B(x) > 0$, there is $j \in \omega$ such that $j \cdot \phi_B(x) > 1$. Therefore $g_{nj}(x) = \min\big(1, j \cdot f_n(x)\big) = \min\big(1, j \cdot \phi_B(x)\big) = 1 > r_\alpha$, hence $x \in g_{nj}^{-1}(r_\alpha, 1] \cap B \subseteq B \subseteq U$. As $g_{nj}^{-1}(r_\alpha, 1] \cap B \in \mathcal{D}_\alpha$, we are done.

c) For each $\alpha < \omega_1$, G_α *is zero–dimensional.* By *b)*, the family $\{D \cap G_\alpha : D \in \mathcal{D}_\alpha\}$ is a σ-discrete base for G_α, so it suffices to show that it is relatively clopen in G_α, i.e. that the boundary of each $D \in \mathcal{D}_\alpha$ is contained in F_α. Let $n, j \in \omega$ and $B \in \mathcal{B}_n$. It follows from (2.1), (2.2) and the definition of ϕ_B that $g_{nj}^{-1}(r_\alpha, 1] \cap B = \phi_B^{-1}(r_\alpha/j, 1]$, hence $\overline{g_{nj}^{-1}(r_\alpha, 1] \cap B} = \overline{g_{nj}^{-1}(r_\alpha, 1]} \cap B$. Consequently,

$$\begin{aligned}\overline{g_{nj}^{-1}(r_\alpha, 1] \cap B} &\setminus (g_{nj}^{-1}(r_\alpha, 1] \cap B)\\ &= \big(\overline{g_{nj}^{-1}(r_\alpha, 1]} \setminus g_{nj}^{-1}(r_\alpha, 1]\big) \cap B \subseteq g_{nj}^{-1}(r_\alpha) \cap B \subseteq g_{nj}^{-1}(r_\alpha) \subseteq F_\alpha.\end{aligned}$$

d) For each σ*-additive saturated ideal* $\mathcal{J}$ *in* X *there is* $\alpha < \omega_1$ *such that* $F_\alpha \in \mathcal{J}$. For $n, j \in \omega$, put $\mathcal{F}_{nj} = \{g_{nj}^{-1}(r_\alpha) : \alpha < \omega_1\}$. The family $\mathcal{F}_{nj}$ is closed and disjoint. Since $\mathcal{J}$ is saturated, there is a countable set $A_{nj} \subseteq \omega_1$ such that $g_{nj}^{-1}(r_\alpha) \in \mathcal{J}$ for each $\alpha \notin A_{nj}$. The set $A = \bigcup_{n,j\in\omega} A_{nj}$ is then countable, thus $\omega_1 \setminus A$ is nonempty, and, as $\mathcal{J}$ is σ-additive, for each $\alpha \in \omega_1 \setminus A$ we have

$$F_\alpha = \bigcup_{n,j\in\omega} g_{nj}^{-1}(r_\alpha) \in \mathcal{J}.$$

a) shows that $\{G_\alpha : \alpha < \omega_1\}$ is a cover, *c)* shows (i) and *d)* shows (ii). We are done. □

Remark 2.2. First, the above proof actually shows that $\{G_\alpha : \alpha \in I\}$ is a cover of X for each uncountable $I \subseteq \omega_1$ and that condition (ii) of the theorem can be strengthened to *"for each* σ*-additive saturated ideal* $\mathcal{J}$ *in* X, $X \setminus G_\alpha \in \mathcal{J}$ *for all but countably many* $\alpha < \omega_1$*"*. Second, if X is supposed to be finitely dimensional or second countable, then the proof can be considerably simplified.

If μ is a finite Borel measure in X, then the ideal of μ-negligible sets is obviously σ-additive and saturated. So we have the following corollary to Theorem 2.1.

Corollary 2.3. *For each finite Borel measure* μ *in* X *there is* $\alpha < \omega_1$ *such that* $\mu(X \setminus G_\alpha) = 0$. *In particular, there is a zero–dimensional* G_δ*–set* $G \subseteq X$ *such that* $\mu(X \setminus G) = 0$.

Recall that a subset of a topological space is called *perfect* if it is closed and has no isolated points, and *meager* if it can be covered by countably many nowhere dense sets.

Corollary 2.4. *If P is a second countable, perfect subset of X, then there is $\alpha < \omega_1$ such that $P \setminus G_\alpha$ is meager in P.*

Proof. The family $\mathcal{J}_P = \{A \subseteq X : A \cap P \text{ is meager in } P\}$ is obviously a σ-additive ideal in X. It is also saturated, for otherwise there would be an uncountable disjoint family of sets in P with nonempty interior. Thus the assertion follows directly from Theorem 2.1. □

Sometimes one can replace F_σ–sets with G_δ–sets. The following corollary to Theorem 2.1 is obvious:

Corollary 2.5. *If $\mathcal{J}$ is a σ-additive, saturated ideal in a metric space X and for each $A \in \mathcal{J}$ there is a G_δ–set $G \in \mathcal{J}$ such that $A \subseteq G$, then there exists a zero–dimensional F_σ–set F such that $X \setminus F \in \mathcal{J}$.*

This obviously applies to a measure ideal. For such an ideal one can do a little better. We prepare two lemmas. If $\langle \mathbb{P}, \leqslant \rangle$ is a p.o. set and $A \subseteq \mathbb{P}$, then A is *cofinal* in $\mathbb{P}$ if for each $p \in \mathbb{P}$ there is $q \in A$ such that $p \leqslant q$. Provide the set ω^ω of all functions from ω to ω with a pointwise order. Recall that the cardinal $\mathfrak{d}$ is defined by

$$\mathfrak{d} = \min\{|A| : A \subseteq \omega^\omega \text{ is cofinal in } \omega^\omega\}.$$

Clearly $\omega_1 \leqslant \mathfrak{d} \leqslant \mathfrak{c}$. See [2] or [10] for more information.

Lemma 2.6. *Let $\mathbb{P} = (0,1]^{\omega\times\omega}$ be the set of all $(0,1]$–valued double sequences. Order $\mathbb{P}$ by $\phi \leqslant \psi$ iff $\phi(n,j) \geqslant \psi(n,j)$ for each $n, j \in \omega$. Then*

$$\min\{|A| : A \textit{ is cofinal in } \langle \mathbb{P}, \leqslant \rangle\} = \mathfrak{d}.$$

Proof. Pick a bijection $\sigma : \omega \to \omega \times \omega$ and consider a pair of mappings $G : \mathbb{P} \to \omega^\omega$, $H : \omega^\omega \to \mathbb{P}$ defined by

$$G(\phi)(n) = \min\{i \in \omega : (\phi \circ \sigma)(n) \geqslant 1/i\}, \quad H(f)(n,j) = \frac{1}{(f \circ \sigma^{-1})(n,j)}.$$

It is routine to verify that $\phi \leqslant H(f) \Leftrightarrow G(\phi) \leq f$ and $\phi \geqslant H(f) \Leftrightarrow G(\phi) \geq f$ hold for each $\phi \in \mathbb{P}$ and $f \in \omega^\omega$, which in turn implies that *a)* if $C \subseteq \mathbb{P}$ is cofinal in $\langle \mathbb{P}, \leqslant \rangle$, then $H[C]$ is cofinal in $\langle \omega^\omega, \leq \rangle$ and *b)* if $C \subseteq \omega^\omega$ is cofinal in $\langle \omega^\omega, \leq \rangle$, then $G[C]$ is cofinal in $\langle \mathbb{P}, \leqslant \rangle$. The rest is trivial. □

Lemma 2.7. *Let X be a metric space. Let $\langle F_n : n \in \omega \rangle$ be a sequence of closed subsets of X and $F = \bigcup_{n\in\omega} F_n$. Then there is a family $\{G_\alpha : \alpha < \mathfrak{d}\}$ of G_δ–sets such that $\bigcap_{\alpha<\mathfrak{d}} G_\alpha = F$ and for each finite Borel measure μ there is $\alpha < \mathfrak{d}$ such that $\mu(G_\alpha \setminus F) = 0$.*

Proof. Consider the p.o. set $\mathbb{P}$ of Lemma 2.6 and a cofinal subset $\mathbb{D} \subseteq \mathbb{P}$ of cardinality $\mathfrak{d}$ that exists by the lemma.

For $A \subseteq X$ and $\varepsilon > 0$, denote by A^ε the open ε–neighborhood of A. For $\phi \in \mathbb{P}$ set $U(\phi) = \bigcap_{j\in\omega} \bigcup_{n\in\omega} F_n^{\phi(n,j)}$ and consider the families

$$\mathcal{U} = \{U(\phi) : \phi \in \mathbb{P}\}, \qquad \mathcal{V} = \{U(\phi) : \phi \in \mathbb{D}\}.$$

Each $U(\phi) \in \mathcal{U}$ is obviously G_δ and $G(\phi) \supseteq F$. Thus $F \subseteq \bigcap \mathcal{U}$. Conversely, if $x \in X \setminus F$, then, as F_n's are closed, for each $n \in \omega$ there is $\varepsilon_n > 0$ such that the lower distance $d(x, F_n) > \varepsilon_n$. Set $\phi(n,j) = \varepsilon_n$ for each $n, j \in \omega$. Then obviously

$\phi \in \mathbb{P}$ and $x \notin U(\phi)$, whence $x \notin \bigcap \mathcal{U}$. It follows that $F = \bigcap \mathcal{U}$. As $\mathbb{D}$ is cofinal in $\mathbb{P}$ and clearly $\phi \leqslant \psi$ implies $U(\phi) \supseteq U(\psi)$, we infer that $F = \bigcap \mathcal{V}$ as well.

Let μ be a finite Borel measure in X and let $\varepsilon > 0$ be given. The sets F_n are closed. Therefore for each $n \in \omega$ there exists $\delta(n, \varepsilon) > 0$ such that $\mu\big(F_n^{\delta(n,\varepsilon)} \setminus F_n\big) < \varepsilon \cdot 2^{-n}$. So if we set $\phi(n, j) = \delta(n, \frac{1}{j})$, we have $\phi \in \mathbb{P}$ and $\mu(U(\phi) \setminus F) < \frac{2}{j}$ for each $j \in \omega$. Thus $\mu(U(\phi) \setminus F) = 0$. The cofinality of $\mathbb{D}$ in $\mathbb{P}$ yields that ϕ can be taken from $\mathbb{D}$.

So $F = \bigcap \mathcal{V}$ and for each μ there is $V \in \mathcal{V}$ such that $\mu(V \setminus F) = 0$. Since $|\mathcal{V}| = |\mathbb{D}| = \mathfrak{d}$, the lemma is proved. □

The following theorem now easily follows from Lemma 2.7, de Morgan's laws, Corollary 2.3 and the evident fact that $|\omega_1 \times \mathfrak{d}| = \mathfrak{d}$.

Theorem 2.8. *In each metric space X there exists a cover $\{F_\alpha : \alpha < \mathfrak{d}\}$ of X by F_σ–subsets such that*

(i) *for each $\alpha < \mathfrak{d}$, the set F_α is zero–dimensional,*
(ii) *for each finite Borel measure μ in X there is $\alpha < \mathfrak{d}$ such that $\mu(X \setminus F_\alpha) = 0$.*

3. Dimension zero vs universal measure zero

We show that under a set–theoretic assumption—namely the Martin's Axiom for countable p.o. sets—there are separable metric spaces of universal measure zero, yet of positive topological dimension.

Recall that a topological space X is said to be of *universal measure zero* if each finite, diffused (i.e. vanishing on singletons) Borel measure in X is identically zero. Equivalently, if $Y \supseteq X$ is another space and μ is a finite Borel measure in Y, then $\mu^\sharp X = 0$.

Since in this section we consider exclusively separable metric spaces, it does not matter which definition of topological dimension we use, for the covering dimension, the small inductive dimension and the large inductive dimension coincide in the realm of separable metrizable spaces. We use $\dim X$ to denote the dimension of a space X and instead of definitions we rather appeal on the following theorems that can be found in any textbook on topological dimension, e.g. in [3].

X denotes a separable metrizable space. *The Enlargement Theorem:* Each subspace of X is contained in a G_δ–subspace of X of the same dimension. *The Addition Theorem:* If $A, B \subseteq X$, then $\dim(A \cup B) \leqslant \dim A + \dim B + 1$. *The Decomposition Theorem:* If $\dim X \leqslant n$, then there is a cover $\{G_0, G_1, \ldots, G_n\}$ of X by zero–dimensional G_δ–sets. *The Countable Sum Theorem:* If $\{F_i : i \in \omega\}$ is a cover of X by F_σ–sets and $\dim F_i \leqslant n$ for each $i \in \omega$, then $\dim X \leqslant n$.

Recall the countable Martin's Axiom, in its topological (and perhaps most edible) form. Let κ be a cardinal. The *countable Martin's Axiom* for κ, denoted by $\mathrm{MA}_{\mathrm{countable}}(\kappa)$, is the assertion "*In a second countable compact space, an intersection of κ many dense open sets is nonempty.*" Equivalently, $\mathbb{R}$ cannot be covered by κ many nowhere dense subsets. The *countable Martin's Axiom*, denoted by $\mathrm{MA}_{\mathrm{countable}}$, is the assertion "$\mathrm{MA}_{\mathrm{countable}}(\kappa)$ *for each* $\kappa < \mathfrak{c}$." We refer to [4], [6] or [10] for more information.

We make use of the following lemmas. Lemma 3.1 is [4, 22H(d), B1B], and Lemma 3.2 is [9, Theorem 3].

Lemma 3.1. *Let κ be a cardinal. Assume $\mathrm{MA}_{\mathrm{countable}}(\kappa)$. Each metrizable space of cardinality κ is of universal measure zero.*

Lemma 3.2. *Let κ be a cardinal. Assume* $\mathrm{MA}_{\mathrm{countable}}(\kappa)$. *Then each cover of ω^ω by κ many closed sets has a countable subcover.*

Recall that a metric space X is *analytic* if it is a continuous image of a nonempty completely metrizable separable space; equivalently, of the space ω^ω of irrationals. Each analytic space is separable. A closed subspace of an analytic space is analytic.

Lemma 3.3. *Let κ be a cardinal. Assume* $\mathrm{MA}_{\mathrm{countable}}(\kappa)$. *Let $n \in \omega$ and let X be an analytic metric space. If $\{F_\alpha : \alpha < \kappa\}$ is a cover of X by F_σ–sets, such that* $\dim F_\alpha \leqslant n$ *for each $\alpha < \kappa$, then* $\dim X \leqslant n$.

Proof. We can assume that F_α's are closed. As X is analytic, there is a continuous surjective mapping $f : \omega^\omega \to X$. The family $\{f^{-1}F_\alpha : \alpha < \kappa\}$ is a closed cover of ω^ω. Infer from Lemma 3.2 that there is a countable set $I \subseteq \kappa$ such that $\{f^{-1}F_\alpha : \alpha \in I\}$ covers ω^ω. As f is onto, it follows that $\{F_\alpha : \alpha \in I\}$ covers X. So X is covered by countably many closed sets of dimension at most n. By the Countable Sum Theorem, $\dim X \leqslant n$. □

Theorem 3.4 (Assume $\mathrm{MA}_{\mathrm{countable}}$)**.** *Each analytic metric space X contains a subspace Y such that*

(i) $|Y \cap F| < \mathfrak{c}$ *for each F_σ zero–dimensional set $F \subseteq X$,*
(ii) $Y \cap F \neq \emptyset$ *for each closed set $F \subseteq X$ of positive dimension.*

Proof. When $\dim X = 0$ then it is enough to put $Y = \emptyset$, so we shall assume that $\dim X > 0$. As X is separable, there are only $\mathfrak{c}$ many F_σ–subsets of X. Therefore it is possible to arrange all F_σ–sets that have dimension zero in a sequence $\langle F_\alpha : \alpha < \mathfrak{c}\rangle$ and all closed sets that have positive dimension in a sequence $\langle H_\alpha : \alpha < \mathfrak{c}\rangle$.

For each $\alpha < \mathfrak{c}$ set

$$Y_\alpha = H_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta.$$

The Martin's Axiom and Lemma 3.3 ensure that the set $\bigcup_{\beta<\alpha} F_\beta$ has dimension zero. As the dimension of H_α is positive, it follows that Y_α is nonempty. Pick a point $y_\alpha \in Y_\alpha$ and set

$$Y = \{y_\alpha : \alpha < \mathfrak{c}\}.$$

The set Y obviously satisfies (ii). Since $\alpha < \beta < \mathfrak{c}$ yields $y_\beta \notin F_\alpha$, it follows that $Y \cap F_\alpha \subseteq \{y_\beta : \beta \leqslant \alpha\}$ for all $\alpha < \mathfrak{c}$. Therefore $|Y \cap F_\alpha| \leqslant |\alpha| < \mathfrak{c}$ and (i) follows. The proof is complete. □

Theorem 3.5 (Assume $\mathrm{MA}_{\mathrm{countable}}$)**.** *Each analytic metric space X contains a subspace Z of universal measure zero such that* $\dim Z = \dim X - 1$.

Proof. Consider the space Y of Theorem 3.4. We first prove that Y is of universal measure zero. Let μ be a finite, diffused Borel measure in X. By virtue of Theorem 2.8, there is $\alpha < \mathfrak{c}$ such that $\mu(X \setminus F_\alpha) = 0$. Hence $\mu^\sharp(Y) = \mu^\sharp(Y \cap F_\alpha)$ (recall that $\mu^\sharp$ denotes the outer measure). But $|Y \cap F_\alpha| \leqslant |\alpha| < \mathfrak{c}$ and we infer from Lemma 3.1 that $Z \cap F_\alpha$ is of universal measure zero. In particular, $\mu^\sharp(Y \cap F_\alpha) = 0$. Thus Y is μ–negligible.

Next we prove that $\dim Y \geqslant \dim X - 1$. Let $n \in \omega$ be such that $\dim X > n$. If $\dim Y < n$, then, by the Enlargement Theorem, there is a G_δ–set $E \supseteq Y$ such that $\dim E = \dim Y < n$. The complement of E in X is then an F_σ–set of dimension at least 1: Otherwise $\dim X \leqslant \dim E + \dim(X \setminus E) + 1 < n + 0 + 1 = n$ by the

Addition Theorem. Therefore, according to the Countable Sum Theorem, there is a closed set $H \subseteq X$ disjoint with E, and *a fortiori* with Y, of positive dimension, which is impossible by condition (ii) of Theorem 3.4. Thus $\dim Z \geqslant n$.

The required set Z is easily constructed from Y: If $\dim X < \infty$ and $\dim Y = \dim X - 1$, or $\dim X = \infty$, put $Z = Y$. If $\dim X < \infty$ and $\dim Y = \dim X$, then take for Z a subset of Y that satisfies $\dim Z = \dim Y - 1$. Such a set exists by [3, 1.5.1]. □

The equality $\dim Z = \dim X - 1$ clearly cannot be improved to $\dim Z = \dim X$: If a subset of $\mathbb{R}^n$ has dimension n, then it has a nonempty interior and thus contains a copy of the Cantor set which is not of universal measure zero. The space Z cannot be analytic (except the case $\dim X \leqslant 1$): Since it is of positive dimension it is of cardinality $\mathfrak{c}$. Each analytic metric space of cardinality $\mathfrak{c}$ contains a copy of the Cantor set.

So under $\mathrm{MA}_{\mathrm{countable}}$ there exists a separable metric space X of universal measure zero such that $\dim X = \infty$, and for each $n \in \omega$ there exists a space $X \subseteq \mathbb{R}^{n+1}$ of universal measure zero such that $\dim X = n$. Here are some corollaries to Theorem 3.4. Recall that a projection $\pi : X \to X$ of a Banach space X is a linear bounded mapping that satisfies $\pi \circ \pi = \pi$.

Corollary 3.6 (Assume $\mathrm{MA}_{\mathrm{countable}}$)**.** *Let X be a separable Banach space. There exists a set $Y \subseteq X$ of universal measure zero such that $\pi Y = \pi X$ for all nontrivial projections $\pi : X \to X$.*

Proof. Let $y \in \pi X$. As π is nontrivial, the fiber $\pi^{-1}y$ is a nontrivial affine subspace of X and therefore it contains a homeomorphic copy of the real line. The space Y of Theorem 3.4 meets every closed one–dimensional set and thus it meets also $\pi^{-1}y$. Hence $y \in \pi Y$. By the proof of Theorem 3.5, Z is of universal measure zero. □

Corollary 3.7 (Assume $\mathrm{MA}_{\mathrm{countable}}$)**.** *There exists a set $Z \subseteq \mathbb{R}^2$ of universal measure zero that projects onto each line.*

The set Z of Theorem 3.4 is strange. It is so thin that almost all (with respect to any probabitily) shots will miss it, and yet it is so dense that any path between distinct points outside Z will pass through Z.

Corollary 3.8 (Assume $\mathrm{MA}_{\mathrm{countable}}$)**.** *Each analytic metric space contains a set of universal measure zero that meets each nontrivial curve in X.*

A brief inspection of the proof of Theorem 3.4 shows that under the Continuum Hypothesis one can drop the analyticity of X.

Proposition 3.9 (Assume the Continuum Hypothesis)**.** *Each nonempty separable metric space X contains a subspace Z of universal measure zero such that* $\dim Z = \dim X - 1$.

We will now show that Theorem 3.5 and Proposition 3.9 are not theorems of ZFC, the Zermelo–Fraenkel theory of sets including the axiom of choice. I am greatly indebted to Stevo Todorčević who pointed me to the idea of the proof.

Theorem 3.10. *It is relatively consistent with* ZFC *that $\mathfrak{c} = \omega_2$ and that each separable metric space of universal measure zero is zero–dimensional.*

Proof. By a result of Baumgartner and Laver (see [2] or [8]), in the random reals model forcing $\mathfrak{c} = \omega_2$, each subset of ω^ω of universal measure zero has cardinality ω_1 or less. Let X be a separable metric space. Denote by $\tilde{X}$ its completion. Since $\tilde{X}$ is analytic, there is a continuous surjection $f : \omega^\omega \to \tilde{X}$. For each $x \in X$ pick a point $\bar{x} \in f^{-1}(x)$ and put $A = \{\bar{x} : x \in X\}$. The mapping $\phi = f \restriction A : A \to X$ is clearly a continuous bijection onto X. If there is a nontrivial, finite, diffused measure μ in A, then the image measure $\mu\phi^{-1}$ is a nontrivial, finite, diffused measure in X. Therefore, if X is of universal measure zero, then so is A. By virtue of the mentioned property of the model, the set A has cardinality at most ω_1. As ϕ is bijective, $|X| < \omega_2 = \mathfrak{c}$. But each metric space of positive dimension has cardinality at least $\mathfrak{c}$. It follows that $\dim X = 0$. □

Note that the proof also shows that Corollaries 3.6 and 3.7 fail in the model.

4. Dimension zero vs. perfectly meager

We now examine if there is an analogy for perfectly meager spaces. By the usual definition, a set of reals is called perfectly meager iff for each perfect set $P \subseteq \mathbb{R}$ the set $X \cap P$ is meager relative to the topology of P. Following the pattern, we extend this definition to an arbitrary separable metric space X, defining X to be *perfectly meager* iff each perfect set $P \subseteq X$ is meager in itself. The following lemma shows that the extension is meaningful. It is easy to prove: (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (ii) are obvious, (ii) $\Longrightarrow$ (i) is trivial and (i) $\Longrightarrow$ (iii) follows from the Cantor–Bendixson theorem.

Lemma 4.1. *For a separable metric space X, the following are equivalent:*

(i) *X is perfectly meager,*
(ii) *there is a separable metric space $Y \supseteq X$ such that $X \cap P$ is meager in P for each perfect set $P \subseteq Y$,*
(iii) *for each separable metric space $Y \supseteq X$, $X \cap P$ is meager in P for each perfect set $P \subseteq Y$.*

The main result on the existence of a perfectly meager space of non–zero dimension is derived from the following theorem.

Theorem 4.2 (Assume the Continuum Hypothesis)**.** *There is a separable metric space of infinite dimension that has no proper saturated σ-additive ideals.*

Proof. Hurewicz [5] constructed under the Continuum Hypothesis an uncountable separable metric space X such that each finitely–dimensional subspace of X was countable. We show that X is the required space. Assume that $\mathcal{J}$ is a proper, σ-additive, saturated ideal in X. According to Theorem 2.1 there exists a G_δ–set $G \subseteq X$ of dimension zero such that $X \setminus G \in \mathcal{J}$. So G is countable and since $\mathcal{J}$ is proper, it follows that $G \in \mathcal{J}$. Thus X is a union of two elements of $\mathcal{J}$ and therefore $X \in \mathcal{J}$: a contradiction. □

Theorem 4.3 (Assume the Continuum Hypothesis)**.** *There is a separable metric space of infinite dimension that is both perfectly meager and of universal measure zero.*

Proof. Let X be the space of Theorem 4.2. If μ is a finite, diffused, nontrivial Borel measure in X, then the ideal of μ–negligible sets is σ-additive, saturated and proper. If $P \subseteq X$ is a perfect set, then the family $\mathcal{J} = \{A \subseteq X : A \cap P \text{ is meager in } P\}$ is a σ-additive, saturated ideal that contains all singletons. Apply Theorem 4.2. □

Theorems 4.2 and 4.3, like Theorems 3.5–3.9, are not theorems of ZFC:

Theorem 4.4. *It is relatively consistent with* ZFC *that* $\mathfrak{c} = 2^{\omega_1}$ *and that each perfectly meager separable metric space is zero–dimensional.*

Proof. As shown in [7], in the iterated perfect set model introduced in [1], every perfectly meager set of reals has cardinality at most ω_1. Also $\mathfrak{c} = 2^{\omega_1}$ in the model. If X is a perfectly meager space and $\dim X > 0$, then obviously $|X| = \mathfrak{c}$. By virtue of Theorem 2.1 there is a family $\langle G_\alpha : \alpha < \omega_1 \rangle$ of zero–dimensional subsets of X that covers X. As $\mathfrak{c} = 2^{\omega_1}$, it follows that $\operatorname{cf} \mathfrak{c} > \omega_1$. Thus there is $\alpha < \omega_1$ such that $|G_\alpha| = \mathfrak{c}$. Since G_α has dimension zero, it embeds into the real line. So G_α is a perfectly meager set of reals of cardinality $\mathfrak{c}$: a contradiction. □

5. Other small spaces

According to Theorem 4.3, it may happen that there is a space of positive dimension that is both perfectly meager and of universal measure zero. A strong measure zero is a property (due to Borel) that implies both perfectly meager and universal measure zero. Recall that a metric space X has *strong measure zero*, if, for each sequence $\langle \varepsilon_n \rangle$ of positive reals, X can be covered by a sequence $\langle E_n \rangle$ of subsets of X such that $\operatorname{diam} E_n \leqslant \varepsilon_n$ for all n. See [8] for more details.

Proposition 5.1. *Each space of strong measure zero has dimension zero.*

Proof. Let X be a metric space of strong measure zero. Denote by ρ its metric. Assume that X has positive dimension. Using the definition of the small inductive dimension, it follows that there is a point $x \in X$ and $r > 0$ such that for each $t \in [0, r]$ there exists $y \in X$ satisfying $\rho(x, y) = t$. Therefore the mapping $f(y) = \rho(x, y)$ from X to $\mathbb{R}$ covers $[0, r]$. The triangle inequality implies that f is Lipschitz. Hence the f–image of X, and *a fortiori* $[0, r]$, is of strong measure zero: a contradiction. □

A space X is a *σ-set* if each G_δ–subset of X is F_σ, and X is a *Q–set* if each subset of X is F_σ. Each Q-set is a σ-set of universal measure zero. Each σ-set is perfectly meager. It is not known if Q-sets have strong measure zero. See [8] for details.

Proposition 5.2. *Each Q-set has dimension zero. Each σ-set of finite dimension has dimension zero. In particular, each σ-set $X \subseteq \mathbb{R}^n$ has dimension zero.*

Proof. A separable space X has $2^{|X|}$ many subsets and $\mathfrak{c}$ many F_σ–subsets. So if X is a Q-set, then $|X| < \mathfrak{c}$. Thus $\dim X = 0$. If X is a σ-set of finite dimension, then, by the Decomposition Theorem, there is a finite cover of X by zero–dimensional G_δ–sets. Each of them is F_σ, so X is covered by countably many zero–dimensional closed sets. By the Countable Sum Theorem, $\dim X = 0$. □

Here are some questions that remain open.

- Is it consistent that there is a perfectly meager space of positive, finite dimension? Cf. Theorem 4.3.
- Has every σ-set dimension zero? Cf. Proposition 5.2.
- A space is a λ–set if each countable subset is G_δ. Is it consistent that there is a λ–set of positive dimension?

- Can the Continuum Hypothesis be replaced with the Martin's Axiom in Proposition 3.9?
- Can the Continuum Hypothesis be replaced with the Martin's Axiom in Theorem 4.2?

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Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic

E-mail address: zindulka@mat.fsv.cvut.cz