PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 6, Pages 1825–1832
S 0002-9939(99)05237-5
Article electronically published on November 29, 1999

RIGIDITY OF AUTOMORPHISMS AND SPHERICAL CR STRUCTURES

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(Communicated by Steven R. Bell)

Abstract. We establish Bochner-type formulas for operators related to CR automorphisms and spherical CR structures. From such formulas, we draw conclusions about rigidity by making assumptions on the Tanaka-Webster curvature and torsion.

1. Statement of results

It is now clear (e.g. [CL1], [CL2], [CT], [Rum]) that certain distinguished second-order partial differential operators and their fourth-order "Laplacians" play important roles in the study of three-dimensional CR geometry. In this paper we will establish Bochner-type formulas for operators related to CR automorphisms and spherical CR structures. From such formulas, we can draw conclusions about rigidity by making assumptions on the so-called Tanaka-Webster curvature and torsion.

To be precise, let (M, J, θ) be a smooth, closed (compact without boundary) 3-dimensional strictly pseudoconvex pseudohermitian manifold (see, e.g. [Web], [Le1]). Here J denotes a CR structure and θ is a contact form, i.e. a nonvanishing real 1-form defining the underlying contact structure. Associated to (M, J, θ), we have a canonical affine connection and notions of Tanaka-Webster scalar curvature and torsion, denoted R and A (a tensor with coefficient A_{11} or $A_1{}^{\bar{1}}$) respectively ([Le1], [Tan], [Web]).

Let T be the characteristic vector field of θ defined by $\theta(T) = 1, \mathcal{L}_T\theta = 0$. (Some authors call T the Reeb vector field.) By choosing suitable complex vector fields $Z_1, Z_{\bar{1}}$ such that $JZ_1 = iZ_1, JZ_{\bar{1}} = -iZ_{\bar{1}}$, we form a (unitary) frame $\{Z_1, Z_{\bar{1}}, T\}$ $(h_{1\bar{1}} = 1)$. The covariant derivatives are taken with respect to this frame and indicated by $_{,1}, _{,\bar{1}}, _{,0}$ and so on. Let $Aut_0(J)$ denote the identity component of the CR automorphism group with respect to J.

Now we can state our first result.

Theorem A. *Let (M, J, θ) be a smooth, connected, closed 3-dimensional strictly pseudoconvex pseudohermitian manifold.*

(a) Suppose $R < 0, \sqrt{3}R_{,0} - 2Im(A_{11,\bar{1}\bar{1}}) > 0$.

Received by the editors August 7, 1998.
1991 *Mathematics Subject Classification.* Primary 32G07; Secondary 32F40, 32C16.
Key words and phrases. Spherical CR structure, Tanaka-Webster curvature, pseudohermitian manifold.
Research supported in part by National Science Council grant NSC 87-2115-M-001-006 (R.O.C.).

Then $Aut_0(J)$ consists of only the identity diffeomorphism.

(b) Suppose $R < 0, \sqrt{3}R_{,0} - 2Im(A_{11,\bar{1}\bar{1}}) = 0$.
Then $dimAut_0(J) \leq 1$.

We remark that the torsion $A = 0$ implies the second condition in (b) holds due to the Bianchi identity: $R_{,0} = A_{11,\bar{1}\bar{1}} + A_{\bar{1}\bar{1},11}$. And the result is compatible with Proposition 4.8(a) in [CT]. On the other hand, we do not know any examples satisfying curvature conditions in (a).

The idea of proving Theorem A goes as follows. Consider a certain second-order linear partial differential operator $\mathcal{D}_J$ and its "Laplacian" $\mathcal{D}_J^\star \mathcal{D}_J$ acting on functions. Here $\mathcal{D}_J^\star$ denotes the adjoint of $\mathcal{D}_J$. The kernel of $\mathcal{D}_J$ parametrizes infinitesimal CR automorphisms (see [CL1] where $\mathcal{D}_J$ was denoted $B_J^{'}$). Establish a suitable Bochner-type formula for $\langle \mathcal{D}_J^\star \mathcal{D}_J f, f \rangle$ by using commutation relations and integration by parts repeatedly (see section 2 for details). Theorem A then follows easily from the final formula. Note that the kernel of $\mathcal{D}_J^\star$ parametrizes the infinitesimal slice in the study of CR moduli spaces ([CL2]).

Next we consider deformation of spherical CR structures. A CR structure or CR manifold is called spherical if it is locally CR-equivalent to the unit sphere with the standard CR structure (e.g. [BS], [CL1]). In dimension 3 it can be characterized quantitatively by the vanishing of a certain fourth-order partial differential operator, the so-called Cartan (curvature) tensor, denoted Q_J (while in higher dimensions, Chern's curvature tensor plays the similar role ([CM]), which is of second order).

A spherical CR structure J is called rigid if there is no infinitesimal deformation up to diffeomorphisms, i.e. for any smooth family of spherical CR structures $J_{(t)}$ on the base manifold M with $J_{(0)} = J$, $d/dt|_{t=0} J_{(t)}$ equals $\mathcal{L}_X J$ for some vector field X of M.

To study the rigidity of spherical CR structures, we consider the linearization of Q_J plus a symmetry-breaking term provided by $\mathcal{D}_J \mathcal{D}_J^\star$. In section 3 we work out a Bochner-type formula and analyze it to obtain pointwise conditions for J to be rigid.

Theorem B. *Let (M, J) be a smooth, closed, spherical CR 3-manifold. Suppose there is a contact form θ such that $R > 0$, (3.11) and (3.12) hold. Then J is rigid.*

Note that conditions (a),(b) in Theorem A and (3.11),(3.12) in Theorem B are independent of positive constant multiples of θ. When the torsion vanishes, we have the simplified expression as follows.

Corollary C. *Let (M, J) be a smooth, closed, spherical CR 3-manifold. Suppose there is a contact form θ such that the torsion $A = 0$ and*

$$R > 0, 4R(5R^2 + 3\triangle_b R) - 3|\nabla_b R|_\theta^2 > 0.$$

Then J is rigid.

The sublaplacian and subgradient operators $\triangle_b, \nabla_b$ acting on (smooth) functions are defined by $\triangle_b f = -f_{,1}{}^1 - f_{,\bar{1}}{}^{\bar{1}}$ and $\nabla_b f = f_{,}{}^1 Z_1 + f_{,}{}^{\bar{1}} Z_{\bar{1}}$ respectively (cf. [Le1] or [Che]). Also we define $|\nabla_b f|_\theta^2 = 2 f_{,1} f_{,}{}^1$ for real f.

Observe that $A = 0$ and R being a positive constant satisfy conditions in Corollary C. In this case the universal cover of (M, J) must be compact by Rumin's pseudohermitian version of Myers' theorem ([Rum]), and hence CR-equivalent to

the standard S^3. It follows that the fundamental group Γ of M is finite. Hence the group cohomology $H^1(\Gamma, \mathcal{G})$ in deformation theory (p.232 in [BS]) vanishes. So in this special case our result is compatible with the result obtained by "Lie theoretical" argument. Note that a small perturbation of $A = 0$ and R being a positive constant still satisfies the conditions in Theorem B.

2. Proof of Theorem A

Let $\{\theta^1, \theta^{\bar{1}}, \theta\}$ be the coframe dual to the "unitary" frame $\{Z_1, Z_{\bar{1}}, T\}$ (with $h^{1\bar{1}} = h_{1\bar{1}} = 1$ in mind, hereafter, we'll write tensors with only lower indices). Recall ([CL2] or [CL1]) that $\mathcal{D}_J f = 2Re[(f_{,11} + iA_{11}f)\theta^1 \otimes Z_{\bar{1}}]$ and the adjoint operator $\mathcal{D}_J^\star$ acts on a deformation tensor $E = 2Re(E_{11}\theta^1 \otimes Z_{\bar{1}})$ by

$$\mathcal{D}_J^\star E = E_{11,\bar{1}\bar{1}} + iA_{11}E_{\bar{1}\bar{1}} + \text{ conjugate.} \tag{2.1}$$

Also the generalized Folland-Stein operator L_α is defined by $L_\alpha f = \triangle_b f + i\alpha f_{,0}$ for a function f. By Lemma 2.1 in [CL2], we have

$$\mathcal{D}_J^\star \mathcal{D}_J = (1/2)L_\alpha^\star L_\alpha + \mathcal{O}_2 \tag{2.2}$$

with $\alpha = i\sqrt{3}$, where $\mathcal{O}_2$ is an operator of weight ≤ 2. In the rest of this section, we'll look into the details of $\mathcal{O}_2$. (Note that L_α in the leading term of (2.2) is subelliptic. This implies the existence of the "infinitesimal slice decompositions" in the study of CR moduli spaces without knowing details of the lower-weight term ([CL2]).)

A direct computation shows that for a real-valued function f, we have

$$\begin{aligned} \mathcal{D}_J^\star \mathcal{D}_J f = {} & f_{,11\bar{1}\bar{1}} + f_{,\bar{1}\bar{1}11} + 2Re[2iA_{11}f_{,\bar{1}\bar{1}} \\ & +2iA_{11,\bar{1}}f_{,\bar{1}} + iA_{11,\bar{1}\bar{1}}f + |A_{11}|^2 f]. \end{aligned} \tag{2.3}$$

We'll frequently use the following commutation relations.

Lemma 2.1 (Ricci identities in pseudohermitian geometry)**.** *Let c_I be a coefficient of some tensor with multi-indices I. Suppose I consists of only 1 and $\bar{1}$, and $\alpha = (\#$ of 1 in $I) - (\#$ of $\bar{1}$ in $I)$. Then*

$$c_{I,1\bar{1}} - c_{I,\bar{1}1} = ic_{I,0} + \alpha c_I R, \tag{2.4}$$

$$c_{I,01} - c_{I,10} = c_{I,\bar{1}}A_{11} - \alpha c_I A_{11,\bar{1}}, \tag{2.5}$$

$$c_{I,0\bar{1}} - c_{I,\bar{1}0} = c_{I,1}A_{\bar{1}\bar{1}} + \alpha c_I A_{\bar{1}\bar{1},1} \tag{2.6}$$

(this lemma generalizes Lemma 2.3 in [Le2] *for the three-dimensional case).*

By using (2.4),(2.5),(2.6) repeatedly, we obtain

$$\frac{1}{2}L_\alpha^\star L_\alpha f = f_{,11\bar{1}\bar{1}} + f_{,\bar{1}\bar{1}11} + 2Re[(\sqrt{3} - i)(A_{\bar{1}\bar{1}}f_{,1})_{,1} - (Rf_{,1})_{,\bar{1}}] \tag{2.7}$$

with the choice of $\alpha = i\sqrt{3}$ eliminating terms having covariant derivative in the direction T. Comparing (2.3) with (2.7) and taking the L^2-inner product with f

gives

$$\begin{aligned}\|\mathcal{D}_J f\|^2 &= \frac{1}{2}\|L_{i\sqrt{3}}f\|^2 - \int_M R|\nabla_b f|_\theta^2 dv_\theta \\ &+2\int_M [Re(\frac{\sqrt{3}+i}{2}A_{11,\bar{1}\bar{1}}) + |A_{11}|^2]f^2 dv_\theta \\ &+2\int_M Re[(-i-\sqrt{3})A_{\bar{1}\bar{1}}f_{,11}]f dv_\theta.\end{aligned} \tag{2.8}$$

Here the volume form $dv_\theta = \theta\wedge d\theta$. With respect to dv_θ, we have the divergence theorem and hence integration by parts in calculus of pseudohermitian geometry ([Le2], [Che]). For instance,

$$\begin{aligned}\textstyle\int_M A_{\bar{1}\bar{1},1}f_{,1}\bar{f}dv_\theta &= \frac{1}{2}\int_M A_{\bar{1}\bar{1},1}(f^2)_{,1}dv_\theta \text{ (f being real)} \\ &= -\frac{1}{2}\int_M A_{\bar{1}\bar{1},11}f^2 dv_\theta \text{(integration by parts)}\end{aligned}$$

was used in deducing (2.8). Now suppose $\phi_t \in Aut_0(J)$ is a smooth family of CR automorphisms with $\phi_0 =$ identity. Then $X = \frac{d}{dt}|_{t=0}\phi_t \in$ Lie $Aut_0(J)$ (=Lie algebra of $Aut_0(J)$) is an infinitesimal CR automorphism, in particular, an infinitesimal contact automorphism. According to Lemma 3.4 and 3.5 in [CL1], $X = X_f$ is determined by a function $f = -\theta(X)$ and satisfies the relation: $\mathcal{L}_X J = 2\mathcal{D}_J f$. Since $\mathcal{L}_X J = 0$, we get

$$0 = \mathcal{D}_J f = f_{,11} + iA_{11}f, \tag{2.9}$$

and hence

$$A_{\bar{1}\bar{1}}f_{,11} = -i|A_{11}|^2 f. \tag{2.10}$$

Substituting (2.9), (2.10) in (2.8), we finally obtain

$$\begin{aligned}0 &= \|L_{i\sqrt{3}}f\|^2 - 2\int_M R|\nabla_b f|_\theta^2 dv_\theta \\ &+\int_M [\sqrt{3}R_{,0} + i(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11})]f^2 dv_\theta.\end{aligned} \tag{2.11}$$

Now it is easy to see from (2.11) that the condition in (a) of Theorem A implies $f = 0$. Therefore $X = X_f = 0$. For (b), the condition implies $\nabla_b f = 0$. So $f_{,0} = 0$ by (2.4). Thus f is constant since M is connected. It follows that $dim(Aut_0(J)) = dim(Lie\, Aut_0(J)) \leq 1$.

3. Proof of Theorem B

Recall ([CL1]) that the Cartan tensor $Q_J = iQ_{11}\theta^1 \otimes Z_{\bar{1}} - iQ_{\bar{1}\bar{1}}\theta^{\bar{1}} \otimes Z_1$ where

$$Q_{11} = \frac{1}{6}R_{,11} + \frac{i}{2}RA_{11} - A_{11,0} - \frac{2i}{3}A_{11,\bar{1}1}$$

and J is spherical if and only if $Q_J = 0$. (Note that we have lowered indices using $h_{1\bar{1}} = 1$; also Q_J changes "tensorially" when we make a different choice of contact form.) Let $\tilde{J}_{(t)}$ be a smooth family of spherical CR structures with $\tilde{J}_{(0)} = J$. By a theorem of Gray ([Gra] or [Ham]), there exists a smooth family of diffeomorphisms

ϕ_t with $\phi_0 = identity$ so that, for all t, $J_{(t)} = \phi_t^\star \tilde{J}_{(t)}$ has the same underlying contact structure as J does. Write the infinitesimal deformation

$$\frac{d}{dt}|_{t=0} J_{(t)} = 2E = 4Re(E_{11}\theta^1 \otimes Z_{\bar{1}})$$

and compute $DQ_J(2E) = \partial_t Q_{J_{(t)}}|_{t=0}$ as we did in [CL1]. There appears a "bad" term $E_{\bar{1}\bar{1},1111}$ in the formula, so we add a "symmetry-breaking" term $\mathcal{D}_J\mathcal{D}_J^\star E$ to cancel it. The final formula including terms of lower weights reads

(3.1)

$$\begin{aligned}
&- DQ_J(2E) + \frac{1}{6}\mathcal{D}_J\mathcal{D}_J^\star E \\
&\quad = 2Re\{\frac{1}{3}E_{11,\bar{1}\bar{1}11}E_{11,00} - \frac{2i}{3}E_{11,0\bar{1}1} + \frac{i}{3}(A_{11}E_{\bar{1}\bar{1}})_{,11} \\
&\quad - \frac{1}{6}E_{11}R_{,1\bar{1}} + \frac{1}{6}E_{11,\bar{1}}R_{,1} - \frac{1}{6}(E_{11}R_{,\bar{1}})_{,1} \\
&\quad + \frac{1}{2}A_{11}(iE_{11,\bar{1}\bar{1}} - iE_{\bar{1}\bar{1},11} - A_{11}E_{\bar{1}\bar{1}} - A_{\bar{1}\bar{1}}E_{11}) + \frac{i}{2}RE_{11,0} \\
&\quad + 2A_{11}(A_{11}E_{\bar{1}\bar{1}} + A_{\bar{1}\bar{1}}E_{11}) + \frac{2i}{3}E_{11}A_{11,\bar{1}\bar{1}} - \frac{2i}{3}E_{11,\bar{1}}A_{11,\bar{1}} - \frac{2i}{3}(E_{\bar{1}\bar{1}}A_{11,1})_{,1} \\
&\quad - \frac{4i}{3}(E_{\bar{1}\bar{1},1}A_{11})_{,1} + \frac{i}{6}A_{11}(E_{11,\bar{1}\bar{1}} + E_{\bar{1}\bar{1},11} + iA_{11}E_{\bar{1}\bar{1}} - iA_{\bar{1}\bar{1}}E_{11})\}\theta^1 \otimes Z_{\bar{1}}.
\end{aligned}$$

The right-hand side of (3.1) can be written as $\frac{1}{12}L_\alpha^\star L_\alpha E + \mathcal{O}_2(E)$ with $\mathcal{O}_2$ being an operator of weight ≤ 2 and $\alpha = 4 + i\sqrt{3}$. Since L_α is subelliptic, the above expression was used in [CL1] to show the short- time solution of a certain regularized evolution equation. Using Lemma 2.1 repeatedly, we can write the highest-weight term of (3.1) as follows:

$$\begin{aligned}
\frac{1}{3}E_{11,\bar{1}\bar{1}11} - E_{11,00} - \frac{2i}{3}E_{11,0\bar{1}1} &= \frac{1}{3}E_{11,\bar{1}1\bar{1}1} - E_{11,00} - iE_{11,0\bar{1}1} \\
&+ \frac{i}{3}(E_{11,1}A_{\bar{1}\bar{1}})_{,1} + \frac{2i}{3}(E_{11}A_{\bar{1}\bar{1},1})_{,1} - \frac{1}{3}(RE_{11,\bar{1}})_{,1}.
\end{aligned} \tag{3.2}$$

On the other hand, we compute

$$\begin{aligned}
\int_M R|E_{11,1}|^2 dv_\theta &= -\int_M (RE_{11,1\bar{1}}E_{\bar{1}\bar{1}} + R_{,\bar{1}}E_{11,1}E_{\bar{1}\bar{1}})dv_\theta \\
&\text{(by integration by parts)} \\
&= -\int_M [R(E_{11,\bar{1}1} + iE_{11,0} + 2RE_{11})E_{\bar{1}\bar{1}} + R_{,\bar{1}}E_{11,1}E_{\bar{1}\bar{1}}]dv_\theta \text{ (by (2.4))}.
\end{aligned} \tag{3.3}$$

To see how we treat (3.1) in general, we first deal with the torsion$= 0$ case. By the local slice theorem ([CL2]), there exists a smooth family of contact diffeomorphisms ψ_t with $\psi_0 =$ identity so that $J_{(t)}^{'} = \psi_t^\star J_{(t)}$ lies in the local slice passing through J. Since the infinitesimal slice at J is parametrized by the kernel of $\mathcal{D}_J^\star$, we have $\mathcal{D}_J^\star E = 0$ for $2E = \frac{d}{dt}|_{t=0} J_{(t)}^{'}$.

Now applying (3.1) to such a deformation tensor E: $\mathcal{D}_J^\star E = 0$, substituting (3.2) in (3.1), and taking the L^2-inner product with E, we obtain

$$\begin{aligned}(3.4)\qquad 0 &= \langle -DQ_J(2E), E\rangle \text{ (since } Q_{J'_{(t)}} = 0)\\ &= \int_M 2Re\{\frac{1}{3}|E_{11,\bar{1}1}|^2 + |E_{11,0}|^2 - iE_{11,0}E_{\bar{1}\bar{1},1\bar{1}} + \frac{1}{6}R|E_{11,1}|^2\\ &+\frac{2i}{3}RE_{11,0}E_{\bar{1}\bar{1}} - \frac{1}{6}RE_{11,\bar{1}1}E_{\bar{1}\bar{1}} + \frac{1}{6}R_{,\bar{1}}E_{11,1}E_{\bar{1}\bar{1}}\\ &+[\frac{11}{6}R_{,\bar{1}1} - 2R_{,1\bar{1}} + \frac{1}{3}R^2]|E_{11}|^2\}dv_\theta\end{aligned}$$

by putting $A_{11} = 0$, using (3.3) and integration by parts repeatedly.

Note that we reduce (3.4) to the formula (6.3) in [CL1] for R being a constant $\hat{R}$. Furthermore, if $\hat{R} > 0$, the right-hand side of (3.4) is a positive definite quadratic hermitian form in $E_{11,\bar{1}1}$, $iE_{11,0}$, and $\hat{R}E_{11}$.

It follows that $E = 0$ and $0 = \frac{d}{dt}|_{t=0}J'_{(t)} = \frac{d}{dt}|_{t=0}(\phi_t\circ\psi_t)^\star\tilde{J}_{(t)} = \mathcal{L}_X J + \frac{d}{dt}|_{t=0}\tilde{J}_{(t)}$ where the vector field $X = \frac{d}{dt}|_{t=0}(\phi_t\circ\psi_t)$. So J is rigid. For general R, we require that the integrand in (3.4) is a (pointwise) positive definite quadratic hermitian form in $E_{11,\bar{1}1}$, $iE_{11,0}$, $E_{11,1}$, E_{11}. Now the conditions in Corollary C can be deduced from basic linear algebra.

When the torsion does not vanish, the formula for $\langle -DQ_J(2E), E\rangle$ with $\mathcal{D}_J^\star E = 0$ reads

$$\begin{aligned}(3.5)\qquad 0 = \langle -DQ_J(2E), E\rangle &= \int_M \{\frac{2}{3}|E_{11,\bar{1}1}|^2 + 2|E_{11,0}|^2\\ &+\frac{1}{3}R|E_{11,1}|^2 + [\frac{2}{3}R^2 + \frac{1}{6}\triangle_b R + 6|A_{11}|^2 + \frac{8i}{3}(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11})]|E_{11}|^2\\ &+2Re[-iE_{11,0}E_{\bar{1}\bar{1},1\bar{1}} + \frac{2i}{3}RE_{11,0}E_{\bar{1}\bar{1}} - \frac{1}{6}RE_{11,\bar{1}1}E_{\bar{1}\bar{1}}\\ &+(\frac{1}{6}R_{,\bar{1}} - 2iA_{\bar{1}\bar{1},1})E_{11,1}E_{\bar{1}\bar{1}} - \frac{2i}{3}A_{11,1}E_{\bar{1}\bar{1},1}E_{\bar{1}\bar{1}} - \frac{5i}{3}A_{11}E_{11,\bar{1}}E_{\bar{1}\bar{1},\bar{1}}]\}dv_\theta.\end{aligned}$$

There are non-cross terms like $A_{11,1}E_{\bar{1}\bar{1},1}E_{\bar{1}\bar{1}}$ and $A_{11}E_{11,\bar{1}}E_{\bar{1}\bar{1},\bar{1}}$ in (3.5). We need an inequality to deal with $A_{11}E_{11,\bar{1}}E_{\bar{1}\bar{1},\bar{1}}$.

Lemma 3.1. *Let λ, ρ be real numbers. Then*

$$\begin{aligned}(3.6)\qquad 2\lambda\rho\int_M R|E_{11,\bar{1}}|^2dv_\theta \quad &\leq \quad \lambda^2\int_M |E_{11,\bar{1}1}|^2dv_\theta + \rho^2\int_M R^2|E_{11}|^2dv_\theta\\ &\quad -\lambda\rho\int_M (R_{,1}E_{11,\bar{1}}E_{\bar{1}\bar{1}} + R_{,\bar{1}}E_{\bar{1}\bar{1},1}E_{11})dv_\theta.\end{aligned}$$

For the term $A_{11,1}E_{\bar{1}\bar{1},1}E_{\bar{1}\bar{1}}$, we use the following estimate:

$$(3.7)\qquad 2Re(-\frac{2i}{3}A_{11,1}E_{\bar{1}\bar{1},1}E_{\bar{1}\bar{1}}) \geq -\frac{2}{3}|A_{11,1}|^{\frac{2}{3}}|E_{\bar{1}\bar{1},1}|^2 - \frac{2}{3}|A_{11,1}|^{\frac{4}{3}}|E_{11}|^2.$$

(To deduce (3.7), replace a, b by ωa, $\omega^2 b$ in the basic inequality $2Re(ab) \geq -|a|^2 - |b|^2$ with $a = -iE_{\bar{1}\bar{1},1}$, $b = E_{\bar{1}\bar{1}}$, and $\omega^3 = A_{11,1}$.)

The reason for taking fractional exponents in (3.7) is to make our conditions invariant under the scale change of contact form by a positive constant multiple as we'll see later. Take a small amount of $\int_M |E_{11,\bar{1}1}|^2dv_\theta$ and $\int_M R^2|E_{11}|^2dv_\theta$ to gain the term $\int_M R|E_{11,\bar{1}}|^2dv_\theta$ by (3.6) in the right-hand side of (3.5) while keeping

the quadratic hermitian form in $E_{11,\bar{1}1}$, $iE_{11,0}$, $E_{11,1}$, $E_{11,\bar{1}}$, E_{11} positive definite at least for $R = constant > 0$, $A_{11} = 0$. For instance, we can take $\lambda = \rho = \frac{1}{4}$ in (3.6) and then use it and (3.7) in estimating the right-hand side of (3.5). The final result reads

$$
\begin{aligned}
(3.8) \qquad 0 \;\geq\; & \int_M \{\frac{29}{48}|E_{11,\bar{1}1}|^2 + 2|E_{11,0}|^2 + \frac{1}{3}R|E_{11,1}|^2 \\
& +(\frac{1}{8}R - \frac{2}{3}|A_{11,1}|^{\frac{2}{3}})|E_{11,\bar{1}}|^2 + [\frac{29}{48}R^2 + \frac{11}{48}\triangle_b R \\
& +6|A_{11}|^2 + \frac{8i}{3}(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11}) - \frac{2}{3}|A_{11,1}|^{\frac{4}{3}}]|E_{11}|^2 \\
& +2Re[-iE_{11,0}E_{\bar{1}\bar{1},1\bar{1}} + \frac{2}{3}iRE_{11,0}E_{\bar{1}\bar{1}} - \frac{1}{6}RE_{11,\bar{1}1}E_{\bar{1}\bar{1}} \\
& +(\frac{5}{48}R_{,\bar{1}} - 2iA_{\bar{1}\bar{1},1})E_{11,1}E_{\bar{1}\bar{1}} - \frac{5i}{3}A_{11}E_{11,\bar{1}}E_{\bar{1}\bar{1},\bar{1}}]\}dv_\theta.
\end{aligned}
$$

The integrand in (3.8) is a quadratic hermitian form in $E_{11,\bar{1}1}$, $iE_{11,0}$, $E_{11,1}$, $E_{11,\bar{1}}$, and E_{11}. By basic linear algebra it is positive definite if and only if $R > 0$,

$$
(3.9) \qquad \begin{vmatrix} \frac{29}{48} & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & \frac{1}{3}R & \frac{5i}{3}A_{\bar{1}\bar{1}} \\ 0 & 0 & \frac{-5i}{3}A_{11} & \frac{1}{8}R - \frac{2}{3}|A_{11,1}|^{\frac{2}{3}} \end{vmatrix} > 0, \text{ and}
$$

$$
(3.10) \qquad \begin{vmatrix} \frac{29}{48} & -1 & 0 & 0 & -\frac{1}{6}R \\ -1 & 2 & 0 & 0 & \frac{2}{3}R \\ 0 & 0 & \frac{1}{3}R & \frac{5i}{3}A_{\bar{1}\bar{1}} & \frac{5}{48}R_{,\bar{1}} - 2iA_{\bar{1}\bar{1},1} \\ 0 & 0 & \frac{-5i}{3}A_{11} & \frac{1}{8}R - \frac{2}{3}|A_{11,1}|^{\frac{2}{3}} & 0 \\ -\frac{1}{6}R & \frac{2}{3}R & \frac{5}{48}R_{,1} + 2iA_{11,\bar{1}} & 0 & \mathcal{M} \end{vmatrix}
$$

is larger than 0. Here $\mathcal{M} = \frac{29}{48}R^2 + \frac{11}{48}\triangle_b R + 6|A_{11}|^2 - \frac{2}{3}|A_{11,1}|^{\frac{4}{3}} + \frac{8i}{3}(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11})$. A straightforward computation shows that (3.9) is equivalent to

$$
(3.11) \qquad \frac{3}{8}R^2 - 2R|A_{11,1}|^{\frac{2}{3}} - 25|A_{11}|^2 > 0
$$

while (3.10) is equivalent to

$$
\begin{aligned}
(3.12) \qquad & (\frac{3}{8}R^2 - 2R|A_{11,1}|^{\frac{2}{3}} - 25|A_{11}|^2)\{\frac{83}{3456}R^2 \\
& +\frac{55}{1152}\triangle_b R + \frac{5}{4}|A_{11}|^2 - \frac{5}{36}|A_{11,1}|^{\frac{4}{3}} + \frac{5i}{9}(A_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1},11})\} \\
& -\frac{15}{8}(\frac{1}{8}R - \frac{2}{3}|A_{11,1}|^{\frac{2}{3}})|\frac{5}{48}R_{,\bar{1}} - 2iA_{\bar{1}\bar{1},1}|^2 > 0.
\end{aligned}
$$

Observe that if θ changes by a positive constant multiple k, R and A_{11} change by multiplying k^{-1} while $A_{11,1}$,$A_{\bar{1}\bar{1},1}$, and $R_{,\bar{1}}$ change by multiplying $k^{-\frac{3}{2}}$. Similarly $A_{11,\bar{1}\bar{1}}$, $A_{\bar{1}\bar{1},11}$, and $\triangle_b R$ change by multiplying k^{-2}. So the conditions (3.11),(3.12) are invariant under the change of contact form by a positive constant multiple. Now *Theorem B* follows from (3.8) under the conditions (3.11),(3.12).

Acknowledgment

This work was carried out during the author's visit at Harvard University. He would therefore like to thank the members of the Mathematics Department, and especially Professor Shing-Tung Yau, for their hospitality during his stay.

References

[BS] D. Burns and S. Shnider, *Spherical hypersurfaces in complex manifolds*, Invent. Math. 33 (1976),223-246 MR **54:**7875

[Che] J.-H. Cheng, *Curvature functions for the sphere in pseudohermitian geometry*, Tokyo J. Math. 14(1991), 151-163 MR **92k:**32016

[CL1] J.-H. Cheng and J. M. Lee, *The Burns-Epstein invariant and deformation of CR structures*, Duke Math. J. 60(1990), 221-254 MR **91a:**32023

[CL2] *A local slice theorem for 3-dimensional CR structures*, Amer. J. Math. 117(1995), 1249-1298 MR **96f:**32030

[CM] S.-S. Chern and J. K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. 133(1974), 219-271 MR **54:**13112

[CT] J.-H. Cheng and I-H. Tsai, *Deformation of spherical CR structures and the universal Picard variety*, to appear in Communications in Analysis and Geometry

[Gra] J. W. Gray, *Some global properties of contact structures*, Ann. Math. 69(1959), 421-450 MR **22:**3016

[Ham] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7(1982), 65-222 MR **83j:**58014

[Le1] J. M. Lee, *The Fefferman metric and pseudohermitian invariants*, Trans. AMS 296(1986), 411-429 MR **87j:**32063

[Le2] J. M. Lee, *Pseudo-Einstein structures on CR manifolds*, Am. J. Math. 110(1988), 157-178 MR **89f:**32034

[Rum] M. Rumin, *Formes différentielles sur les variétés de contact*, J. Diff. Geom., 39(1994), 281-330 MR **95g:**58221

[Tan] N. Tanaka, *A Differential Geometric Study on Strongly Pseudo-Convex Manifolds*, 1975, Kinokuniya Co. Ltd., Tokyo MR **53:**3361

[Web] S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Diff. Geom. 13(1978), 25-41 MR **80e:**32015

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