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AN UNCERTAINTY PRINCIPLE FOR CONVOLUTION OPERATORS ON DISCRETE GROUPS

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Abstract. Consider a discrete group G and a bounded self-adjoint convolution operator T on $l^2(G)$; let $\sigma(T)$ be the spectrum of T. The spectral theorem gives a unitary isomorphism U between $l^2(G)$ and a direct sum $\bigoplus_n L^2(\Delta_n, \nu)$, where $\Delta_n \subset \sigma(T)$, and ν is a regular Borel measure supported on $\sigma(T)$. Through this isomorphism T corresponds to multiplication by the identity function on each summand. We prove that a nonzero function $f \in l^2(G)$ and its transform Uf cannot be simultaneously concentrated on sets $V \subset G$, $W \subset \sigma(T)$ such that $\nu(W)$ and the cardinality of V are both small. This can be regarded as an extension to this context of Heisenberg's classical uncertainty principle.

1. Introduction

Roughly speaking *Heisenberg's uncertainty principle* says that a function $f \in L^2(\mathbb{R}^m)$ and its Fourier–Plancherel transform $\hat{f}$ cannot be both concentrated on small intervals. An extension of this principle was proved quite recently by D.L. Donoho and P.B. Stark ([DO–ST]) who replaced intervals with arbitrary Lebesgue measurable sets. This new formulation led to an extension of the principle in the context of locally compact abelian groups. More precisely K.T. Smith ([S]) proved the following:

Theorem 1.1 (Smith). *Let G be a locally compact abelian group and $\hat{G}$ its character group; denote by μ and ν the Haar measure of G and the Haar measure of $\hat{G}$ respectively (with any normalization). Let $V \subseteq G$ and $W \subseteq \hat{G}$ be measurable sets such that $\mu(V) \neq 0$, $\nu(W) \neq 0$. Suppose that, for some choice of ε and δ in $(0, \infty)$, there exists a nonzero function $f \in L^2(G)$ which is* ε–limited *to V and* δ–bandlimited *to W. That is,*

$$\|f - \chi_V f\|_2 \leq \varepsilon \|f\|_2 \tag{1–1}$$

and

$$\|\hat{f} - \chi_W \hat{f}\|_2 \leq \delta \|f\|_2. \tag{1–2}$$

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Then

$$\mu(V) \cdot \nu(W) \geq (1 - \varepsilon - \delta)^2. \tag{1–3}$$

A further extension of this result was obtained by J.A.Wolf ([W]), who considered the case of a *Gelfand pair* (G, K), that is, a locally compact group G together with a compact subgroup K such that the convolution algebra $L^1(K\backslash G/K)$ of left and right invariant integrable functions is commutative. Then (1–3) extends to right K–invariant functions. In this case, the rôle of the Haar measure on $\hat{G}$ is played by *Plancherel measure* on the set of equivalence classes of unitary K–invariant irreducible representations.

In this paper we apply the techniques of Donoho and Stark to extend Smith's inequality (1–3) to a different context. We consider an arbitrary *discrete* group and a right convolution operator on $l^2(G)$. That is, an operator T such that

$$Tf = f * \mu \tag{1–4}$$

where μ belongs to $l^1(G)$. We assume that μ is *symmetric* (that is, $\mu(x^{-1}) = \mu(x)$ for every x) and real. Under these hypotheses T is self–adjoint. Then, by the spectral theorem, $l^2(G)$ decomposes as a direct sum of cyclic subspaces; the restriction of T to each summand is equivalent to a multiplication operator on a suitable L^2 space. In this context an analogue of inequality (1–3) is still true if W is a Borel subset of $\sigma(T)$ and the Haar measure is replaced by a suitable diagonal coefficient of the *spectral measure* of T. In [STE], which was prompted by [W], the author proved a similar result for the special case in which G is assumed to be a finitely generated free group, and μ is supported on the set of the generators and their inverses.

In order to describe in greater detail the Hilbert space isomorphism which provides the relevant realization of $l^2(G)$, we shall review in the next section some basic facts about ordered representations.

2. Ordered representations of Hilbert spaces

Definition 2.1. Let T be a normal bounded linear operator on a separable Hilbert space $\mathcal{H}$; let $\sigma(T)$ be the spectrum of T. Consider a regular Borel measure ν, supported on $\sigma(T)$, and a decreasing sequence $\{\Delta_n\}$ of Borel subsets of $\sigma(T)$. Then a linear mapping

$$U : \mathcal{H} \longrightarrow \bigoplus_{n \in \mathbb{N}} L^2(\Delta_n, \nu)$$

is called an *ordered representation relative to* T when

1. U is unitary,
2. for every bounded Borel function f on $\sigma(T)$ and every $x \in \mathcal{H}$, $n \in \mathbb{N}$, the equality

$$\left(U\left(f(T)x\right)\right)_n(\lambda) = f(\lambda)\left(Ux\right)_n(\lambda) \tag{2–1}$$

holds ν–almost everywhere on Δ_n.

The following result is a consequence of the spectral theorem for normal operators.

Theorem 2.2 ([DS], X.5.10). *A separable Hilbert space $\mathcal{H}$ admits a spectral representation with respect to any normal bounded linear operator defined on it. Moreover, if $U_i : \mathcal{H} \longrightarrow \bigoplus_{n\in\mathbb{N}} L^2(\Delta_n^i, \nu_i)$ $(i = 1, 2)$ are two ordered representations relative to the same operator, then U_1 and U_2 are equivalent in the following sense:*

$$\begin{cases}\nu_1 \cong \nu_2, \\ \nu_1(\Delta_n^1 \triangle \Delta_n^2) = \nu_2(\Delta_n^1 \triangle \Delta_n^2) = 0 \qquad \textit{for every } n.\end{cases} \tag{2–2}$$

The crucial step in the proof of Theorem 2.2 is the following result.

Lemma 2.3 ([DS], X.5.8). *Let T be a normal bounded linear operator on a separable Hilbert space $\mathcal{H}$; let E be its spectral resolution. Then there exist a sequence $\{x_n\}$ of vectors in $\mathcal{H}$ and a decreasing sequence $\{\Delta_n\}$ of Borel subsets of $\mathbb{C}$ such that*

$$\mathcal{H} = \bigoplus_{k\in\mathbb{N}} \mathcal{H}(x_k) \tag{2–3}$$

(*where $\mathcal{H}(x_k)$ is the cyclic subspace* $\operatorname{span}(\{T^m(T^*)^r x_k : m, r \in \mathbb{N}\})$) *and, for every Borel set $\Delta \subset \mathbb{C}$ and $k \in \mathbb{N}$,*

$$(E(\Delta)x_n,\ x_n) = (E(\Delta \cap \Delta_n)x_1,\ x_1)\,. \tag{2–4}$$

Theorem 2.2 can then be deduced from Lemma 2.3 by letting

$$\nu(A) := \nu_{x_1}(A) := (E(A)x_1,\, x_1) = \chi_A(T) \tag{2–5}$$

for every Borel subset A of $\sigma(T)$.

Remark. Inspection of the proof of Lemma 2.3 shows that one can choose x_1 arbitrarily among the set of vectors such that the measure $(E(\cdot)x_1,\, x_1)$ is *basic*, that is,

$$(E(\cdot)y,\, y) \ll (E(\cdot)x_1,\, x_1) \qquad \text{for every } y \in \mathcal{H}; \tag{2–6}$$

this condition is certainly fulfilled if x_1 is cyclic for the commutator algebra of T (see [DIX] I.7.2).

3. Uncertainty principle for convolution operators

Recall that in our case T is the operator of right convolution by $\mu \in l^1(G)$ acting on $l^2(G)$. Clearly T commutes with translations and linear combinations of translates of δ_{1_G} are dense in $l^2(G)$. It follows from Lemma 2.3, and the remark which follows it, that we can choose once and for all an ordered representation $\mathcal{F}$ of $l^2(G)$, relative to T, such that $x_1 = \delta_{1_G}$. We denote by ν the corresponding measure on $\sigma(T)$. Note that, for any Borel measurable $A \subseteq \sigma(T)$, formula (2–5) gives

$$\nu(A) = (\chi_A(T)\delta_{1_G}, \delta_{1_G})_{l^2(G)} = (\chi_A(T)\delta_{1_G})\,(1_G);$$

moreover, for this choice of x_1, the map

$$\begin{aligned} C\,(\sigma(T)) &\longrightarrow l^2(G) \\ f &\longmapsto f(\mu) \end{aligned} \tag{3–1}$$

is unitary if $C\,(\sigma(T))$ is regarded as a subspace of $L^2\,(\sigma(T),\, d\nu)$.

Given $\mathcal{F}$, fix a finite set $V \subset G$ and a Borel set $W \subseteq \sigma(T)$. Define two projections P_V, Q_W on $l^2(G)$ in the following way:

$$P_V f = \chi_V f, \tag{3–2}$$

$$Q_W f = \mathcal{F}^{-1}\left(\chi_W \mathcal{F} f\right). \tag{3–3}$$

Thus a function $f \in l^2(G)$ is P_V–invariant if and only if it is supported on V; f is Q_W–invariant if and only if every component of its transform $\mathcal{F}f$ is supported on W. We will extend to our case, in terms of such projections, the concept of ε–limited and δ–bandlimited functions introduced in Theorem 1.1.

Definition 3.1. A function $f \in l^2(G)$ is *ε–limited* to V (resp. *δ–bandlimited* to W), for some ε, $\delta \in (0, 1)$, when

$$\|f - P_V f\|_2 \leq \varepsilon \|f\|_2 \qquad (\text{resp. } \|f - Q_W f\|_2 \leq \delta \|f\|_2). \tag{3–4}$$

Remark. The definition above is independent of the choice of $\mathcal{F}$ among the ordered representations of $l^2(G)$ relative to T. Indeed, by Theorem 2.2, all such representations are equivalent in the sense of (2–2). So, while Q_W seemingly depends on the choice of $\mathcal{F}$, the second inequality in (3–4) is unaffected when one replaces $\mathcal{F}$, in formula (3–3), by another ordered representation relative to T. This will also follow immediately from (3–5).

We can now state our result.

Theorem 3.2 (Uncertainty Principle)**.** *Consider V, W as above and suppose there exists a nonzero function $f \in l^2(G)$ which is simultaneously ϵ–limited to V and δ–bandlimited to W. Then*

$$|V| \cdot \nu(W) \geq (1 - \epsilon - \delta)^2$$

(*where $|V|$ denotes the cardinality of V*).

As in Donoho and Stark's theorem, the result will be obtained by an estimate of the norm of the operator $P_V Q_W$.

Lemma 3.3. *Let $V \subset G$ be a finite set and $W \subseteq \sigma(T)$ a Borel set. Define P_V, Q_W as above. Then*

$$\|P_V Q_W\|_{\mathcal{L}(l^2(G))} \leq (|V| \cdot \nu(W))^{1/2}.$$

Proof. Recall that $\nu(W) = (\chi_W(T)\delta_{1_G}, \delta_{1_G})_{l^2(G)}$. Since ν is a regular Borel measure, there exists a sequence $\{r_n\}$ of polynomials such that

$$r_n \to_{n\to\infty} \chi_W \qquad \nu\text{–a.e. on } \sigma(T)$$

and $\|r_n\|_{L^\infty}$ is uniformly bounded. Thus, by a well-known property of Borel functional calculus ([DS], Corollary X.2.8), we have:

$$r_n(T) \to_{n\to\infty} \chi_W(T) \qquad \text{strongly},$$

that is, for every $h \in l^2(G)$,

$$r_n(\mu) * h \to_{n\to\infty} \chi_W(T)h$$

where $r_n(\mu)$ is a convolution polynomial. For $h \in l^2(G)$ we get:

$$\begin{aligned}
Q_W h &= \mathcal{F}^{-1}\left(\chi_W \cdot \mathcal{F}h\right) \\
&= \mathcal{F}^{-1}\left(\bigoplus_{n\in\mathbb{N}} \left(\lambda \longmapsto \chi_W(\lambda)\left(\mathcal{F}h\right)_n(\lambda)\right)\right) \\
&= \mathcal{F}^{-1}\left(\mathcal{F}\left(\chi_W(T)h\right)\right) = \chi_W(T)h. \qquad (3\text{–}5)
\end{aligned}$$

This implies that, for $x \in G$,

$$\begin{aligned}
\left|\left(Q_W h\right)(x)\right| = \left|\left(\chi_W(T)h\right)(x)\right| &= \left|\left(\chi_W(T)h,\, \delta_x\right)_{l^2(G)}\right| \\
&= \lim_{n\to\infty} \left|\left(r_n(\mu) * h\right)(x)\right| \\
&\leq \overline{\lim}_{n\to\infty} \|r_n(\mu)\|_{l^2(G)} \cdot \|h\|_{l^2(G)} \qquad \text{by Hölder's inequality.}
\end{aligned}$$

Now isometry condition for the mapping (3–1) gives

$$\begin{aligned}
\left|\left(Q_W h\right)(x)\right| &\leq \overline{\lim}_{n\to\infty} \|r_n\|_{L^2(\sigma(T),\, d\nu)} \cdot \|h\|_{l^2(G)} \\
&= \|\chi_W\|_{L^2(\sigma(T),\, d\nu)} \cdot \|h\|_{l^2(G)} \qquad \text{by Lebesgue's Theorem} \\
&= \left(\nu(W)\right)^{1/2} \|h\|_{l^2(G)}
\end{aligned}$$

and, by definition of P_V,

$$\begin{aligned}
\|P_V Q_W h\|_2 &\leq \|\chi_V\|_2 \left(\nu(W)\right)^{1/2} \|h\|_{l^2(G)} \\
&= \left(|V| \cdot \nu(W)\right)^{1/2} \|h\|_{l^2(G)}\,.
\end{aligned}$$

□

Proof of Theorem 3.2. Let f be simultaneously ε–limited to V and δ–bandlimited to W. Then, by Lemma 3.3,

$$\begin{aligned}
\left(|V| \cdot \nu(W)\right)^{1/2} \cdot \|f\|_2 &\geq \|P_V Q_W f\|_2 \\
&\geq \left|\, \|P_V Q_W f - f\|_2 - \|f\|_2 \right| \\
&\geq \|f\|_2 - \varepsilon \|f\|_2 - \delta \|f\|_2
\end{aligned}$$

and the thesis follows since $\|f\|_2 \neq 0$. □

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