AN INDUCTIVE EXPLICIT CONSTRUCTION OF $\ast$-PRODUCTS ON SOME POISSON MANIFOLDS

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ABSTRACT. We extend the Cahen Gutt coboundary construction on cotangent bundles of $n$-dimensional parallelisable manifolds to manifolds which admit $n$ global vector fields defining a parallelisation on a dense open set. This result is used to give an inductive explicit construction of $\ast$-products on certain Poisson manifolds.

INTRODUCTION

The theory of formal deformation quantization of Poisson manifolds was introduced by Bayen et al. in [2]. The main notion of this theory is the concept of a $\ast$-product. The general question of the existence of such a product for symplectic manifolds has been completely solved by several authors, using various techniques [5], [14], [11]. Recently, M. Kontsevitch has proved the existence of $\ast$-products on arbitrary finite-dimensional Poisson manifolds [9]. Nevertheless, since Kontsevitch’s result is not given by a simple geometrical construction, it has increased the interest of having a simple geometrical of $\ast$-products on non-regular Poisson manifolds.

Since every Poisson manifold splits into a collection of symplectic submanifolds, known as the leaves of the symplectic foliation, one naturally asks whether a $\ast$-product on a Poisson manifold restricts to give a $\ast$-product on the symplectic leaves. Lately, in [4], [13] it has been proved that such $\ast$-products do not always exist. When they exist we called them tangential. In particular, the dual of the so-called “book algebra” with the Lie Poisson structure admits a tangential $\ast$-product [1]. Furthermore, this example provides the basic idea to construct explicit $\ast$-products on some other Poisson manifolds (see Theorem 1).

The goal of this paper will be to give such a construction. In order to do this, we first generalize a coboundary construction due to Cahen and Gutt [3], and use this result to construct explicitly the $\ast$-product.

Using a different method a similar result has been obtained in [7]. However, the approach used there does not provide an explicit construction of a $\ast$-product.
**-PRODUCTS**

A Poisson structure on a manifold $M$ is a Lie algebra structure $\{\cdot, \cdot\}$ on $C^\infty(M)$ which satisfies the derivation property

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \quad \forall f, g, h \in C^\infty(M).$$

The operation $\{\cdot, \cdot\}$ determines a contravariant skew-symmetric 2-tensor $\Lambda$ such that $\{f, g\} = \Lambda(df, dg)$. A Poisson structure may also be defined by such a tensor (the Poisson tensor); the Jacobi identity for the Poisson structure is equivalent to the vanishing of the so-called Schouten bracket $[\Lambda, \Lambda]_s = 0$ (see [2]).

If $(M, \Lambda)$ is a Poisson manifold we set $N = C^\infty(M)$. Let $N[[\lambda]]$ be the space of formal power series in a parameter $\lambda$, with coefficients in $N$.

**Definition 1** ([2]). A *-product on $(M, \Lambda)$ is a bilinear map $N^2 \rightarrow N[[\lambda]]$ defined by

$$f \ast g = \sum_{n=0}^{\infty} \lambda^n C_n(f, g),$$

where the so-called cochains $C_n$, are bilinear maps with values in $N$ and satisfy the following axioms:

1. $C_0(f, g) = fg, \quad C_1(f, g) = \{f, g\}, \quad \forall f, g \in C^\infty(M)$;
2. $C_n(f, g) = (-1)^n C_n(g, f), \quad \forall f, g \in C^\infty(M), \quad \forall n \geq 1$;
3. $C_n(f, k) = 0, \quad \forall f \in C^\infty(M), \quad \forall k \in \mathbb{R}, \quad \forall n \geq 1$;
4. $\sum_{r+s+k} C_r(C_s(f, g), h) = \sum_{r+s+k} C_r(f, C_s(g, h)), \quad k \geq 0$.

The theory of deformations in the sense of [6] relates the deformations of an associative algebra to the corresponding Hochschild cohomology.

**Definition 2.** A (Hochschild) $p$-cochain is a $p$-linear map $N^p \rightarrow N$. The **Hochschild coboundary** of a $p$-cochain is the $(p + 1)$-cochain $\partial C$ given by

$$\partial C(u_0, \ldots, u_p) = u_0 C(u_1, \ldots, u_p) - C(u_0 u_1, u_2, \ldots, u_p) + C(u_0, u_1 u_2, \ldots, u_p)$$

$$+ \cdots + (-1)^p C(u_0, u_1, \ldots, u_{p-1} u_p) + (-1)^{p+1} C(u_0, \ldots, u_{p-1}) u_p.$$

A cochain $C$ is called differential if it is defined by multi-differential operators in each argument. A *-product is called differential if all its cochains are differential.

In [12] it has been proved that if $E$ is a $p$-cocycle (differential and null on the constants), then there exist a skew-symmetric contravariant smooth $p$-tensor $A$ and a $(p - 1)$-cochain $C$ such that

$$E(f_1, \ldots, f_p) = \partial C(f_1, \ldots, f_p) + A(df_1, \ldots, df_p), \quad f_i \in C^\infty(M).$$

A bilinear map [1] is said to be an associative formal deformation up to the order $k$ if

$$\sum_{r+s=t, \ r, s \geq 1} C_r(C_s(f, g), h) - C_r(f, C_s(g, h)) = \partial C_t(f, g, h).$$

Thus, an associative formal deformation up to the order $k$ can be extended to one of order $k + 1$ provided that the cocycle $E_{k+1}$ is a 3-coboundary.
TANGENTIAL $\ast$-PRODUCTS

Let $(M, \Lambda)$ be a Poisson manifold, and let $O$ be a symplectic leaf.

**Definition 3.** Let $x \in O$. A differential operator $D$ on $M$ is **tangential** to $O$ at $x$, if there exist a neighbourhood $V$ of $x$ in $O$ and a neighbourhood $U$ of $V$ in $M$, such that when $\varphi_1, \varphi_2 \in C^\infty(U)$ with $\varphi_1|_V = \varphi_2|_V$, then

\[ D(\varphi_1)|_V = D(\varphi_2)|_V. \]

A bi-differential operator $C$ on $M$ is said to be **tangential** to $O$, if for any function $f \in C^\infty(M)$, the differential operators $C(f, \cdot)$ and $C(\cdot, f)$ are tangential to $O$, at $x$ for all $x \in O$.

**Definition 4.** A differential $\ast$-product is called tangential to $O$, if all its cochains $C_n$, $n \geq 1$, are tangential.

A COBOUNDARY CONSTRUCTION

In what follows, we shall use the summation convention on pairs of upper and lower indices. Let $(M, \Lambda)$ be a Poisson manifold of dimension $n$, and let $T^1, \ldots, T^n$ be smooth vector fields on $M$ such that they are pointwise linearly independent on a dense open set of $M$. The following proposition is a simple generalization of Proposition 2 in [3]. The argument given in [3] is combinatorial, and is based on 3 lemmas which in fact only require independence of the vector fields $T^1$ on a dense open set.

**Proposition 1 ([3]).** Let $E$ be a differential 3-cocycle (null on the constants), of the form

\[ E(f, g, h) = \sum_{0 < a, b, c \leq K} E_{i_1 \ldots i_a, j_1 \ldots j_b, k_1 \ldots k_c} T^{i_1} \cdots T^{i_a} f T^{j_1} \cdots T^{j_b} g T^{k_1} \cdots T^{k_c} h, \]

where $f, g, h \in C^\infty(M)$, and $E_{i_1 \ldots i_a, j_1 \ldots j_b, k_1 \ldots k_c}$ are smooth functions on $M$ symmetric in the $i$’s, in the $j$’s and in the $k$’s. Then, there is a 2-cochain $C$ completely determined by $E$ of the form

\[ C(f, g) = \sum_{0 < p, q \leq K} C_{i_1 \ldots i_p, j_1 \ldots j_q} T^{i_1} \cdots T^{i_p} f T^{j_1} \cdots T^{j_q} g, \]

such that $E = \partial C + A$, where $A$ is the completely antisymmetric part of $E$, i.e., a $\ast$-contravariant smooth tensor. Moreover, the coefficients $C_{i_1 \ldots i_p, j_1 \ldots j_q}$ are constant (rational) linear combinations of the coefficients $E_{k_1 \ldots k_a, l_1 \ldots l_b, m_1 \ldots m_c}$ of $E$.

**Remark 1.** Note that if the $C_r$’s $(r \leq k)$ satisfy the symmetry properties of Definition [1], then the cocycle $E_{k+1}$ satisfies $E_{k+1}(f, g, h) = (-1)^k E_{k+1}(h, g, f)$. Thus, if $E_{k+1} = \partial C_{k+1}$ we can always assume that $C_{k+1}(f, g) = (-1)^{k+1} C_{k+1}(g, f)$ just by replacing $C_{k+1}$ by its symmetrization or antisymmetrization.

**Theorem 1.** Let $(M, \Lambda)$ be a Poisson manifold, and let us assume that there exist $T^1, T^2$ smooth vector fields on $M$ such that they are pointwise linearly independent on a dense open set of $M$, and such that $\Lambda$ can be written as $\Lambda = T^1 \wedge T^2$. Then, there is a $\ast$-product on $(M, \Lambda)$ with 2-cochains $C_r$ of the form [2].
Proof. By assumption the Poisson structure $C_1$ on $M$ admits the expression $C_1(f, g) = T^1 f T^2 g - T^2 f T^1 g$. Let us assume that there exist $k$ ($k \geq 1$) 2-cochains $C_1, \ldots, C_k$ constructed recursively (using Proposition 1) from the equations

$$E_t = \partial C_t \quad t = 1, \ldots, k,$$

defining a deformation (up to the order $k$) on $M$ so that

$$C_t(f, g) = \sum_{0 < p, q \leq K_t} C_{1, \ldots, p, j_1, \ldots, j_q} T^{i_1} \cdots T^{i_p} f T^{j_1} \cdots T^{j_q} g,$$

where all the $T^i$'s and $T^j$'s are $T^1$ or $T^2$.

Since $[T^1 \wedge T^2, T^1 \wedge T^2] = 0$, it follows that $[T^1, T^2] = f_1 T^1 + f_2 T^2$ ($f_1, f_2 \in C^\infty(M)$), and so that $E_{k+1}$ expressed as in Proposition 1 only includes $T^1$'s and $T^2$'s. Let $C_{k+1}$ and $A_{k+1}$ be the 2-cochains constructed by means of Proposition 1 such that $E_{k+1} = \partial C_{k+1} + A_{k+1}$. Then, since $A_{k+1}$ is a 3-contravariant (skew-symmetric) tensor only including $T^1$ and $T^2$, it follows that $A_{k+1}$ vanishes, i.e., $E_{k+1} = \partial C_{k+1}$. Thus, the theorem follows by induction on $k$.

Remark 2. The $*$-product constructed is tangential to the 2-dimensional symplectic leaves.

**Examples**

**Example 1.** Let $g$ be the Lie algebra (book algebra) with basis $(e_1, e_2, e_3)$, such that $[e_1, e_2] = 0$, $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$. Let $(x_1, x_2, x_3)$ be a coordinate system on $g^*$ determined by the dual basis $(e_1, e_2, e_3)$. The Lie-Poisson structure $\Lambda$ can be expressed in terms of the above global coordinate system as

$$\Lambda = (x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) \wedge \frac{\partial}{\partial x_3}.$$ 

Therefore, using Theorem 1 we get an explicit $*$-product (in fact, the Gutt $*$-product [8]).

**Example 2.** Let us consider the Lie group $SU(2)$, and let us choose

$$e_2 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

as a basis of its Lie algebra $su(2)$.

Let $Sp(1)$ be the group of unitary quaternions. We identify $Sp(1)$ and $SU(2)$ as Lie groups by means of

$$\psi: Sp(1) \rightarrow SU(2),$$

$$(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & x_1 - x_2 i \end{pmatrix}.$$ 

As usual, we denote by $R_g$ ($L_g$) the right translation (left translation) map. Let $r = e_3 \wedge e_4 \in \wedge^2 su(2)$; then the Iwasawa-Poisson-Lie structure $\pi$ on $SU(2)$ is defined by [10]

$$\pi(g) := dR_g r - dL_g r, \quad g \in SU(2).$$
The linearization (at the identity) of this Poisson structure is isomorphic to the book algebra, and therefore this Poisson structure can be considered as the non-linear version of that in Example 1. Let $X_i$ ($i = 2, 3, 4$) be the right invariant vector fields on $SU(2)$ corresponding to $e_i$. We define two vector fields on $SU(2)$ by setting

\[ T^1 := x_2 X_2 + x_3 X_3 + x_4 X_4, \]
\[ T^2 := (-2x_1) X_2 + (-2x_4) X_3 + (2x_3) X_4. \]

A straightforward computation shows that the Poisson structure $\pi$ on $SU(2)$ can be written as $\pi = T^1 \wedge T^2$. Hence, using Theorem 1 one can construct an explicit $*$-product on this Poisson manifold (the non-linear version of the one constructed in the previous example).

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